High-dimensional Gaussian fields
with isotropic increments seen through spin glasses

Anton Klimovsky†

Abstract
We study the free energy of a particle in (arbitrary) high-dimensional Gaussian random potentials with isotropic increments. We prove a computable saddle-point variational representation in terms of a Parisi-type functional for the free energy in the infinite-dimensional limit. The proofs are based on the techniques developed in the course of the rigorous analysis of the Sherrington-Kirkpatrick model with vector spins.

Keywords: Gaussian random fields; isotropic increments; random energy model; hierarchical replica symmetry breaking; Parisi Ansatz.

AMS MSC 2010: Primary 60K35, Secondary 82B44; 82D30; 60G15; 60G60; 60F10.
Submitted to ECP on August 26, 2011, final version accepted on April 26, 2012.

1 Introduction
Recently, considerable (renewed) attention in the theoretical physics literature has been devoted to Gaussian random fields with isotropic increments viewed as random potentials, see, e.g, the works by Fyodorov and Sommers [8], Fyodorov and Bouchaud [7], and references therein. In particular, it was heuristically argued in these works that Parisi’s theory of hierarchical replica symmetry breaking (Parisi Ansatz, cf. [11]) is applicable in this context. In the probabilistic context, these results provide rather sharp information about the extremes of the strongly correlated fields with high-dimensional correlation structures, which is a challenging area of probability theory [14, 4, 1, 2, 17, 18].

In this note, we initiate the rigorous derivation of the results of [8, 7]. We concentrate on the computation of the free energy of a particle subjected to arbitrary high-dimensional Gaussian random potentials with isotropic increments. In the high-dimensional limit, we derive a computable saddle-point representation for the free energy, which is similar to the Parisi formula for the Sherrington-Kirkpatrick (SK) model of a mean-field spin glass. Our proofs are based on the local comparison arguments for Gaussian fields with non-constant variance developed in [5], which are, in turn, based on the ideas of Guerra [9], Guerra and Toninelli [10], Talagrand [16] and Panchenko [12].

*Research supported in part by the European Commission (Marie Curie fellowship, project PIEF-GA-2009-251200); by DFG and NWO through the bilateral Dutch-German research group “Random Spatial Models from Physics and Biology”. The author was a guest of the Hausdorff Research Institute for Mathematics (Bonn) in the Fall of 2010 (Junior Trimester Program on Stochastics).
†EURANDOM, Eindhoven University of Technology, The Netherlands. E-mail: ak@aklimovsky.net
Gaussian fields with isotropic increments seen through spin glasses

This note is organized as follows. We state our results in Section 2. The proofs are given in Sections 3 and 4. In Section 5, we give an outlook and announce some important consequences of the results of this note. In the Appendix, we provide some complementary information for the reader’s convenience.

2 Setup and main results

Consider the Gaussian random field with isotropic increments $X = X_N = \{X_N(u) : u \in \mathbb{R}^N\}, N \in \mathbb{N}$. The adjective “isotropic” means here that the law of the increments of the field $X$ is invariant under rigid motions (= translations and rotations) in $\mathbb{R}^N$. We are interested in the case $N \gg 1$ and in the case of strongly correlated fields with high-dimensional correlation structure. Therefore, we assume that the field $X_N$ satisfies

$$
\mathbb{E} [(X_N(u) - X_N(v))^2] = D \left( \frac{1}{N} \|u - v\|_2^2 \right) =: D_N(\|u - v\|_2^2), \quad u, v \in \mathbb{R}^N,
$$

(2.1)

where $\| \cdot \|_2$ denotes the Euclidean norm on $\mathbb{R}^N$ and the correlator $D : \mathbb{R}_+ \to \mathbb{R}_+$ is any admissible function. Complete characterization of all correlators $D$ that are admissible in (2.1), for all $N$, is known, see Theorem A.1. Note that the law of the field $X_N$ is determined by (2.1) only up to an additive shift by a Gaussian random variable. In what follows, without loss of generality, we assume that $X_N(0) = 0$.

We are interested in the asymptotic behavior of the extremes of the random field $X_N$ on the sequence of the particle state spaces $S_N \subset \mathbb{R}^N$ as $N \uparrow +\infty$. The state spaces are assumed to be equipped with a sequence of a priori reference measures $\{\mu_N \in \mathcal{M}_{\text{finite}}(S_N) | N \in \mathbb{N}\}$. We now define the main quantities of interest in this work. Consider the partition function

$$
Z_N(\beta) := \int_{S_N} \mu_N(du) \exp \left( \beta \sqrt{N} X_N(u) \right), \quad \beta \in \mathbb{R}.
$$

(2.2)

We view (2.2) as an exponential functional of the field $X_N$, which is parametrized by the inverse temperature $\beta$. Heuristically, for large $\beta$ (i.e., $\beta \uparrow +\infty$), the maxima of the field $X_N$ give substantial contribution to the integral (2.2). The $N$-scalings in (2.2), (2.1) and the “size” of $S_N$ are tailored for studying the large-$N$ limit of the random log-partition function:

$$
p_N(\beta) := \frac{1}{N} \log Z_N(\beta), \quad \beta \in \mathbb{R}.
$$

(2.3)

For comparison with the theoretical physics literature, let us note that there one conventionally substitutes $\beta \mapsto -\beta$ in (2.2) (this has no effect on the distribution of $Z_N$ due to the symmetry of the centered Gaussian distribution of the field $X_N$), and considers instead of (2.3) the free energy

$$
f_N(\beta) := -\frac{1}{\beta} p_N(\beta), \quad \beta \in \mathbb{R}_+.
$$

(2.4)

Assumptions. Informally, we require the particle state space $S_N$ to have an exponentially growing in $N$ volume (respectively, cardinality, if $S_N$ is discrete). In particular, using physics parlance, this assures that the entropy competes with the energy (given by the random field $X_N$) on the same scale. More formally, we assume

$$
S_N := S^N, \quad S \subset \mathbb{R}.
$$

(2.5)

Let $\mu \in \mathcal{M}_{\text{finite}}(S)$ be such that the origin is contained in the interior of the convex hull of the support of $\mu$. Define $\mu_N := \mu \otimes N \in \mathcal{M}_{\text{finite}}(S_N)$. A canonical example is the
Gaussian fields with isotropic increments seen through spin glasses

discrete hypercube $S_N := \{-1;1\}^N$ equipped with the uniform a priori measure, i.e.,
$\mu\{u\} := 2^{-N}$, for all $u \in S_N$.

