

# Explicit Presentations for Exceptional Braid Groups

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We give presentations for the braid groups associated with the complex reflection groups  $G_{24}$  and  $G_{27}$ . For the cases of  $G_{29}$ ,  $G_{31}$ ,  $G_{33}$ , and  $G_{34}$ , we give (strongly supported) conjectures. These presentations were obtained with VKCURVE, a GAP package implementing Van Kampen's method.

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## 1. INTRODUCTION

To any complex reflection group  $W \subset \mathrm{GL}(V)$ , one may attach a braid group  $B(W)$ , defined as the fundamental group of the space of regular orbits for the action of  $W$  on  $V$  [Broué et al. 98].

The “ordinary” braid group on  $n$  strings, introduced by [Artin 47], corresponds to the case of the symmetric group  $\mathfrak{S}_n$ , in its monomial reflection representation in  $\mathrm{GL}_n(\mathbb{C})$ . More generally, any Coxeter group can be seen as a complex reflection group, by complexifying the reflection representation. It is proved in [Brieskorn 71] that the corresponding braid group can be described by an *Artin presentation*, obtained by “forgetting” the quadratic relations in the Coxeter presentation.

Many geometric properties of Coxeter groups still hold for arbitrary complex reflection groups. Various authors, including Coxeter himself, have described “Coxeter-like” presentations for complex reflection groups. Of course, one would like to have not just a “Coxeter-like” presentation for  $W$ , but also an “Artin-like” presentation for  $B(W)$ .

The problem can be reduced to the irreducible case. Irreducible complex reflection groups have been classified by [Shephard et al. 54]: there is an infinite family  $G(de, e, r)$  (which contains the infinite families of Coxeter groups), plus 34 exceptional groups  $G_4, \dots, G_{37}$  (among them are the exceptional Coxeter groups).

Before this note, presentations were known for all but 6 exceptional groups (see the tables of [Broué et al. 98]):

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- the braid group of  $G(de, e, r)$  is computed in [Broué et al. 98]. The proof makes use of fibration arguments, taking advantage of the fact that  $G(de, e, r)$  is monomial.
- the first exceptional groups ( $G_4$  to  $G_{22}$ ) are two-dimensional. The spaces of regular orbits are complements of (fairly elementary) complex algebraic curves; the braid groups have been computed by [Bannai 76], using Zariski/Van Kampen method.
- among the 15 higher-dimensional exceptional groups, six are Coxeter groups; Brieskorn’s theorem applies to them. In addition, three more groups happen to have orbit spaces isomorphic to orbit spaces of certain Coxeter groups (this was observed by [Orlik and Solomon 88]).
- the six remaining groups are  $G_{24}$ ,  $G_{27}$ ,  $G_{29}$ ,  $G_{31}$ ,  $G_{33}$ , and  $G_{34}$ . No presentations for their braid groups are given in [Broué et al. 98] (except a conjectural one for  $G_{31}$ ).

In the present note, we describe presentations for the first two of the six remaining groups and conjectural presentations for the last four. The evidence for our conjectures is very strong, and only a minute step of the proof is missing.

**2. THE PRESENTATIONS**

Before listing the individual presentations, it is worth noting that they share some common features: the number of generators is equal to the rank of the group (except for  $G_{31}$ , where an additional generator is needed); the generators correspond geometrically to *generators-of-the-monodromy* (in the sense of [Broué et al. 98] and [Bessis 01]) or equivalently *braid reflections* (this nicer terminology was introduced in [Broué 00]); the relations are positive and homogeneous; by adding quadratic relations to the presentation, one gets a presentation for the reflection group; the product of the generators, taken in a certain order, has a central power. Existence of such presentations was proved in [Bessis 01]. All presentations below satisfy all these properties.

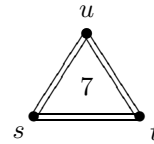
**2.1 The Three-Dimensional Group  $G_{24}$**

**Theorem 2.1.** *The braid group associated with the complex reflection group  $G_{24}$  admits the presentation*

$$\left\langle s, t, u \mid \begin{array}{l} stst = tsts, susu = usus, tutu = utut, \\ stustus = tustust = ustustu \end{array} \right\rangle.$$

*These relations imply that  $(stu)^7$  is central.*

We suggest representing this presentation by the following diagram:



Playing with the above presentation, one may obtain other ones that are less symmetrical but also interesting. For example, replacing  $t$  by  $usts^{-1}u^{-1}$  gives (after simplification)

$$\left\langle s, t, u \mid \begin{array}{l} sts = tst, tut = utu, susu = usus, \\ sustustus = ustustust \end{array} \right\rangle.$$

Also, replacing  $t$  by  $susts^{-1}u^{-1}s^{-1}$  yields

$$\left\langle s, t, u \mid \begin{array}{l} sts = tst, tutu = utut, susu = usus, \\ sutsuts = usutsut \end{array} \right\rangle.$$

**2.2 The Three-Dimensional Group  $G_{27}$**

For  $G_{27}$ , we couldn’t find a nice symmetrical presentation, involving only classical braid relations and cyclic three-terms relations.

