A Note on Pseudo-Anosov Maps with Small Growth Rate

Peter Brinkmann

CONTENTS

1. Introduction
2. Train Tracks
3. Motivation
4. The Sequence
Acknowledgments
References

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We present an explicit sequence of pseudo-Anosov maps $\phi_k : S_{2k} \to S_{2k}$ of surfaces of genus $2k$ whose growth rates converge to one.

1. INTRODUCTION

In this note, we present an explicit sequence ϕ_k of pseudo-Anosov maps of surfaces of genus $2k$ whose growth rates converge to one. This answers a question of Joan Birman, who had previously asked whether such growth rates are bounded away from one. Norbert A'Campo, Mladen Bestvina, and Klaus Johannson independently communicated this question to me. McMullen previously obtained a similar result using quite different techniques [McMullen 00].

The growth of the genus is not an artifact of our construction. For a surface S of fixed genus g, the growth rates of pseudo-Anosov maps of S are clearly bounded away from one, for the rates are Perron-Frobenius eigenvalues of irreducible integral $m \times m$ matrices, with $m \leq 6g - 3$ [Bestvina and Handel 95]. Finding the smallest possible growth rate for each genus is an interesting problem that remains open.

One curious observation, due to Norbert A'Campo, is that for each k, the mapping torus of ϕ_k is the complement of a w-slalom knot B_k (Figure 1).

In Section 2, we review the part of the theory of train tracks [Bestvina and Handel 92, Bestvina and Handel 95] that we use in this paper. Section 3 explains the intuition that led to the result, and Section 4 contains the statement and proof of the main results (Theorem 4.2 and Corollary 4.3).

The results of this paper grew out of massive computer experiments with my software package XTrain [Brinkmann 00, Brinkmann and Schleimer 01] in the context of the REU program at the University of Illinois at Urbana-Champaign.

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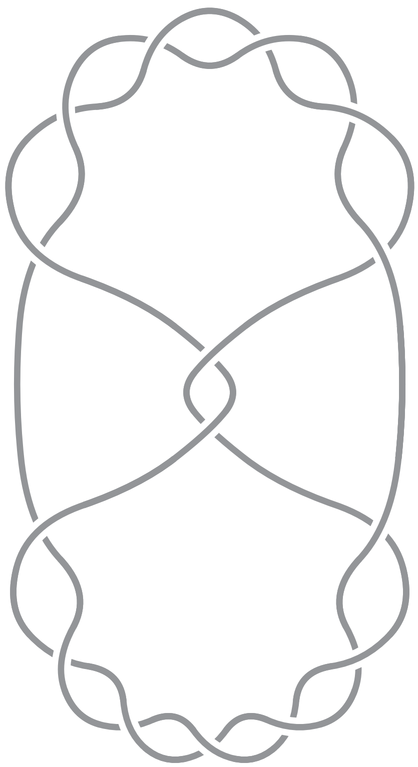

FIGURE 1. The knot B_3, drawn by Knotscape. For $k \geq 1$, B_k is a knot similar to the one in the picture, but with $2k$ crossings on top and $2k+1$ crossings at the bottom.

2. TRAIN TRACKS

We present a brief review of train tracks as defined in [Bestvina and Handel 92]. Let G be a finite graph without vertices of valence one or two, and let $f\colon G \to G$ be a homotopy equivalence of G that maps vertices to vertices. The map f is said to be a *train track map*, if for every integer $n \geq 1$ and every edge e of G, the restriction of f to the interior of e is an immersion.

If $E_1, \cdots, E_m$ is the collection of edges of G, the *transition matrix* of f is the nonnegative $m \times m$ matrix M whose ijth entry is the number of times the f-image of E_j crosses E_i, regardless of orientation. The matrix M is said to be *irreducible* if, for every tuple $1 \leq i$ and $j \leq m$, there exists some exponent $n > 0$ such that the ijth entry of M^n is nonzero. If M is irreducible, then it has a maximal real eigenvalue $\lambda \geq 1$ (see [Seneta 73]). We call λ the *growth rate* of f.