**Parisi-type functional.** To formulate our results on the limiting log-partition function,
we need the following definitions. Given $r \in \mathbb{R}_+$, consider the space of the functional
order parameters

$$
\mathcal{X}(r) := \{x : [0;r] \to [0;1] \mid x \text{ is non-decreasing càdlàg}, x(0) = 0, x(r) = 1\},
$$

(2.6)

It is convenient to work with the space of the discrete order parameters

$$
\mathcal{X}'_n(r) := \{x \in \mathcal{X}(r) \mid x \text{ is piece-wise constant with at most } n \text{ jumps}\}.
$$

(2.7)

Let us denote the effective size of the particle state space by

$$
d := \sup_N \left(\frac{1}{N} \sup_{u \in S_N} \|u\|_2^2\right).
$$

(2.8)

For what follows, it is enough to assume that $r \in [0;d]$ in (2.6). Note that, in case (2.5),
$d = \sup_{u \in S} u^2$.

Now, let us define the non-linear functional that appears in the variational formula
of our main result. We do it in three steps:

1. Given large enough $M \in \mathbb{R}_+$, define the regularized derivative $D^{r,M} : \mathbb{R}_+ \to \mathbb{R}$ of
the correlator $D$ as

$$
D^{r,M}(r) := \begin{cases}
D'(r), & r \in [1/M; +\infty), \\
M, & r \in [0;1/M).
\end{cases}
$$

(2.9)

Given $r, M \in \mathbb{R}_+$, define the function $\theta^{(M)}_r : [-r;r] \to \mathbb{R}$ as

$$
\theta^{(M)}_r(q) := q D^{r,M}(2(r-q)) + \frac{1}{2} D(2(r-q)), \quad q \in [-r;r].
$$

(2.10)

2. Given $r \in \mathbb{R}_+$, $x \in \mathcal{X}(r)$ and the (sufficiently regular) boundary condition $h : \mathbb{R} \to \mathbb{R}$,
consider the semi-linear parabolic Parisi’s terminal value problem:

$$
\begin{align*}
\partial_t f(y,q) + \frac{1}{2} D^{r,M}(2(r-q)) \left(\partial^2_{qq} f(y,q) + x(q) (\partial_q f(y,q))^2\right) &= 0, \quad (y,q) \in \mathbb{R} \times (0,r), \\
f(y,1) &= h(y),
\end{align*}
$$

(2.11)

Let $f^{(M)}_{r,x,h} : [0;1] \times \mathbb{R}_+ \to \mathbb{R}$ be the unique solution of (2.11). Solubility of the
Parisi terminal value problem (2.11), its relation to the Hamilton-Jacobi-Bellman
equations and stochastic control problems is discussed in a more general multi-
dimensional context in [5, Section 6].

3. Given the family of the (sufficiently regular for (2.11) to be solvable) boundary
conditions

$$
g := \{g_\lambda : \mathbb{R} \to \mathbb{R} \mid \lambda \in \mathbb{R}\},
$$

(2.12)

and given $r \in [0;d]$, define the local Parisi functional $\mathcal{P}(\beta, r, g) : \mathcal{X}(r) \to \mathbb{R}$ as

$$
\mathcal{P}(\beta, r, g)[x] := \lim_{M \to +\infty} \inf_{\lambda \in \mathbb{R}} \left( \int_{\mathbb{R}} f^{(M)}_{r,x,g}\lambda(0,0) - \lambda r \right) - \frac{\beta^2}{2} \int_0^1 x(q) d\theta^{(M)}_r(q), \quad x \in \mathcal{X}(r).
$$

(2.13)

In (2.13), the integral with respect to $\theta^{(M)}_r$ is understood in the Lebesgue-Stieltjes
sense.
Main results. Let us start by recording the basic convergence result for the log-partition function.

**Theorem 2.1** (Existence of the limiting free energy). For any $\beta > 0$, the large $N$-limit of the log-partition function exists and is a.s. deterministic:

$$p_N(\beta) \xrightarrow{N \to \infty} p(\beta), \text{ almost surely and in } L^1. \quad (2.14)$$

In addition, for any $N \in \mathbb{N}$, the following concentration of measure inequality holds

$$\mathbb{P}\{|p_N(\beta) - \mathbb{E}[p_N(\beta)]| > t\} \leq 2 \exp\left(-\frac{Nt^2}{4D(d)}\right), \quad t \in \mathbb{R}_+. \quad (2.15)$$

The main result of this work is the following variational representation for the limiting log-partition function in terms of the Parisi functional (2.13).

**Theorem 2.2** (Free energy variational representation, comparison with cascades). Assume (2.5). Let the family of boundary conditions (2.12) be defined as

$$g_\lambda(y) := \log \int_{\mathcal{S}} \mu(du) \exp(\beta uy + \lambda u^2), \quad y \in \mathbb{R}. \quad (2.16)$$

Then, for all $\beta \in \mathbb{R}$,

$$p(\beta) := \sup_{r \in [0; d]} \inf_{x \in \mathcal{X}(r)} \mathcal{P}(\beta, r, g)[x] - \mathcal{R}(r)[x], \text{ almost surely and in } L^1, \quad (2.17)$$

where the remainder term $\mathcal{R}(r) : \mathcal{X}(r) \to \mathbb{R}_+$ is a functional on $\mathcal{X}(r)$ taking non-negative values (see (4.23) for the definition).

The sign-definiteness of the remainder term $\mathcal{R}(r)$ immediately implies the following bound.

**Corollary 2.3** (Log-partition function upper bound). For all $\beta \in \mathbb{R}$,

$$p(\beta) \leq \sup_{r \in [0; d]} \inf_{x \in \mathcal{X}(r)} \mathcal{P}(\beta, r, g)[x], \text{ almost surely.} \quad (2.18)$$

**Remark 2.4.** In the case (A.4), the field (2.20) has a feature, which is not within the assumptions typically found in the literature [9, 10, 16, 15, 12]: the correlator $D$ is not of class $C^1$, namely, $D$ can have a singular derivative at 0. To deal with the singularity, we need a regularization procedure, cf. (2.9) and (2.13).