**Theorem 2.2.** *The braid group associated with the complex reflection group  $G_{27}$  admits the presentations*

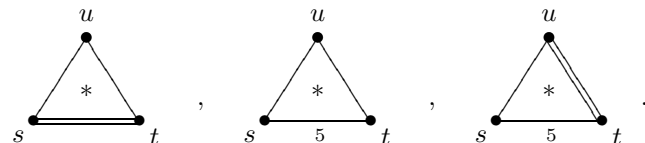
$$\left\langle s, t, u \mid \begin{array}{l} stst = tsts, tut = utu, sus = usu, \\ stustustusts = tstustustust \end{array} \right\rangle,$$

$$\left\langle s, t, u \mid \begin{array}{l} ststs = tstst, tut = utu, sus = usu, \\ ststustust = tstustustu \end{array} \right\rangle,$$

$$\left\langle s, t, u \mid \begin{array}{l} ststs = tstst, tutu = utut, sus = usu, \\ stustut = ustustu \end{array} \right\rangle.$$

*In each of these presentations, the element  $(stu)^5$  is central.*

These presentations could be symbolized by the following diagrams:



**2.3 The Four-Dimensional Group  $G_{29}$**

The presentation for  $G_{29}$  given in [Broué et al. 98] was not conjectured to give (by forgetting the quadratic relations) a presentation for the braid group. Surprisingly,

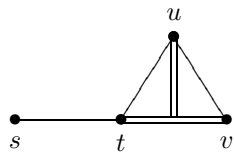
our computations happened to give precisely this presentation.

**Conjecture 2.3.** *The braid group associated with the complex reflection group  $G_{29}$  admits the presentation*

$$\left\langle s, t, u, v \left| \begin{array}{l} sts = tst, tut = utu, uvu = vuv, \\ tvtv = vtvt, su = us, sv = vs, \\ utvutv = tvutvu \end{array} \right. \right\rangle.$$

*These relations imply that  $(stuv)^5$  is central.*

Broué, Malle, and Rouquier used the following diagram to symbolize this presentation:



### 2.4 The Four-Dimensional Group $G_{31}$

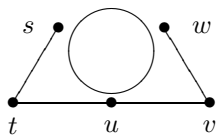
The following is consistent with the conjectural presentation from [Broué et al. 98], but this time there is some computational evidence behind the conjecture.

**Conjecture 2.4.** *The braid group associated with the complex reflection group  $G_{31}$  admits the presentation*

$$\left\langle s, t, u, v, w \left| \begin{array}{l} sts = tst, tut = utu, uvu = vuv, \\ vuv = wvw, sv = vs, tv = vt, \\ tw = wt, suw = uws = wsu \end{array} \right. \right\rangle.$$

*These relations imply that  $(stuvw)^6$  is central.*

The corresponding Broué-Malle-Rouquier diagram is:



**Remark 2.5.** Since our generators are braid reflections, they map to generating reflections in the reflection group. It is well known that, even though it is four-dimensional,  $G_{31}$  cannot be generated by fewer than 5 reflections.

### 2.5 The Five-Dimensional Group $G_{33}$

The relations in the presentation below do not coincide with the homogeneous part of the Broué-Malle-Rouquier presentation of  $G_{33}$ . However, the relations involving  $t, u,$  and  $w$  coincide with the Broué-Malle-Rouquier relations

for the braid group of  $G(3, 3, 3)$  (the similar remarks also apply to  $G_{34}$ ).

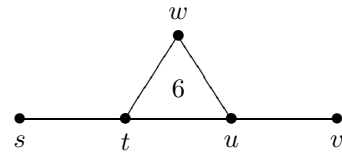
**Conjecture 2.6.** *The braid group associated with the complex reflection group  $G_{33}$  admits the presentation*

$$\left\langle s, t, u, v, w \left| \begin{array}{l} sts = tst, tut = utu, uvu = vuv, \\ wtw = twt, wuw = uwu, su = us, \\ sv = vs, tv = vt, ws = sw, vw = vw \\ twwtw = uwtwt = wtuwtu \end{array} \right. \right\rangle.$$

*These relations imply that  $(stuvw)^9$  is central.*

The relation  $uwtwt = wtuwtu$  is redundant.

We suggest representing this presentation by the following diagram:



Following [Broué et al. 98] where a second diagram for  $G_{33}$  is given (to account for some parabolic subgroups missing in their first diagram), it is not difficult to obtain the equivalent presentation

$$\left\langle s, t, u, v, w \left| \begin{array}{l} vt = tv, uv = vu, tu = ut, \\ wu = uw, wsw = sws, sus = usu, \\ svs = vsv, sts = tst, vuv = wvw, \\ twt = twt, twvstw = wstvw \end{array} \right. \right\rangle,$$

which contains a parabolic subdiagram of type  $D_4$ . (A similar diagram may be derived from the conjectural presentation for  $B(G_{34})$  given below).

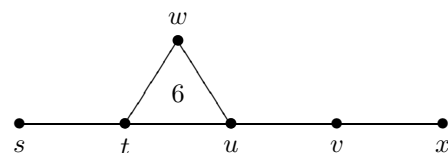
### 2.6 The Six-Dimensional Group $G_{34}$

**Conjecture 2.7.** *The braid group associated with the complex reflection group  $G_{34}$  admits the presentation*

$$\left\langle s, t, u, v, w, x \left| \begin{array}{l} \text{relations of } G_{33} + \\ xvx = vxv, xs = sx, xt = tx, \\ xv = vx, xw = wx \end{array} \right. \right\rangle.$$

*These relations imply that  $(stuvw)^7$  is central.*

We suggest representing this presentation by the following diagram:



### 3. DEFINITIONS AND PRELIMINARY WORK

Our strategy of proof is, basically, brute force. Let  $V$  be a  $\mathbb{C}$ -vector space of dimension  $r$ , and let  $W \subset \mathrm{GL}(V)$  be a complex reflection group. The algebra  $\mathbb{C}[V]^W$  of invariant polynomial functions is isomorphic to a polynomial algebra [Shephard et al. 54]; let  $f_1, \dots, f_r$  be homogeneous polynomials such that  $\mathbb{C}[V]^W = \mathbb{C}[f_1, \dots, f_r]$ .