The following theorem from [Bestvina and Handel 92] will be our main tool. Recall that an outer automorphism ω of a free group F is called *reducible* if there are proper free factors $F_1, \ldots, F_r$ of F such that ω permutes the conjugacy classes of the F_is and $F_1 * \cdots * F_r$ is a free factor of F; ω is *irreducible* if it is not reducible. Also, note that $\pi_1 G$ is a finitely generated free group, and that a homotopy equivalence $f\colon G \to G$ induces an outer automorphism of $\pi_1 G$.

Theorem 2.1. [Bestvina and Handel 92, Theorem 4.1] *Let ω be an outer automorphism of a finitely generated free group F. Suppose that each positive power of ω is irreducible and that there is a nontrivial word $s \in F$ such that ω preserves the conjugacy class of s (up to inversion). Then ω is geometrically realized by a pseudo-Anosov homeomorphism $\phi\colon S \to S$ of a surface with one puncture.*

Remark 2.2. If $f\colon G \to G$ is a train track map that induces an outer automorphism ω as in Theorem 2.1, then the transition matrix of f is irreducible, and the growth rate of f is the same as the pseudo-Anosov growth rate of ϕ.

Moreover, if $f\colon G \to G$ is a train track map such that all positive powers of its transition matrix M are irreducible, then all positive powers of the induced outer automorphism ω are irreducible [Bestvina and Handel 92].

Remark 2.3. The proof of Corollary 4.3 uses an explicit construction of invariant foliations for pseudo-Anosov maps. This construction is straightforward but too long to be reviewed in this note; we point the reader to [Bestvina and Handel 95] for details.

3. MOTIVATION

Warning 3.1. The discussion in this section is not supposed to present any rigorous mathematical reasoning. Rather, the purpose of this section is to explain the origin of the technical definitions and computations of Section 4.

One crucial tool in the development of the intuition behind Theorem 4.2 was XTrain [Brinkmann 00, Brinkmann and Schleimer 01], a software package that implements algorithms from [Bestvina and Handel 92, Bestvina and Handel 95], among others. In particular, the software allows users to define homeomorphisms of surfaces with one puncture as a composition of Dehn twists with respect to the curves shown in Figure 2. When computing Dehn twists, we adopt the following convention: we equip the surface with an outward pointing normal vector field. When twisting with respect to a

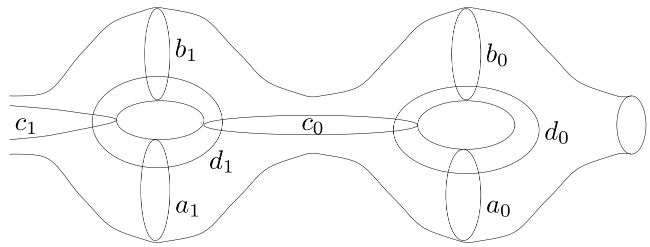

FIGURE 2. Generators of the mapping class group.

curve c, we turn *right* whenever we hit c. We denote by D_c the twist with respect to c.

The software represents a surface homeomorphism ϕ of a punctured surface S as a homotopy equivalence f of a graph G that is embedded in (as well as homotopy equivalent to) S. There exists a loop σ in G that corresponds to a short loop around the puncture of S. In particular, f preserves the free homotopy class of σ (up to orientation).

The first ingredient is the observation that a homeomorphism of a surface of genus g given by

$$\phi_g = D_{c_0} \cdots D_{c_{g-1}} D_{d_0} \cdots D_{d_{g-1}} \tag{3–1}$$

can be represented by a train track map of a graph H_g, as in Figure 3, such that $x_0 \mapsto x_1, x_1 \mapsto x_2, \ldots, x_{2g} \mapsto x_0^{-1}$ with $\sigma_g = x_0 x_1 \cdots x_{2g} x_0^{-1} x_1^{-1} \cdots x_{2g}^{-1}$. Note, in particular, that this map cyclically permutes the edges of H_g (up to orientation).