Heuristics. It is natural to ask the following questions: Why is Parisi’s theory of hierarchical replica symmetry breaking [11] (which is usually behind the functionals of the type (2.13)) applicable to Gaussian fields with isotropic increments satisfying (2.1)? Where are the “interacting spins” in the present context?

A hint is given by the following observation. Define

$$\langle u, v \rangle_N := \frac{1}{N} \sum_{i=1}^{N} u_i v_i, \quad u, v \in \mathbb{R}^N. \quad (2.19)$$

Let us fix $r \in [0; d]$. By (A.6), the restriction of the field $X_N$ with isotropic increments to a sphere with radius $r$ centered at the origin, leads to the mixed $p$-spin spherical SK model (cf. [15]) with the following covariance structure

$$\mathbb{E}[X_N(u)X_N(v)] = D(r) - \frac{1}{2}D(2(r - \langle u, v \rangle_N)) =: G_r(\langle u, v \rangle_N), \quad \|u\|_2^2 = \|v\|_2^2 = rN, \quad (2.20)$$
Given a random field \( C \) and the covariance structure of the field state space as (2.19) only, and, moreover, the correlator \( D \) the representation (A.2). Therefore, [10, Theorem 1] is applicable with by function \( \{N_k\} \) in independent copies then readily follows. Consider \( \{N_k\} \) we prove the convergence of (2.14) along the subsequences \( N \), case is covered by [10, Theorem 1]. Indeed, in that case, the covariance of the field is smooth way. Therefore, using the terminology of Remark A.2, only the short-range potential depends on the scalar product (overlap) of the particle configurations in a smooth way. Therefore, using the terminology of Remark A.2, only the short-range case is covered by [10, Theorem 1]. Indeed, in that case, the covariance of the field \( X_N \) satisfies (A.1), where the function \( B \) is analytic and convex, which follows from the representation (A.2). Therefore, [10, Theorem 1] is applicable with \( Q_N(u, v) := N^{-1}||u - v||_2^2 \), for \( u, v \in \mathbb{R}^N \).

In the long-range case (A.6), the proof of the [10] requires some care, because the covariance structure of the field \( X_N \) (cf. (A.6)) does not depend on the scalar product (2.19) only, and, moreover, the correlator \( D \) is not of class \( C^1 \) (cf. Remark 2.4). For the reader’s convenience, we now retrace the main parts of this argument. Given \( N \in \mathbb{N} \), we prove the convergence of (2.14) along the subsequences \( \{N_K := N^K\} \). Convergence along other subsequences then readily follows. Consider \( N \) independent copies \( \{X^{(k)}_{N_{K-1}} \mid k \in [N]\} \) of the field \( X_{N_{K-1}} \). Given an interval \( V \subset [0, d] \), define the localized state space as

\[
S_N(V) := \{u \in S_N : ||u||_2^2 \leq N \cdot V\}.
\]

Given a random field \( C = \{C_N(u) \mid u \in \mathbb{R}^N\} \), denote the corresponding local partition function by

\[
Z_N(\beta, V)[C] := \int_{S_N(V)} \mu_N(du) \exp \left( \beta \sqrt{N} C_N(u) \right).
\]

In what follows, for \( u \in \mathbb{R}^N, v \in \mathbb{R}^M \), we denote by \( u \parallel v \) the vector in \( \mathbb{R}^{N+M} \) obtained by concatenation of \( u \) and \( v \). Let us define the Gaussian field \( Y \) as

\[
Y_{N,K}(u^{(1)} \parallel u^{(2)} \parallel \ldots \parallel u^{(N)}) := \frac{1}{\sqrt{N}} \sum_{k=1}^{N} X^{(k)}_{N_{K-1}}(u), \quad u^{(k)} \in \mathbb{R}^{N_{K-1}}, \quad k \in [N].
\]
Let us define $D$. Using (A.6), the smoothness of the correlator $\rho^{\ell}$ assures that $D$, $u, v \in [N]$. Let us note that the product structure (3.7) and independence (3.4) imply

$$Z_{Nk}(\beta, V) \geq \tilde{Z}_{Nk}(\beta, V).$$

The product structure (3.7) and independence (3.4) imply

$$\frac{1}{N_k} \mathbb{E} \left[ \log \tilde{Z}_{Nk}(\beta, V)[Y_{Nk}] \right] = \frac{1}{N_k} \mathbb{E} \left[ \log \prod_{k=1}^{N} Z_{Nk-1}(\beta, V)[X_{Nk-1}] \right] = \frac{1}{N_k-1} \mathbb{E} \left[ \log Z_{Nk-1}(\beta, V)[X_{Nk-1}] \right].$$

For $\varepsilon > 0$, set $V_i := [i \varepsilon; (i + 1)\varepsilon]$, $i \in \mathbb{N}$. By the Gaussian comparison formula [5, Proposition 2.5],

$$\frac{1}{N_k} \mathbb{E} \left[ \log \tilde{Z}_{Nk}(\beta, V_i)[X_{Nk}] \right] = \frac{1}{N_k} \mathbb{E} \left[ \log Z(\beta, V_i)[Y_{Nk}] \right] + \frac{\beta^2}{2} \int_0^1 dt \int_{\tilde{S}_{Nk}(V_i)} \tilde{G}_{Nk}(t)(du) \int_{\tilde{S}_{Nk}(V_i)} \tilde{G}_{Nk}(t)(dv) \left[ \text{Var} \left( X_{Nk}(u) - \frac{1}{N} \sum_{k=1}^{N} \text{Var} \left( X_{Nk-1}(u^{(k)}) \right) \right) \right].$$

where $\tilde{G}_{Nk}(t) \in \mathcal{M}_1(\tilde{S}_{Nk})$ is the interpolating Gibbs measure with the density

$$\frac{d\tilde{G}_{Nk}(t)}{d\mu_{Nk}}(u) = \exp \left( \beta \sqrt{N_k} \left( \sqrt{1} X_{Nk}(u) + \sqrt{1 - t} Y_{Nk}(u) \right) \right), \quad u \in \tilde{S}_{Nk}(V_i).$$