Let  $\mathcal{A}$  be the set of all reflecting hyperplanes. For each  $H \in \mathcal{A}$ , the pointwise stabilizer  $W_H$  of  $H$  in  $W$  is a cyclic subgroup of order  $e_H$ ; choose  $l_H$  a linear form with kernel  $H$ . Let  $V^{\mathrm{reg}} := V - \bigcup_{H \in \mathcal{A}} H$ . The *regular orbits space* is  $V^{\mathrm{reg}}/W$ . We have  $\prod_{H \in \mathcal{A}} l_H^{e_H} \in \mathbb{C}[V]^W$ , so there is a unique polynomial  $\Delta \in \mathbb{C}[X_1, \dots, X_r]$  such that  $\prod_{H \in \mathcal{A}} l_H^{e_H} = \Delta(f_1, \dots, f_r)$ . We call  $\Delta$  the *discriminant* of  $W$  (with respect to  $f_1, \dots, f_r$ ). Clearly,  $V^{\mathrm{reg}}/W$  is isomorphic, as an algebraic variety, to the complement of the hypersurface  $\mathcal{H}$  defined in  $\mathbb{C}^r$  by the equation  $\Delta = 0$ .

There is a general method, though not always practically tractable, to compute the fundamental group of such a space. First, choose a 2-plane  $P$  such that the embedding  $P \cap (\mathbb{C}^r - \mathcal{H}) \hookrightarrow \mathbb{C}^r - \mathcal{H}$  is a  $\pi_1$ -isomorphism (by a Zariski theorem, this should hold for a *generic* choice of  $P$ —how exactly this choice can be made is a difficult issue, which we will discuss later on). Then, use the Zariski/Van Kampen method to compute the fundamental group of  $P \cap (\mathbb{C}^r - \mathcal{H})$ . The computations involved in the second step are far beyond human capabilities (or at least beyond *our* capabilities), especially if one wants to avoid imprecise arguments. Therefore, we designed a software package, VKCURVE [Bessis and Michel 03], to carry out the computations.

#### 3.1 General Remarks about the Implementation

Our computations are performed using the computer algebra software GAP3, which is designed to handle cyclic numbers, matrices over these numbers, permutations, presentations, and all sorts of algebraic objects and algorithms involving exact computations. The source of its mathematically advanced functions is public (and in a rather intelligible language), and any user is free to check their validity.

Our package VKCURVE builds on the older package CHEVIE, which implements (among other) complex reflection groups, Coxeter groups, and Artin groups.

#### 3.2 Computing the Discriminant

For each of the six groups, the discriminant can be recovered from the data given in Appendix B of [Orlik and Terao 92], where they explain how to construct the

matrix  $M \in M_r(\mathbb{C}[X_1, \dots, X_r])$  of logarithmic vector fields (a.k.a., basic derivations) for the quotient singularity (called the discriminant matrix in [Orlik and Terao 92, Definition 6.67]). The polynomial  $\Delta$  is simply the determinant of  $M$ .

To prevent typos, we actually re-checked all needed computations.

We summarize their method. Let  $d_i = \deg f_i$ , and let  $d_1^*, \dots, d_j^*$  be the codegrees of  $W$ . We assume that the degrees are ordered in increasing order (but we *do not* assume the same for codegrees). The matrix  $M$  is an  $r \times r$ -matrix whose  $(i, j)$  entry is an homogeneous invariant polynomial of degree  $d_i + d_j^*$ .

The six groups have the property that  $d_1 < d_2$ , so  $f_1$  is unique up to a scalar, and if  $H = (\partial_j \partial_i f_1)_{ij}$  is the Hessian matrix of  $f_1$ , then  $\det H$  may be chosen as one of the basic invariants  $f_k$  (which we assume). Then, if  $J = (\partial_j f_i)_{ij}$  is the Jacobian matrix of the  $f_i$ , we have the following matrix equation ([Orlik and Terao 92, Equation (1), page 280]):

$$M = (d_1 - 1)JH^{-1}tJC \quad (3-1)$$

where  $C$  is a matrix of homogeneous invariant polynomials such that  $\deg C_{ij} = d_1 + d_j^* - d_i$ . Orlik and Terao note that there exists an ordering of the  $d_j^*$  such that  $C$  is the identity matrix, except for some line  $q$  where  $C_{iq} = 0$  for  $i < q$ ,  $C_{qq} = f_k$ , and  $C_{iq}$  is a polynomial in  $f_1, \dots, f_{k-1}, f_{k+1}, \dots, f_r$  for  $i > q$  (the degrees of the entries of  $C$  determine the ordering).

Equation (3-1) is used first to determine  $C$ , and then to determine  $M$ . It may be used to determine  $C$  since it implies the polynomial congruence  $0 \equiv JH'^tJC \pmod{f_k}$ , where  $H'$  is the cofactor matrix of  $H$ ; each nonzero entry of  $C$  is a linear combination of (known from their degree) monomials in the basic invariants, and the above polynomial congruence is sufficient to determine the coefficients of the linear combination.

**Example 3.1.** Sufficient data to construct the matrix of basic derivations for  $G_{24}$  is given on page 284 in [Orlik and Terao 92]. Note, however, that the formula given page 264 in [Orlik and Terao 92] for its determinant contains a typo. The correct formula is

$$\begin{aligned} \Delta_{24} = & -2048x^9y + 22016x^6y^3 - 60032x^3y^5 \\ & + 1728y^7 - 256x^7z + 1088x^4y^2z \\ & + 1008xy^4z^2 - 88x^2yz^2 + z^3. \end{aligned}$$

To check that such a formula is correct, it suffices to substitute the invariants: the result should be the product

$$\begin{pmatrix} 8x & 12y + 12xy & 20y + \frac{1}{135}xy & 24 + 24x - \frac{1}{135}y^2 & 1 & 0 \\ 12y & 18x^2 - 97200y^2 + 18x^3 & -36 - 36x + \frac{1}{90}x^3 & -42xy - \frac{1}{90}x^2y & 0 & 1 \\ 20y & -36 - 72x + 60xy^2 - 36x^2 & -\frac{1}{270}x - \frac{1}{270}x^2 - \frac{1}{54}y^2 & \frac{1}{270}y + \frac{1}{270}xy + \frac{1}{54}x^2y & 0 & 1 \\ 24 + 24x & -42xy - 42x^2y - 60y^3 & -\frac{1}{54}y - \frac{1}{54}xy - \frac{1}{135}x^2y & \frac{1}{135}xy^2 + \frac{1}{54}x^2 + 20y^2 + \frac{1}{54}x^3 & 1 & 0 \end{pmatrix}$$

FIGURE 1. Condition for transversality at infinity of the 2-plane for  $G_{31}$ .

of the square of the linear forms defining the reflecting hyperplanes.