The second ingredient comes from certain PV-automorphisms ψ_n [Stallings 82] of a free group $F = \langle y_0, \ldots, y_n \rangle$ given by $y_0 \mapsto y_1, y_1 \mapsto y_2, \ldots, y_n \mapsto y_0 y_1$. Mathematically, these automorphisms are very different

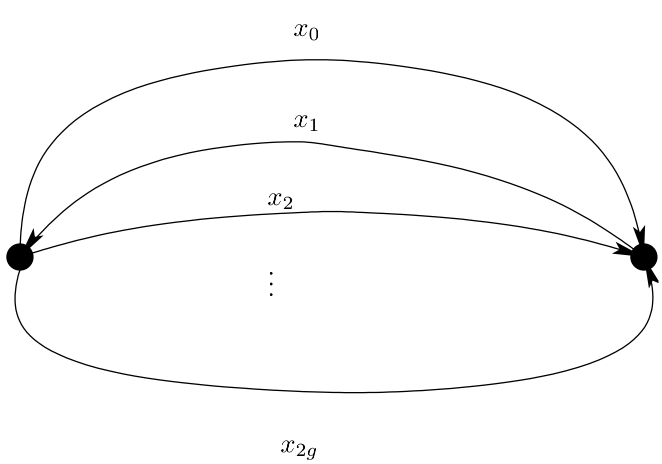

FIGURE 3. The graph H_g.

from the maps constructed earlier in this section (after all, PV automorphisms are nongeometric and of exponential growth, whereas the maps of the previous paragraph are geometric and periodic).

Superficially, though, these two classes of maps look strikingly similar. Moreover, the growth rates of the maps ψ_n converge to one. These two observations prompted me to investigate maps that are built from blocks as in Equation (3–1). Maps of surfaces of genus $2k$ of the form

$$\begin{aligned}\phi_k = & D_{c_0} \cdots D_{c_{k-1}} D_{d_0} \cdots \\ & D_{d_{k-1}} (D_{c_k} \cdots D_{c_{2k-1}} D_{d_k} \cdots D_{d_{2k-1}})^{-1}\end{aligned}$$

turned out to be pseudo-Anosov with rather small growth rates. Computer experiments suggested that the growth rates of these maps converge to one, and the same experiments suggested that train tracks representing these maps conform to a describable pattern, which gave rise to Definition 4.1 and Theorem 4.2. Notice how Definition 4.1 seems reminiscent of both PV automorphisms as well as homeomorphisms as in Equation (3–1).

4. THE SEQUENCE

Motivated by the discussion of Section 3, we now define a sequence of surface homeomorphisms.

Definition 4.1. Let $k \geq 1$ be an integer, and let the graph G_k be as in Figure 4. We define a map $f_k\colon G_k \to G_k$ by letting

$$\begin{aligned} a &\mapsto a x_0 y_0 \\ b &\mapsto b y_0^{-1} x_0^{-1} \\ c &\mapsto d \\ d &\mapsto d y_1 x_0 \\ x_0 &\mapsto x_1 \\ x_1 &\mapsto x_2 \\ &\vdots \\ x_{2k-1} &\mapsto a^{-1} b y_0^{-1} \\ y_0 &\mapsto y_1 \\ y_1 &\mapsto y_2 \\ &\vdots \\ y_{2k-1} &\mapsto c^{-1} b. \end{aligned}$$

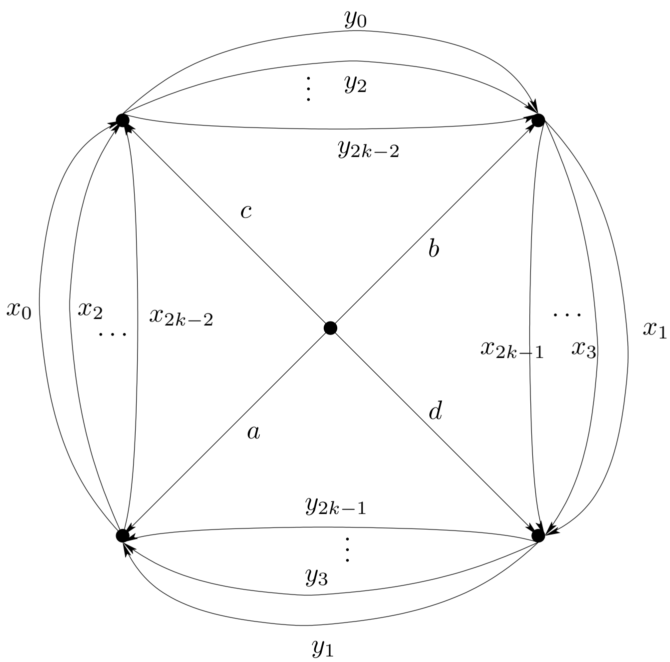

FIGURE 4. The graph G_k.