Using (A.6), the smoothness of the correlator $D$ on $(0; +\infty)$, the fact that $D$ is non-decreasing, continuous at 0, and $D(0) = 0$, we get

$$\sup_{u \in \tilde{S}_{Nk}(V_i)} \left| \text{Var} \left( X_{Nk}(u) - \frac{1}{N} \sum_{k=1}^{N} \text{Var} \left( X_{Nk-1}(u^{(k)}) \right) \right) \right| \leq D(\varepsilon), \quad i \in \mathbb{N}. \quad (3.12)$$

As for the covariance terms in (3.10), the concavity of the correlator $D$ (cf., Remark A.3) and the explicit covariance representation (A.6) assure that

$$\sup_{u, v \in \tilde{S}_{Nk}(V_i)} \left( \text{Cov} \left[ X_{Nk}(u), X_{Nk}(v) \right] - \frac{1}{N} \sum_{k=1}^{N} \text{Cov} \left[ X_{Nk-1}(u^{(k)}), X_{Nk-1}(v^{(k)}) \right] \right) \leq D(\varepsilon). \quad (3.13)$$
Gaussian fields with isotropic increments seen through spin glasses

Therefore, combining (3.8), (3.9), (3.10), (3.12) and (3.13) we get

\[
\frac{1}{N_K} \mathbb{E} \left[ \log Z_{N_K}(\beta, V_i)[X_{N_K}] \right] \geq \frac{1}{N_{K-1}} \mathbb{E} \left[ \log Z_{N_{K-1}}(\beta, V_i)[X_{N_{K-1}}] \right] - CD(\varepsilon), \quad i \in \mathbb{N}.
\]

(3.14)

The proof is finished by using the concentration inequality (2.15) to remove the localization in (3.14), as in [10, Theorem 1].

\[\square\]

4 Comparison with cascades

In this section, we prove Theorem 2.2. The proof follows the strategy that was previously implemented in [5, Section 5]. The appearance of the auxiliary structures below can be made more transparent by the “cavity” arguments, as is done in the seminal work of Aizenman et al. [3].

4.1 Auxiliary structures

Consider the auxiliary index space \( A = \mathcal{A}_{n} := \mathbb{N}^n, n \in \mathbb{N} \). Let us define the projection operator \( A \ni \alpha \mapsto [\alpha]_k := (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k, \text{ for } k \in [n] \). It is useful to treat the elements of \( A \) as the leaves of the tree of depth \( n \). We use the convention that \( [\alpha]_0 = \emptyset \), where \( \emptyset \) denotes the root of the tree. Given a leaf \( \alpha \in \mathcal{A} \), we think of \([\alpha_k] : k \in [n]\) as of the sequence of branches connecting the leaf \( \alpha \) to the root \( \emptyset \). We equip \( \mathcal{A} \) with a random measure called Ruelle’s probability cascade (RPC). Let us briefly recall the construction of the RPC, see, e.g., [3] for more details. Note that each function \( x \in \mathcal{X}^*_n(r) \) can be represented as

\[
x(q) = \sum_{i=0}^{n} x_i \mathbb{I}_{[q_i,q_{i+1})}(r),
\]

(4.1)

where \( \bar{x} = \{x_k\}_{k=0}^{n+1} \) and \( \bar{q} = \{q_k\}_{k=0}^{n+1} \) satisfy

\[
0 =: x_0 < x_1 < \ldots < x_n < x_{n+1} := 1,
0 =: q_0 < q_1 < \ldots < q_n < q_{n+1} := r.
\]

(4.2)

To define the RPC, we need only the sequence \( \bar{x} \) as in (4.2). Consider the family of the independent (inhomogeneous) Poisson point processes \( \{\xi_k, [\alpha]_k, \alpha \in \mathcal{A}, k \in [n]\} \) on \( \mathbb{R}_+ \) with intensity

\[
R_+ \ni t \mapsto x_k t^{-x_k-1} \in \mathbb{R}_+, \quad k \in [1; n] \cap \mathbb{N}.
\]

(4.3)

To each branch \( [\alpha]_k, \alpha \in \mathcal{A}, k \in [n] \) of the tree we associate the position of the \( \alpha_k \)-th atom (e.g., according to the decreasing enumeration) of the Poisson point process \( \xi_k, [\alpha]_{k-1} \). The RPC is the point process \( \text{RPC} = \text{RPC}(x_1, \ldots, x_n) := \sum_{\alpha \in \mathcal{A}} \delta_{\text{RPC}(\alpha)} \), where \( \text{RPC}(\alpha), \alpha \in \mathcal{A} \) is obtained by multiplying the random weights attached to the branches along the path connecting the given leaf \( \alpha \in \mathcal{A} \) with the root of the tree:

\[
\text{RPC}(\alpha) := \prod_{k=1}^{n} \xi_k, [\alpha]_{k-1}(\alpha_k).
\]

(4.4)

Since \( \sum_{\alpha \in \mathcal{A}} \text{RPC}(\alpha) < \infty \), the RPC can be thought of as a finite random measure on \( \mathcal{A} \) with (abusing the notation) \( \text{RPC}([\alpha]) := \text{RPC}(\alpha), \alpha \in \mathcal{A} \). To lighten the notation, we keep the dependence of the RPC on \( \bar{x} \) implicit.
Recall (3.2). Given the sequence \( \bar{x} \) as in (4.2) and any suitable Gaussian field \( C := \{ C(u, \alpha) \mid u \in S_N, \alpha \in \mathcal{A} \} \), let us define the extended log-partition functional \( \Phi_N(\bar{x}, V) \) as

\[
\Phi_N(\bar{x}, V)[C] := \frac{1}{N} \mathbb{E} \left[ \log \left( \int_{S_N(V)} \mu(du) \int_{\mathcal{A}} \text{RPC}(da) \exp \left( \beta \sqrt{N} C(u, \alpha) \right) \right) \right],
\]

(4.5)

where the RPC is induced by \( \bar{x} \).