In Section 6, we list basic derivations for all examples (except  $G_{34}$ , for which the matrix is too large to be printed).

### 4. CHOOSING THE 2-PLANE

#### 4.1 A General Strategy

An explicit *genericity* criterion is given in [Dimca 92, Chapter 4, Theorem 1.17]: it suffices that  $P$  is transverse to all the strata of a Whitney stratification of the hypersurface. The theorem applies to a projective context. We replace  $\Delta$  by a homogeneous polynomial  $\tilde{\Delta} \in \mathbb{C}[X_0, \dots, X_r]$ . The equation  $\tilde{\Delta} = 0$  defines a projective hypersurface  $\tilde{\mathcal{H}}$ ; we are interested in the complement  $\mathbb{C}^r - \mathcal{H} = \mathbb{C}\mathbb{P}^r - \tilde{\mathcal{H}} \cup \mathbb{C}\mathbb{P}^{r-1}$ .

First, we stratify  $\mathbb{C}^r$  as follows: for all  $k \in \{0, \dots, r\}$ , set  $E_k$  to be the locus where the matrix  $M$  has rank  $k$ . This stratification is the quotient modulo  $W$  of the stratification of  $V$  by the intersection lattice of  $\mathcal{A}$ , and hence is a Whitney stratification. Moreover, the tangent space of the stratum at a given point is spanned by the columns of  $M$ . With the explicit knowledge of this matrix, there is no major difficulty in checking transversality of a given 2-plane.

**Example 4.1.** For  $G_{31}$ , one may check that the transversality at infinity is satisfied by the 2-plane of the equations

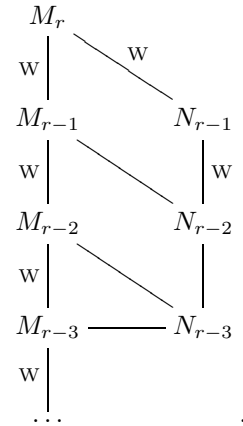
$$\begin{aligned} z &= y \\ t &= 1 + x. \end{aligned}$$

The affine transversality condition for this 2-plane is that, for each value of  $x$  and  $y$ , the matrix in Figure 1 has rank 4 (the matrix of basic derivations for  $G_{31}$  is given in Section 6), where the first four columns generate the tangent vector to the local stratum of the discriminant

and the last two columns generate the tangent vector space to the 2-plane.

To apply [Dimca 92, Chapter 4, Theorem 1.17], we also need a stratification of the hyperplane at infinity  $\mathbb{C}\mathbb{P}^{r-1}$ . Let  $\mathcal{H}_\infty := \tilde{\mathcal{H}} \cap \mathbb{C}\mathbb{P}^{r-1}$ . We view  $\mathcal{H}_\infty$  as an algebraic hypersurface in  $\mathbb{C}\mathbb{P}^{r-1}$ , defined by the equation  $\sqrt{\Delta_\infty} = 0$ , where  $\Delta_\infty$  is the homogeneous part of highest degree of  $\Delta$  and  $\sqrt{\Delta_\infty}$  is a reduced version of  $\Delta_\infty$ . We set  $N_{r-1} := \mathbb{C}\mathbb{P}^{r-1} - \mathcal{H}_\infty$ ,  $N_{r-3} := (\mathcal{H}_\infty)_{\text{sing}} \cup \overline{M_{r-2}}$  and  $N_{r-2} := \mathcal{H}_\infty - N_{r-3}$ .

Together, the  $M_i$ 's and the  $N_i$ 's form a stratification (without border condition), with incidence diagram:



We mark W where we know that the incidence satisfies Whitney's conditions. We have already explained why the first column is a Whitney stratification. It is trivial that  $M_r$  is Whitney over  $N_{r-1}$  and that  $N_{r-1}$  is Whitney over  $N_{r-2}$ . By splitting  $N_{r-3}$  into smaller strata, we may ensure that everything below  $N_{r-2}$  and  $M_{r-2}$  is Whitney (see, for example, the construction explained at the beginning of [Gibson et al. 76]).

**Question 4.2.** Does  $M_{r-1}$  satisfy Whitney's conditions over  $N_{r-2}$ ?

$$\cdots \longrightarrow \pi_2(\overline{E}, y_0) \longrightarrow \pi_1(L - L \cap \mathcal{H}, x_0) \xrightarrow{t_*} \pi_1(E, (x_0, y_0)) \xrightarrow{p_*} \pi_1(\overline{E}, y_0) \longrightarrow 1$$

FIGURE 2. Exact sequence for the fibration of  $E \rightarrow \overline{E}$ .

Note that, since  $\Delta_\infty$  is not reduced, the points of  $N_{r-2}$  are not smooth in  $\mathcal{H}$ , so the answer is not that trivial. It is a pity that no software is available to answer such a question, on specific examples with explicit equations.