Finally, let

$$\sigma_k = x_0y_0x_1y_1\cdots x_{2k-1}y_{2k-1}a^{-1}by_0^{-1}c^{-1}d \\ x_{2k-1}^{-1}b^{-1}cx_{2k-2}^{-1}y_{2k-1}^{-1}x_{2k-3}^{-1}y_{2k-2}^{-1}\cdots x_0^{-1}y_1^{-1}d^{-1}a.$$

We are now ready to state and prove the main result of this note.

Theorem 4.2. *The sequence of maps $f_k\colon G_k \to G_k$ is a sequence of homotopy equivalences induced by pseudo-Anosov maps $\phi_k\colon S_{2k} \to S_{2k}$ of surfaces of genus $2k$ with one puncture. If λ_k is the pseudo-Anosov growth rate of ϕ_k, then*

$$\lim_{k\to\infty} \lambda_k = 1.$$

Proof: A number of tedious but straightforward checks yield the following facts:

1. The maps f_k are train track maps.
2. All positive powers of the transition matrix M_k of f_k are irreducible.
3. The map f_k preserves the free homotopy class of the loop σ_k.

Hence, by Theorem 2.1 and Remark 2.2, the outer automorphism induced by f_k is induced by a pseudo-Anosov map $\phi_k\colon S_k \to S_k$, and a quick computation of Euler characteristics shows that the genus of S_k is $2k$. Finally, a simple induction shows that the characteristic polynomial of the transition matrix M_k is of the form

$$\chi(\lambda) = (\lambda-1)^2(\lambda^{4k+2} - \lambda^{4k+1} - 4\lambda^{2k+1} - \lambda + 1).$$

Solving for the growth rate λ_k, we obtain

$$\lambda_k = 1 + \lambda_k^{4k+2} - \lambda_k^{4k+1} - 4\lambda_k^{2k+1}. \tag{4–1}$$

Note that the polynomial χ is palindromic (this is no surprise, as f_k is induced by a surface homeomorphism), i.e., $\chi(\lambda) = \lambda^{4k+4}\chi(\frac{1}{\lambda})$. Hence, Equation (4–1) also holds for λ_k^{-1}:

$$\lambda_k^{-1} = 1 + \lambda_k^{-(4k+2)} - \lambda_k^{-(4k+1)} - 4\lambda_k^{-(2k+1)} \\ \geq 1 - \lambda_k^{-(4k+1)} - 4\lambda_k^{-(2k+1)}. \tag{4–2}$$

Recall that $\lambda_k^{-1} < 1$. Let $0 < u < 1$ be some real number. We have $\lim_{k\to\infty} 1 - u^{4k+1} - 4u^{2k+1} = 1$, which implies that u only satisfies Inequality (4–2) for finitely many values of k. Hence, for any such u, the set $\{\lambda_k | \lambda_k^{-1} < u\}$ is finite. This immediately implies that $\lim_{k\to\infty} \lambda_k^{-1} = 1$, hence

$$\lim_{k\to\infty} \lambda_k = 1.$$

□

Corollary 4.3. *The maps $\phi_k\colon S_{2k} \to S_{2k}$ from Theorem 4.2 can be extended to pseudo-Anosov maps of closed surfaces. The growth rates of the extended maps are the same as those of the original maps.*

Proof: A lengthy but straightforward computation of invariant foliations (see Remark 2.3) yields that the four outer vertices of the graph in Figure 4 give rise to singularities of index $\frac{1}{2} - k$, while the central vertex does not give rise to any singularity. Hence, the sum of the indices of all singularities coming from vertices of the graph is $2-4k$, which is the Euler characteristic of a closed surface of genus $2k$.

Hence, the foliations have no singularity at the puncture, which implies that the extension of ϕ_k to the closed surface obtained by filling in the puncture is pseudo-Anosov, with the same growth rate as ϕ_k. □

5. NOTE ADDED IN PROOF

I have since learned that Robert Penner previously constructed an explicit sequence of pseudo-Anosov maps whose growth rates converge to one [Penner 91]. It is my hope, however, that the construction in the current article is sufficiently interesting to stand in its own right.

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Peter Brinkmann, Department of Mathematics, University of Illinois at Urbana-Champaign, 273 Altgeld Hall, 1409 W. Green St., Urbana, IL 61801 (brinkman@math.uiuc.edu)

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