Let us use the sequence \( \bar{q} = \{ q_k \}_{k=0}^{n+1} \) as in (4.2) to construct the Gaussian cavity fields indexed by \( S_N \times \mathcal{A} \). To this end, define the lexicographic overlap between the configurations \( \alpha^{(1)}, \alpha^{(2)} \in \mathcal{A} \) as

\[
l(\alpha^{(1)}, \alpha^{(2)}) := \begin{cases} 0, & \text{max} \{ k \in [N] : [\alpha^{(1)}]_k = [\alpha^{(2)}]_k \}, \\ \alpha^{(1)} \neq \alpha^{(2)}, & \text{otherwise.} \end{cases}
\]

(4.6)

Let us define (slightly abusing the notation) the lexicographic overlap \( q : \mathcal{A}^2 \to [0; 1] \) as

\[
q(\alpha^{(1)}, \alpha^{(2)}) := q(\alpha^{(1)}, \alpha^{(2)})
\]

(4.7)

Given \( \bar{q} \) as in (4.2), the cavity field is the Gaussian field \( A = A^{(M)}_N = \{ A_N(u, \alpha) \mid u \in S_N, \alpha \in \mathcal{A} \} \) such that

\[
\text{Cov} \left[ A^{(M)}(u, \alpha^{(1)}), A^{(M)}(v, \alpha^{(2)}) \right] = D^{l,M} \left( 2(r - q(\alpha^{(1)}, \alpha^{(2)})) \right) \langle u, v \rangle_N, \quad \alpha^{(1)}, \alpha^{(2)} \in \mathcal{A}, \quad u, v \in S_N.
\]

(4.8)

The existence of the cavity field \( A \) is guaranteed by the following result.

**Lemma 4.1** (Existence of the cavity field). For any sequence \( q \) as in (4.2) and large enough \( M \in \mathbb{R}_+ \), there exists the unique (in distribution) Gaussian field satisfying (4.8).

**Proof.** Since the distribution of the Gaussian field is completely identified by the covariance, the uniqueness follows once we prove the existence. For this purpose, we first construct the Gaussian field \( a = \{ a^{(M)}(\alpha) \}_{\alpha \in \mathcal{A}} \) with

\[
\text{Cov} \left[ a^{(M)}(\alpha^{(1)}), a^{(M)}(\alpha^{(2)}) \right] = D^{l,M} \left( 2(r - q(\alpha^{(1)}, \alpha^{(2)})) \right), \quad \alpha^{(1)}, \alpha^{(2)} \in \mathcal{A}.
\]

(4.9)

To construct the field \( a^{(M)} \) explicitly, we define

\[
m_k := D^{l,M}(2(r - q_k)), \quad k \in [n + 1].
\]

(4.10)

The representations (A.3) and (A.4), guarantee that the sequence (4.10) is non-decreasing. Therefore, we can set

\[
a^{(M)}(\alpha) := \sum_{k=1}^{n} (m_{k+1} - m_k)^{1/2} g^{(k)}_{\alpha|k}, \quad \alpha \in \mathcal{A},
\]

(4.11)

where \( \{ g^{(k)}_{\alpha|k} \mid \alpha \in \mathcal{A}, \ k \in [n] \} \) are i.i.d. standard normal random variables. A straightforward check shows that the covariance structure of (4.11) satisfies (4.9).

To finish the construction, for \( i \in [N] \), let \( a^{(M)}_i = \{ a_i^{(M)}(\alpha) \}_{\alpha \in \mathcal{A}} \) be the i.i.d. copies of the field \( a^{(M)} = \{ a^{(M)}(\alpha) \}_{\alpha \in \mathcal{A}} \). Define

\[
A^{(M)}_N(u, \alpha) := \frac{1}{\sqrt{N}} \sum_{i=1}^{N} a^{(M)}_i(\alpha) u_i, \quad u \in S_N, \quad \alpha \in \mathcal{A}.
\]

(4.12)

An inspection shows that the field (4.12) satisfies (4.8). \( \square \)
4.2 Interpolation

In this section, we shall apply Guerra’s comparison scheme (cf. [9]) to the Gaussian field with isotropic increments satisfying (2.1). To this end, we restrict the state space of a particle to a thin spherical layer. This assures that the variance of the field \( X_N \) does not change much. We refer to this procedure as localization. Then, we interpolate between the field of interest \( X_N \) and the cavity field (4.12) and compare the corresponding local log-partition functions. We use the auxiliary structures from Section 4.1.

Given \( x \in \mathcal{X}_n^{(r)} \) and large enough \( M \in \mathbb{R}_+ \), let us consider the following interpolating field on the extended configuration space \( S_N \times \mathcal{A} \)

\[
H_t^{(M)}(u, \alpha) := \sqrt{t}X_N(u) + \sqrt{1-t}A_N^{(M)}(u, \alpha), \quad t \in [0; 1], \quad u \in S_N, \quad \alpha \in \mathcal{A},
\]

where \( A_N^{(M)} \) is the cavity field with (4.8). In the usual way, the field (4.13) induces the local log-partition function

\[
\varphi_N^{(M)}(t, x, V) := \Phi_N(x, V)[H_t], \quad V \subset [0; d], \quad x \in \mathcal{X}_n^{(r)}.
\]

At the end-points of the interpolation, we obtain

\[
\varphi_N^{(M)}(0, x, V) = \Phi_N(\bar{x}, V)[A^{(M)}], \quad \text{and} \quad \varphi_N^{(M)}(1, x, V) = \Phi_N(\bar{x}, V)[X] =: \rho_N(\beta, V).
\]

The idea is that \( \Phi_N(\bar{x}, V)[A^{(M)}] \) is computable due to the properties of the RPC and the hierarchical structure of the cavity field. Let us now disintegrate the Gibbs measure on \( V \times \mathcal{A} \) induced by (4.13) into two Gibbs measures acting on \( V \) and \( \mathcal{A} \) separately. To this end, we define the correspondent (random) local free energy on \( V \) as follows

\[
\psi_N^{(M)}(t, x, \alpha, V) := \log \int_{S_N(V)} \exp \left[ \beta \sqrt{N} H_t^{(M)}(u, \alpha) \right] d\mu^\otimes N(u), \quad \alpha \in \mathcal{A}.
\]