**Example 4.3.** For  $G_{31}$ , we represent points of  $\mathbb{CP}^5$  by 5-tuples  $(h, x, y, z, t)$ , with either  $h = 1$  (affine portion) or  $h = 0$  (space at infinity). The strata  $M_i$  have explicit equations, using the matrix given in Section 6. The affine hypersurface  $\mathcal{H}$  is given by  $h = 1$  and  $\Delta(x, y, z, t) = 0$ , where  $\Delta$  is the determinant of the relevant matrix from Section 6. We have  $\Delta_\infty = -\frac{4}{27}x^7z^2t - \frac{8}{81}x^6yz^3$ , thus a reduced equation for  $\mathcal{H}_\infty$  is  $xz(3xt + 2yz)$  (and  $h = 0$ ). One may prove (by means of Gröbner basis) that if a sequence  $(1, x_m, y_m, z_m, t_m)_{m \in \mathbb{N}}$  of points in  $M_2 \cup M_1 \cup M_0$  converges to  $(0, \bar{x}, \bar{y}, \bar{z}, \bar{t})$ , then either  $(\bar{x}, \bar{y}) = (0, 0)$ , or  $(\bar{x}, \bar{z}) = (0, 0)$ , or  $(\bar{t}, \bar{z}) = (0, 0)$ . This locus actually coincides with  $(\mathcal{H}_\infty)_{\text{sing}}$ , thus  $N_{r-3}$  is the complement in  $\mathcal{H}_\infty$  of this locus (this explains why the particular 2-plane given earlier avoids  $N_{r-3}$ : the points at infinity of the 2-plane have the form  $(0, x, y, y, x)$ , where either  $x \neq 0$  or  $y \neq 0$ ). Question 4.2 specializes to: is  $\mathcal{H}_{\text{smooth}}$  Whitney over  $(\mathcal{H}_\infty)_{\text{smooth}}$ ?

We may now explain what we have checked and what is missing to turn our conjectures into theorems:

- for all six examples, our presentations were obtained by applying Van Kampen’s method to the algebraic curves obtained with particular 2-planes.
- for all six examples, we have checked that the 2-planes are transversal to the affine strata  $M_0, \dots, M_r$ .
- for all examples but  $G_{34}$ , we have computed (by means of Gröbner basis) equations for  $N_{r-1}, N_{r-2}$ , and  $N_{r-3}$  and checked that our 2-planes are also transversal to these strata. Transversality implies that the 2-planes do not intersect  $N_{r-3}$  and, therefore, remain transversal to the Whitney refinement of  $N_{r-3}$ . Therefore, if Question 4.2 had a positive answer, our conjectures would be theorems (except for  $G_{34}$ ).
- note that it is easy to check that our 2-planes give generators of the fundamental group, and any ho-

motopy in the 2-plane is a homotopy in  $\mathbb{C}^r$ . Therefore, we know for sure that there are presentations for the braid groups obtained by adding relations to our conjectural presentations. On the other hand, we have checked that adding quadratic relations to our conjectural presentations yields actual presentations for the complex reflection group. Any missed relation should be trivial in this quotient.

### 4.2 A Strategy for Three-Dimensional Groups

Another, more algebraic, approach can be used to find good 2-planes. Although we may start the discussion with any of our examples, it will be conclusive only for three-dimensional groups. We work with the setting and notations from [Bessis 01, Section 2.2]: we have  $\Delta \in \mathbb{C}[X_1, \dots, X_r]$  ( $\Delta$  plays the part of the polynomial  $P$  of *loc. cit.*). We distinguish the variable  $X := X_r$ , we choose a generic (in the sense of *loc. cit.*) line  $L$  of direction  $X$ . Viewed as a polynomial in only the variable  $X$  (with coefficients involving the other variables),  $P$  has a discriminant  $\text{Disc}(P_X)$ . Let  $E := \{v \in \mathbb{C}^r \mid P(v) \neq 0, \text{Disc}(P_X)(v) \neq 0\}$ . We denote by  $p$  the projection  $(x_1, \dots, x_r) \mapsto (x_1, \dots, x_{r-1})$ . Let  $\overline{E} := p(E)$ . The map  $p$  induces a fibration  $E \rightarrow \overline{E}$ , whose exact sequence ends as shown in Figure 2. In our setting,  $\Delta$  is monic in  $X$  (since  $d_r$  is regular, it follows from [Bessis 01, Lemma 1.6]). It is then easy to construct a section  $s : \pi_1(\overline{E}, y_0) \rightarrow \pi_1(E, (x_0, y_0))$  of  $p_*$ .

The basespace  $\overline{E}$  is the complement in  $\mathbb{C}^{r-1}$  of the hypersurface of equation  $\text{Disc}(\Delta_X) = 0$ . In our setting,  $\text{Disc}(\Delta_X)$  is a weighted homogeneous polynomial.

When  $r = 3$ , this implies that  $\pi_2(\overline{E}, y_0) = 0$  (complements of weighted homogeneous curves are  $K(\pi, 1)$ ). We then have a semidirect product structure

$$\pi_1(E, (x_0, y_0)) \simeq \pi_1(L - L \cap \mathcal{H}, x_0) \rtimes \pi_1(\overline{E}, y_0).$$

To obtain a presentation for  $\pi(\mathbb{C}^r - \mathcal{H})$ , one starts with a presentation for  $\pi_1(E, (x_0, y_0))$  and adds relations forcing elements of  $\pi_1(\overline{E}, y_0)$  to become trivial. It is an easy exercise to check, in this setting, that any 2-plane  $P$  satisfying:

- (i) the line  $L$  is contained in  $P$ , and
- (ii) the image line  $p(P)$  is such that  $p(P) \cap \overline{E} \hookrightarrow \overline{E}$  is  $\pi_1$ -surjective,

is good for our purposes. In our examples, it is easy to construct such planes, since  $\text{Disc}(\Delta_X)$  is monic in one of the remaining variables. This is how we obtained, for  $G_{24}$  and  $G_{27}$ , theorems rather than conjectures.