For \( \alpha \in \mathcal{A} \), let us define the (random) local Gibbs measure \( G_N(t, x, \alpha, V) \in \mathcal{M}_1(S_N) \) by specifying its density with respect to the a priori distribution as

\[
\frac{dG_N^{(M)}(t, x, \alpha, V)}{d\mu^\otimes N}(u) = 1_{S_N(V)}(u) \exp \left[ \beta \sqrt{N} H_t^{(M)}(u, \alpha) - \psi_N^{(M)}(t, x, \alpha, V) \right], \quad u \in S_N.
\]

Let us define the re-weighting of the RPC by means of the local free energy (4.16)

\[
\widetilde{\text{RPC}}(\alpha) := \text{RPC}(\alpha) \exp \left( \psi_N^{(M)}(t, x, V, \alpha) \right), \quad \alpha \in \mathcal{A}.
\]

Let us also define the normalization operation \( \mathcal{N} : \mathcal{M}_{\text{finite}}(\mathcal{A}) \to \mathcal{M}_1(\mathcal{A}) \) as

\[
\mathcal{N}(\eta)(\alpha) := \frac{\eta(\alpha)}{\sum_{\alpha' \in \mathcal{A}} \eta(\alpha')}, \quad \alpha \in \mathcal{A}, \quad \eta = (\eta_\alpha)_{\alpha \in \mathcal{A}} \in \mathcal{M}_{\text{finite}}(\mathcal{A}).
\]

We introduce the local Gibbs measure \( G_N^{(M)}(t, x, V) \in \mathcal{M}_1(V \times \mathcal{A}) \) as follows. We equip \( V \times \mathcal{A} \) with the product topology between the Borel topology on \( V \) and the discrete topology on \( \mathcal{A} \). For any measurable \( U \subset V \times \mathcal{A} \), let us put

\[
G_N^{(M)}(t, x, V)[U] := \mathcal{N}(\widetilde{\text{RPC}}(\alpha)) G_N^{(M)}(t, x, \alpha, V)[v \in V \mid (v, \alpha) \in U].
\]

Let us define the remainder term as

\[
\mathcal{R}_N^{(M)}(t, V)[x] := \frac{\beta^2}{2} \mathbb{E} \left[ \int G_N^{(M)}(t, x, V)(du, da^{(1)}) \int G_N^{(M)}(t, x, V)(dv, da^{(2)}) \left( \frac{1}{2} \left( D(2(r - q(\alpha^{(1)}, \alpha^{(2)}))) - D(2(r - \langle u, v \rangle_N)) \right) - D^\cdot M(2(r - q(\alpha^{(1)}, \alpha^{(2)}))(q(\alpha^{(1)}, \alpha^{(2)})) - \langle u, v \rangle_N) \right) \right].
\]
Gaussian fields with isotropic increments seen through spin glasses

Given $r \in (0; d]$, let us denote

$$V_{\varepsilon} := (r - \varepsilon; r + \varepsilon).$$  (4.22)

Define the local remainder term as

$$\mathcal{R}^{(M)}(r)[x] := \lim_{\varepsilon \downarrow +0} \lim_{N \uparrow +\infty} \int_{0}^{1} \mathcal{R}_{N}^{(M)}(t, V_{\varepsilon})dt, \quad x \in X'_{n}(r).$$  (4.23)

The main step in the proof of Theorem 2.2 is the following.

**Lemma 4.2** (Comparison with cascades). Given $r \in (0; d]$, for any $x \in X'_{n}(r)$, as $\varepsilon \downarrow +0$, and $M \uparrow +\infty$,

$$\varphi^{(M)}_{N}(t, x, V_{\varepsilon}) = -\mathcal{R}^{(M)}(r)[x] - \frac{\beta^{2}}{2} \sum_{k=1}^{n} x_{k} \left( \theta^{(M)}_{r}(q_{k+1}) - \theta^{(M)}_{r}(q_{k}) \right) + O(\varepsilon) + O(1/M),$$  (4.24)

where

$$\mathcal{R}^{(M)}(r)[x] \geq 0.$$  (4.25)

**Proof.** Fix some $r \in (0; d]$. Using the notation (2.21) and smoothness of $D$ on $(0; +\infty)$, we have

$$\text{Var} \, X(u) = G_{r}(r) + O(\varepsilon), \quad \text{Var} \, A(u, \alpha) = rG'_{r}(r) + O(\varepsilon), \quad u \in V_{\varepsilon}, \quad \alpha \in A.$$  (4.26)

and

$$\text{Cov} \left[ X(u), X(v) \right] = G_{r}([u, v]_{N}),$$  (4.27)

$$\text{Cov} \left[ A(u, \alpha^{(1)}), A(v, \alpha^{(2)}) \right] = G'_{r}(q(\alpha^{(1)}), \alpha^{(2)})(u, v)_{N}.$$  (4.28)

Applying the abstract Gaussian interpolation formula (see, e.g., [5, Proposition 2.5]) to the field $X_{N}$ and the cavity field (4.12), we obtain

$$\frac{\partial}{\partial t} \varphi^{(M)}(t, x, V_{\varepsilon}) = \frac{\beta^{2}}{2} \mathbb{E} \left[ \int G_{N}(t, x, V)(du, d\alpha^{(1)}) \int G_{N}(t, x, V)(dv, d\alpha^{(2)}) \left( \text{Var} \, X(u) - \text{Var} \, A(u, \alpha) - \text{Cov} \left[ X(u), X(v) \right] + \text{Cov} \left[ A(u, \alpha^{(1)}), A(v, \alpha^{(2)}) \right] \right) \right] + O(\varepsilon).$$  (4.29)

Using (4.26) and (4.27), we get

$$\text{Var} \, X(u) - \text{Var} \, A(u, \alpha) - \text{Cov} \left[ X(u), X(v) \right] + \text{Cov} \left[ A(u, \alpha^{(1)}), A(v, \alpha^{(2)}) \right]$$

$$= G_{r}(r) - rG'_{r}(r) - \left( G_{r}(q(\alpha^{(1)}), \alpha^{(2)}) - q(\alpha^{(1)}), \alpha^{(2)})G'_{r}(q(\alpha^{(1)}), \alpha^{(2})) \right)$$

$$- \left[ G_{r}([u, v]_{N}) - G_{r}(q(\alpha^{(1)}), \alpha^{(2)}) - G'_{r}(q(\alpha^{(1)}), \alpha^{(2)})([u, v]_{N} - q(\alpha^{(1)}), \alpha^{(2})) \right].$$  (4.30)