Note that, for other groups, all assumptions used here (including the monicity of  $\text{Disc}(\Delta_X)$ ) remain valid, except that we do not know whether  $\pi_2(\overline{E}, y_0) = 0$ . Instead of answering Question 4.2, checking that  $\pi_2(\overline{E}, y_0) = 0$  would turn our conjectures into theorems.

## 5. THE PACKAGE VKCURVE

Once a 2-plane  $P$  has been chosen, it is enough to feed VKCURVE with the equation of the curve  $P \cap \mathcal{H}$  to obtain a presentation of  $\pi_1(P - (P \cap \mathcal{H}))$ .

**Example 5.1.** For  $G_{31}$ , when computing the determinant of  $M_{31}$  and evaluating at  $z = y$  and  $t = 1 + x$ , we obtain the following equation for  $P \cap \mathcal{H}$ :

$$\begin{aligned} \Delta'_{31} = & 746496 + 3732480x - 3111936xy^2 \\ & - \frac{93281756}{27}xy^4 + \frac{58341596}{27}xy^6 + 7464960x^2 \\ & - 384y^2 - 9334272x^2y^2 + \frac{17556484}{27}x^2y^4 \\ & + \frac{43196}{27}x^2y^6 + 7464576x^3 - \frac{756138248}{81}x^3y^2 \\ & + \frac{192792964}{81}x^3y^4 + \frac{16}{81}x^3y^6 + 3730944x^4 \\ & - \frac{139967996}{81}y^4 - \frac{84021416}{27}x^4y^2 + \frac{82088}{27}x^4y^4 \\ & + 744192x^5 + \frac{43192}{27}x^5y^2 - \frac{1720}{27}x^5y^4 \\ & - \frac{124412}{81}x^6 + 777600800y^6 + \frac{95896}{81}x^6y^2 \\ & - \frac{8}{81}x^6y^4 - \frac{10364}{27}x^7 - \frac{4}{27}x^7y^2 + \frac{4}{27}x^8 \\ & - \frac{8}{81}y^8 - \frac{4}{27}x^8y^2 + \frac{4}{81}x^9. \end{aligned}$$

On a 3 GHz Pentium IV, VKCURVE needs about one hour to deal with this example.

Writing VKCURVE was of course the most difficult part of our work. This software accepts as input any square-free polynomial in  $\mathbb{Q}[i][X, Y]$  and computes a presentation for the fundamental group of the complement of the corresponding complex algebraic curve. The program does not use floating point computations (even when computing monodromy braids); therefore, there is no issue of numerical accuracy, and the result is “certified” to be correct (provided that our implementation does not contain mathematical errors).

The remainder of this section is an overview of the algorithms used in VKCURVE. We rely on the version of Van Kampen’s method exposed in [Bessis 03, Procedure 4], where it is decomposed into four steps.

### 5.1 Implementing Steps 1 and 2

Starting with our polynomial  $P \in \mathbb{Q}[i][X, Y]$ , we view it as a one-variable polynomial in  $\mathbb{Q}[i][Y][X]$  and compute its discriminant  $\Delta \in \mathbb{Q}[i][Y]$ . The discriminant  $\Delta$  may not be reduced; to compute approximations  $\tilde{y}_1, \dots, \tilde{y}_r \in \mathbb{Q}[i]$  of its complex roots  $y_1, \dots, y_r$ , we apply Newton’s method to the reduced polynomial  $\Delta_0$  obtained by dividing  $\Delta$  by the resultant of  $\Delta$  and  $\Delta'$ . As is proved in the beautiful article [Hubbard et al. 01], Newton’s method can be made into a failsafe algorithm producing arbitrarily good approximations of  $y_1, \dots, y_r$ .

Since we will reuse them later, we recall a few trivialities about complex polynomials. Let  $P \in \mathbb{C}[Z]$ . Let  $\alpha_1, \dots, \alpha_n$  be the complex roots of  $P$ . Let  $z \in \mathbb{C}$ . If  $P'(z) \neq 0$ , we set  $N_P(z) := z - \frac{P(z)}{P'(z)}$ . Considering the first order approximation of  $P$  around  $z$ , we expect  $P(N_P(z))$  to be close to 0. Newton’s method consists of starting with  $z_0 \in \mathbb{C}$  (chosen randomly, or smartly as in [Hubbard et al. 01]) and constructing iteratively  $z_{m+1} := N_P(z_m)$ , hoping that  $(z_m)$  will converge towards a root of  $P$ —which indeed happens for “many” choices of  $z_0$ . How may we decide that a given  $z_n$  is a “good enough” approximation?

**Lemma 5.2.** *Assume  $P$  has  $n$  distinct roots  $\alpha_1, \dots, \alpha_n$ . Let  $z \in \mathbb{C}$ , with  $P'(z) \neq 0$ . Then, there exists  $\alpha \in \{\alpha_1, \dots, \alpha_n\}$  such that  $|z - \alpha| \leq n \left| \frac{P(z)}{P'(z)} \right|$ .*

*Proof:* If  $P(z) = 0$ , the result is trivial. Otherwise, we have  $\frac{P'(z)}{P(z)} = \sum_{i=1}^n \frac{1}{z - \alpha_i}$ . Choose  $i$  such that, for all  $j$ ,  $|z - \alpha_i| \leq |z - \alpha_j|$ . By triangular inequality,  $\frac{1}{|z - \alpha_i|} \geq \left| \frac{P'(z)}{P(z)} \right| - \sum_{j \neq i} \frac{1}{|z - \alpha_j|} \geq \left| \frac{P'(z)}{P(z)} \right| - (n-1) \frac{1}{|z - \alpha_i|}$ . The result follows.  $\square$

Although elementary, this lemma provides a very inexpensive (in terms of computational time) test for deciding whether a tentative list  $\tilde{\alpha}_1, \dots, \tilde{\alpha}_n$  of complex numbers “separates” the roots (i.e., whether there exists  $\varepsilon_1, \dots, \varepsilon_n$  such that the disks  $D(\tilde{\alpha}_i, \varepsilon_i)$  do not overlap and each of them contains a root of  $P$ ).

Instead of working with the exact Newton’s method, we use a truncated version, where  $N_P(z)$  is replaced by an approximate  $(a + ib)10^k$ , where  $a, b \in \mathbb{Z}$  and  $k$  is an integer slightly smaller than  $\log_{10} \left| \frac{P'(z)}{P(z)} \right|$ . This is to avoid

the very fast increase of the denominators when the exact method is carried out in  $\mathbb{Q}[\tilde{i}]$ : the complexity of the exact method is very good from the “abstract” viewpoint (the number of iterations), but in practice really bad (each individual iteration involves costly operations on very big integers). Of course, our modification does not make the method less rigorous, since the test can be performed exactly. The main difference between our implementation and floating point is that  $k$  is modified dynamically and has no preassigned bound.

Once separating approximates  $\tilde{y}_1, \dots, \tilde{y}_r \in \mathbb{Q}[\tilde{i}]$  of the roots of  $\Delta$  have been obtained, Step 2 of [Bessis 03, Procedure 4] is performed as follows: first, we construct the Voronoi cells around  $\tilde{y}_1, \dots, \tilde{y}_r$ ; then, concatenating some of the affine segments bounding the Voronoi cells, we construct, for each  $i$ , a loop  $\gamma_i$  representing a meridian around  $\tilde{y}_i$ ; it is easy to make sure that we recover a meridian around the actual  $y_i$ .

## 5.2 Step 3: Computing Monodromy Braids

[Bessis 03, Procedure 12] decomposes Step 3 into smaller steps a–e. Only Substep *a* is not a straightforward algebraic manipulation, and most of the computational time is spent there. The problem is as follows: let  $[y_0, y_1]$  be one of the affine segments involved in the  $\gamma_i$ . For  $t \in [0, 1]$ , denote by  $P_t \in \mathbb{Q}[\tilde{i}][X]$  the polynomial obtained by evaluating  $P$  at  $Y = (1-t)y_0 + ty_1$ . We want to compute the word in Artin generators corresponding to the real projection of the braid obtained by tracking the roots of  $P_t$  when  $t$  runs over  $[0, 1]$ .

As we have seen above, we may find  $x_1, \dots, x_n \in \mathbb{Q}[\tilde{i}]$  separating the roots of  $P_0$ . Concretely, using Lemma 5.2, we iterate a truncated Newton method until, when we set  $\varepsilon_i := \inf_{j \neq i} \frac{|x_i - x_j|}{2}$  (this is a simple way, though not optimal, to ensure that  $\forall i, j, |x_i - x_j| > \varepsilon_i + \varepsilon_j$ ), we have

$$\forall i, \left| \frac{P_0(x_i)}{P_0'(x_i)} \right| < \frac{\varepsilon_i}{n}.$$

For each  $i$ , consider the polynomial

$$Q_i := \varepsilon_i^2 |P_t'(x_i)|^2 - n^2 |P_t(x_i)|^2 \in \mathbb{Q}[t].$$

By assumption, we have  $\forall i, Q_i(0) > 0$ . Whenever  $t_0 \in [0, 1] \cap \mathbb{Q}$  is such that  $\forall t \in [0, t_0], \forall i, Q_i(t) > 0$ , we know that, for  $t \in [0, t_0]$ , the strings of the monodromy braids will be in the cylinders of radius  $\varepsilon_i$  around the  $x_i$ . This fragment of the monodromy braid can be replaced by the constant braid with strings fixed at the positions given by the  $x_i$ . Set  $y'_0 := (1-t_0)y_0 + y_1$ ,  $x'_i := N_{P_{t_0}}(x_i)$ . Though the  $x_i$  already separate the roots of  $P_{t_0}$ , the  $x'_i$  should be

“better” approximates. We compute new radii  $\varepsilon'_i$  separating the  $x'_i$  and iterate, studying now the monodromy braid over  $[y'_0, y_1]$ , with initial approximates  $x'_1, \dots, x'_n$ . Eventually, we hope that after some number of iterations,  $t_0 = 1$  will suit.

The main difficulty is to find an actual  $t_0$  such that for all  $t \in [0, t_0], \forall i, Q_i(t) > 0$ . On the one hand, we want it to be as large as possible, to avoid unnecessary iterations; on the other hand, computing the largest theoretical value for  $t_0$ , for example using Sturm sequences, is very costly. Finding a good balance is a delicate art. The curious reader may have a look at the source of the VKCURVE function `FollowMonodromy`, where a very naive method is used, together with careful coding and adaptative heuristics (note that, in `FollowMonodromy`, one actually computes a distinct  $t_0$  for each individual string: the above description is simplified for the sake of clarity).

## 5.3 Step 4: Writing and Simplifying the Presentation

Working with GAP, it is then straightforward to write a presentation. However, this presentation is much more complicated than desirable. Since no “normal form” theory exists for arbitrary group presentations, it is not clear how one can simplify it and obtain one of our “pretty” presentations. Fortunately, some natural heuristics (typically, replace a generator by its conjugate by another generator, try to simplify, and iterate in the regions of the tree of all possibilities where the total length of the presentation tends to decrease) happen to be quite effective in dealing with the (highly redundant) presentations obtained with Van Kampen’s method. Playing with these (non-deterministic) heuristics, which are part of VKCURVE, we obtained quite easily a few “simple” presentations. At this point, in the absence of a general combinatorial theory of generalized braid groups, there is some arbitrariness in deciding which one should be retained; in most cases though, one of them clearly emerged as being the “prettiest.”