Comparing (2.21) and (2.10), we note

$$G_{r}(q) - sG'_{r}(q) = D(r) + \theta_{r}(q), \quad q \in \mathbb{R}_{+}.$$  (4.31)

We have (cf. the proof of [5, Lemma 5.2])

$$\mathbb{E} \left[ \int \mathcal{G}^{(M)}_{N}(t, x, V_{\varepsilon})(du, d\alpha^{(1)}) \int \mathcal{G}^{(M)}_{N}(t, x, V_{\varepsilon})(dv, d\alpha^{(2)})(\theta_{r}(r) - \theta_{r}(q(\alpha^{(1)}), \alpha^{(2)))) \right]$$

$$= \mathbb{E} \left[ \int \mathcal{N}(\mathbb{R}PC)(du^{(1)}) \int \mathcal{N}(\mathbb{R}PC)(dv^{(2)})(\tilde{\theta}^{(M)}_{r}(r) - \theta^{(M)}_{r}(q(\alpha^{(1)}), \alpha^{(2)))) \right]$$

$$= \sum_{k=1}^{n} x_{k} (\theta^{(M)}_{r}(q_{k+1}) - \theta^{(M)}_{r}(q_{k})).$$  (4.32)
By (2.21),

\[
G_r((u, v)_N) - G_r(q(\alpha^{(1)}, \alpha^{(2)})) - G'_r(q(\alpha^{(1)}, \alpha^{(2)})) \left( (u, v)_N - q(\alpha^{(1)}, \alpha^{(2)}) \right)
\]
\[
= \frac{1}{2} \left( D(2r - q(\alpha^{(1)}, \alpha^{(2)})) - D(2r - (u, v)_N) \right)
- D'(2r - q(\alpha^{(1)}, \alpha^{(2)}))(q(\alpha^{(1)}, \alpha^{(2)})) - (u, v)_N).
\]

Combining (4.31), (4.29), (4.32) and (4.28), we get (4.24). Due to Remark A.3, the function \(G\) is convex. Therefore,

\[
G_r((u, v)_N) - G_r(q(\alpha^{(1)}, \alpha^{(2)})) - G'_r(q(\alpha^{(1)}, \alpha^{(2)})) \left( (u, v)_N - q(\alpha^{(1)}, \alpha^{(2)}) \right) \geq 0.
\]

Inequality (4.25) follows from (4.33).

\[\square\]

4.3 Regularization and localization

In this section, we finish the proof of Theorem 2.2.

**Lemma 4.3** (Regularization, well-definiteness). For any \(x \in X'_n(r)\),

\[
\lim_{M \uparrow +\infty} \left\{ \lim_{x_n \uparrow 1 - 0} \left( \lim_{\epsilon \downarrow 0} \Phi_N(\bar{x}, V_r)[\tilde{A}] - \frac{\beta^2}{2} \sum_{k=1}^n x_k \left( \theta_r^{(M)}(q_{k+1}) - \theta_r^{(M)}(q_k) \right) \right) \right\} < \infty.
\]

**Proof.** Recall (4.11). Given \(x \in X'_n(r)\), large enough given \(M > 0\), as \(\epsilon \downarrow +0\) and \(x_n \uparrow 1 - 0\), we have

\[
\psi^{(M)}_N(0, x, V_r) = \frac{\beta^2}{2} (M - D'(2(r - q_n))) r + \Phi_N(\bar{x}, V_r)[\tilde{A}] + \mathcal{O}(\epsilon) + \mathcal{O}(1 - x_n),
\]

where \(\bar{A}(u, \alpha) := \frac{1}{\sqrt{N}} \sum_{i=1}^N \tilde{a}^{(M)}_i(\alpha) u_i\), and \(\{\tilde{a}_i\}\) are i.i.d. copies of

\[
\tilde{a}^{(M)}(\alpha) := \sum_{k=1}^{n-1} (m_{k+1} - m_k)^{1/2} g^{(k)}_{[\alpha]_k}, \quad \alpha \in A.
\]

Using the definition (2.10), for large enough given \(M > 0\), as \(x_n \uparrow 1 - 0\), we get

\[
x_n \left( \theta_r^{(M)}(q_{k+1}) - \theta_r^{(M)}(q_k) \right) = (M - D'(2(r - q_n))) r - \frac{1}{2} D(2(r - q_n)) + \mathcal{O}(1 - x_n).
\]

Combining (4.35) and (4.37), we note that the unbounded in \(M\) terms in (4.34) cancel out and therefore (4.34) holds.

\[\square\]

**Lemma 4.4** (Localization, large deviations and cascades). For any \(x \in X'_n(\epsilon)\),

\[
\lim_{\epsilon \downarrow 0} \psi^{(M)}_N(0, x, V_r) = \inf_{\lambda \in \mathbb{R}} \left[ \int_{x, V_r, \phi} (0, 0) - \lambda r \right].
\]

**Proof.** This is a standard computation (cf., e.g., [3, Lemma 6.2]), using the well-known averaging properties of the RPC (see, e.g., [5, (5.27)]) and the quenched large deviations principle as is done in [5, Sections 3-5].

\[\square\]

**Proof of Theorem 2.2.** Combining Lemmata 4.2, 4.4 and 4.3, we obtain Theorem 2.2.
5 Outlook

It is plausible that combining the methods of Talagrand [16] with Theorem 2.2, one can show that the remainder term in (2.17) vanishes at the saddle-point. This implies that, in fact, the equality holds in (2.18). Summarizing, we conjecture that the following holds.