## 6. EXPLICIT MATRICES OF BASIC DERIVATIONS

### 6.1 The Group $G_{24}$

With Klein’s matrices (as in [Orlik and Terao 92]) the first invariant is  $f_1 = x^3y + zy^3 + xz^3$ . The others are  $f_2 = \det(\text{Hessian}(f_1))/108$  and  $\text{bord}(f_1, f_2)/36$ .

The basic derivations are in Figure 3(a).



$$\begin{aligned}
 & \begin{pmatrix} 4x & 6y^2 & 14z - 36x^2y \\ 6y & -z & 128xy^2 - 7x^4 \\ 14z & 128xy^3 - 6x^2z - 7x^4y & 287xyz - 35x^3y^2 - 294y^4 + 7x^6 \end{pmatrix} \\
 & \text{(a)} \\
 & \begin{pmatrix} 6x & 12y^2 & 30z + 234x^3y \\ 12y & -4z & -34xz - 1362x^2y^2 + 156y^3 + 900x^4y - 270x^6 \\ 30z & -34xyz - 1362x^2y^3 + 78x^3z + 156y^4 + 900x^4y^2 - 270x^6y & 4836xy^4 + 330y^2z - 3349x^2yz - 17727x^3y^3 + 2013x^4z + 7110x^5y^2 + 135x^7y - 810x^9 \end{pmatrix} \\
 & \text{(b)} \\
 & \frac{1}{80} \begin{pmatrix} 320x & 640y^2 & 960z + 2xy & 1600t + 8yz \\ 640y & 4096000t + 225280yz + 1280xy^2 & 64xz + 4x^2y & -640tx + 16xyz + 1536z^2 \\ 960z & -12800tx - 640xyz & 200t - 5yz + 3x^2z & -10ty + 10tx^2 - 8xz^2 \\ 1600t & -51200tz - 640txy - 1280yz^2 & -10ty + 8tx^2 - 4xz^2 & 72txz - 96z^3 \end{pmatrix} \\
 & \text{(c)} \\
 & \frac{1}{270} \begin{pmatrix} 2160x & 3240ty & 5400z + 2xy & 6480t - 2y^2 \\ 3240y & 4860tx^2 - 26244000z^2 & -9720t + 3x^3 & -11340xz - 3x^2y \\ 5400z & 16200xz^2 - 9720t^2 & -tx - 5yz & ty + 5x^2z \\ 6480t & -11340txz - 16200yz^2 & -5ty - 2x^2z & 2xyz + 5tx^2 + 5400z^2 \end{pmatrix} \\
 & \text{(d)} \\
 & \frac{1}{128} \begin{pmatrix} 512x & 768yz & 1280z + (\frac{-4}{3})xy & 1536t - 4y^2 & 2304u - 4ty \\ 768y & -663552u + 1152x^2z & 768t - 2x^3 & 8064xz - 6x^2y & -6tx^2 + 23040z^2 \\ 1280z & 768tz - 1152ux & (\frac{-1}{3})tx + 3yz & 576u - ty + 9x^2z & 6uy + 36xz^2 - 4t^2 \\ 1536t & -3456uy + 8064xz^2 & 576u + 3ty - 5x^2z & -15xyz + 9tx^2 + 11520z^2 & -42txz + 18u x^2 + 108yz^2 \\ 2304u & -3456tu + 23040z^3 & 6uy - 4xz^2 & 18ux^2 - 12yz^2 & 90uxz - 48tz^2 \end{pmatrix} \\
 & \text{(e)}
 \end{aligned}$$

FIGURE 3. Basic derivations of (a)  $G_{24}$ , (b)  $G_{27}$ , (c)  $G_{29}$ , (d)  $G_{31}$ , and (e)  $G_{33}$ .

## 6.2 The Group $G_{27}$

With Wiman's matrices (as in [Orlik and Terao 92]), the first invariant is  $f_1 = -135xyz^4 - 45x^2y^2z^2 + 10x^3y^3 + 9x^5z + 9y^5z + 27z^6$ . The others are  $f_2 = \det(\text{Hessian}(f_1))/6750$  and  $\text{bord}(f_1, f_2)/5400$ .

The basic derivations are in Figure 3(b).

## 6.3 The Groups $G_{29}$ and $G_{31}$

For the data relative to  $G_{29}$  and  $G_{31}$ , see [Maschke 87]. The group  $G_{31}$  is generated by the matrices  $T$  and  $Ue^{2i\pi/8}$  in Maschke's notations.

$G_{29}$  is the subgroup that leaves invariant  $\Phi_1$ , which we take as the first invariant. Then, we choose  $(-1/20736) \det(\text{Hessian}(\Phi_1)) = (4F_8 - \Phi_1^2)/3$ . We do not

choose  $F_{12}$  but the simpler  $((\Phi_1^3 - 3\Phi_1F_8)/2 + F_{12})/108$ . We do not choose  $F_{20}$  but the simpler  $(F_{20} - F_8F_{12})/1296$ .

For  $G_{31}$  we choose  $F_8$  and  $F_{12}$ . Then as for  $G_{29}$  we choose  $(F_{20} - F_8F_{12})/1296$ ; the fourth is still (as in [Orlik and Terao 92])  $\det(\text{Hessian}(F_8))/265531392$ .

The basic derivations of  $G_{29}$  and  $G_{31}$  are in Figures 3(c) and 3(d), respectively.

## 6.4 The Group $G_{33}$

We take the matrices and invariants of [Burkhardt 91, pages 208–209] with the corrections indicated in [Orlik 89]. The third invariant is taken to be  $\det(\text{Hessian}(J_4))/63700992$  where  $J_4$  is the first invariant.

The basic derivations of  $G_{33}$  are in Figure 3(e).

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