Conjecture 5.1 (Parisi-type formula). In the case of the product state space (2.5), for all \( \beta \in \mathbb{R} \),

\[
p(\beta) = \sup_{r \in [0,d]} \inf_{x \in \mathcal{X}(r)} \mathcal{P}(\beta, r, g)[x], \quad \text{almost surely.} \tag{5.1}
\]

Parallel to the product state space (2.5), one can consider the rotationally invariant state space:

\[
S_N := \{ u \in \mathbb{R}^N : \|u\|_2 \leq L \sqrt{N} \}, \quad L > 0. \tag{5.2}
\]

In this case, we assume that the a priori measure \( \mu_N \in \mathcal{M}_{\text{finite}}(S_N) \) has the density

\[
\frac{d\mu_N}{d\lambda_N}(u) := \exp\left( \sum_{i=1}^N f(u_i) \right), \quad u = (u_i)_{i=1}^N \in \mathbb{R}^N, \quad f : \mathbb{R} \to \mathbb{R} \tag{5.3}
\]

with respect to the Lebesgue measure \( \lambda \) on \( \mathbb{R}^N \). Let the function \( f \) be of the form \( f(u) := h_1u - h_2u^2 \), where \( h_1 \in \mathbb{R} \) and \( h_2 \in \mathbb{R}_+ \) are given constants. Let us note that in case (5.2), \( d = L^2 \).

In the case of the rotationally invariant state space (5.2), one can obtain a more explicit representation for the Parisi functional (2.13), which does not require any regularization. Given \( x \in \mathcal{X}(r) \), define \( q_{\text{max}} := q_{\text{max}}(x) := \sup \{ q \in [0; r) : x(q) < 1 \} \). Consider the Crisanti-Sommers type functional (cf. [6, (A2.4)] and [8, (47)])

\[
CS(\beta, r)[x] := \frac{1}{2} \left[ \log(r - q_{\text{max}}) + \int_0^{q_{\text{max}}} dq \frac{D'(2(r - q))}{x(q)ds} + h_1^2 \int_0^r x(q)dq - h_2r \right] + \frac{\beta^2}{2} \left( D'(2(r - q_{\text{max}})) + \int_0^{q_{\text{max}}} D'(2(r - q))x(q)dq \right), \quad x \in \mathcal{X}(r). \tag{5.4}
\]

It is plausible that by reducing the case of the rotationally invariant state space to the product state space case using a large deviations argument (an idea exploited in [15]) one can obtain the following result.

Conjecture 5.2 (Fyodorov-Sommers formula). In the case of the rotationally invariant state space (5.2), for all \( \beta \in \mathbb{R}_+ \), \( h_1 \in \mathbb{R} \), \( h_2 \in \mathbb{R}_+ \), there exists unique \( r^* \in [0; d] \) and unique \( x^* \in \mathcal{X}(r) \) such that

\[
p(\beta) = \max_{r \in [0,d]} \min_{x \in \mathcal{X}(r)} CS(\beta, r)[x] = CS(\beta, r^*)[x^*], \quad \text{almost surely.} \tag{5.5}
\]

Resolution of Conjectures 5.1 and 5.2 is beyond the scope of this short communication and will be reported on elsewhere.

Remark 5.3. The Crisanti-Sommers type functional (5.4) corresponds to the a priori distribution (5.3), which represents the linear combination of linear and quadratic external fields. Formula [8, (47)] was derived under the assumption of the quadratic external field, whereas formula [6, (A2.4)] was obtained for the spherical SK model with the linear external field.

Remark 5.4. The explicit form of the functional (5.4) assures that it is strictly convex with respect to \( x \in \mathcal{X}(r) \). In contrast, convexity of the functional (2.13) is (to the author’s best knowledge) open, see [13] and [5, Theorem 6.4] for partial results.
Gaussian fields with isotropic increments seen through spin glasses

A Characterization of the correlators

We recall some facts about high-dimensional Gaussian processes with isotropic increments. The following result can be extracted from the work [20] of A.M. Yaglom (see also [19]).

**Theorem A.1.** If $X$ is a Gaussian random field with isotropic increments that satisfies (2.1), then one of the following two cases holds:

1. **Isotropic field.** There exists the correlation function $B : \mathbb{R}_+ \to \mathbb{R}$ such that

$$
\mathbb{E}[X_N(u)X_N(v)] = B \left( \frac{1}{N} \|u - v\|_2^2 \right), \quad u, v \in \Sigma_N,
$$

where the function $B$ has the representation

$$
B(r) = c_0 + \int_0^{+\infty} \exp \left( -t^2 r \right) \nu(dt),
$$

where $c_0 \in \mathbb{R}_+$ is a constant and $\nu \in \mathcal{M}_{\text{finite}}(\mathbb{R}_+)$ is a non-negative finite measure. In this case, the function $D$ in (2.1) is expressed in terms of the correlation function $B$ as

$$
D(r) = 2(B(0) - B(r)).
$$

2. **Non-isotropic field with isotropic increments.** The function $D$ in (2.1) has the following representation

$$
D(r) = \int_0^{+\infty} \left[ 1 - \exp \left( -t^2 r \right) \right] \nu(dt) + A \cdot r, \quad r \in \mathbb{R}_+,
$$

where $A \in \mathbb{R}_+$ is a constant and $\nu \in \mathcal{M}(\{0; +\infty\})$ is a $\sigma$-finite measure with

$$
\int_{0}^{+\infty} \frac{t^2 \nu(dt)}{t^2 + 1} < \infty.
$$

**Remark A.2.** In Theorem A.1, assuming $c_0 = 0$, case 1 is sometimes referred to as the short-range one which reflects the decay of correlations: $B(r) \downarrow +0$, as $r \uparrow +\infty$. This fact follows from the representation (A.2). Correspondingly, case 2 is called the long-range one, since here, assuming $X(0) = 0$, the correlation structure is

$$
\mathbb{E}[X_N(u)X_N(v)] = \frac{1}{2} \left( D_N(\|u\|_2^2) + D_N(\|v\|_2^2) - D_N(\|u - v\|_2^2) \right), \quad u, v \in \mathbb{R}^N.
$$

Equation (A.6) in combination with the representation (A.4) implies that the correlations of the field $X_N$ do not decay, as $\|u - v\| \to +\infty$.

**Remark A.3.** Theorem A.1 implies that the function $D$ appearing in (2.1) is necessarily concave, infinitely differentiable, and non-decreasing on $(0; +\infty)$.

**References**


Gaussian fields with isotropic increments seen through spin glasses


Acknowledgments. The author is grateful to Prof. Yan V. Fyodorov for useful remarks and his interest in this work. Kind hospitality of the Hausdorff Research Institute for Mathematics, where a part of the present work was done, is gratefully acknowledged.