HOMOTOPY LIE ALGEBRA OF CLASSIFYING SPACES FOR HYPERBOLIC COFORMAL 2-CONES

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Abstract

In this paper, we show that the rational homotopy Lie algebra of classifying spaces for certain types of hyperbolic coformal 2-cones is not nilpotent.

1. Introduction

A simply connected space $X$ is called an $n$-cone if it is built up by a sequence of cofibrations

$$Y_k \xrightarrow{f} X_{k-1} \xrightarrow{g_k} X_k$$

with $X_0 = *$ and $X_n \simeq X$. One can further assume that $Y_k \simeq \Sigma^{k-1} W_k$ is a $(k-1)$-fold suspension of a connected space $W_k$ [3]. In particular a 2-cone $X$ is the cofibre of a map between two suspensions

$\Sigma A \xrightarrow{f} \Sigma B \to X$. (1)

Spaces under consideration are assumed to be 1-connected and of finite type, that is, $H^1(X; \mathbb{Q})$ is a finite-dimensional $\mathbb{Q}$-vector space. To every space $X$ corresponds a free chain Lie algebra of the form $(L(V), \delta)$ [2], called a Quillen model of $X$. It is an algebraic model of the rational homotopy type of $X$. In particular, one has an isomorphism of Lie algebras $H_*(L(V), \delta) \cong \pi_*(\Omega X) \otimes \mathbb{Q}$. The model is called minimal if $\delta V \subset L^{\geq 2}(V)$. A space $X$ is called coformal if there is a map of differential Lie algebras $(L(V), \delta) \to (\pi_*(\Omega X) \otimes \mathbb{Q}, 0)$ that induces an isomorphism in homology. Any continuous map $f : X \to Y$ has a Lie representative $\tilde{f} : (L(W), \delta) \to (L(V), \delta)$ between respective models of $X$ and $Y$.

If $X$ is a 2-cone as defined by (1) and $\tilde{f} : L(W) \to L(V)$ is a model of $f$, then a Quillen model of the cofibre $X$ of $f$ is obtained as the push out of the following diagram:

$$
\begin{array}{ccc}
(L(W), 0) & \xrightarrow{\tilde{f}} & (L(V), 0) \\
\downarrow{i} & & \downarrow{i} \\
(L(W \oplus sW), d) & \xrightarrow{\tilde{f}} & (L(V \oplus sW), \delta)
\end{array}
$$
where \((L(W \oplus sW), d)\) is acyclic. Moreover the differential on \(L(V \oplus sW)\) verifies \(\delta sW \subset L(V)\). Hence a 2-cone \(X\) has a Quillen model of the form \((L(V_1 \oplus V_2), \delta)\) such that \(\delta V_1 = 0\) and \(\delta V_2 \subset L(V_1)\).

A Sullivan model of a space \(X\) is a cochain algebra \((\wedge Z, d)\) that algebraically models the rational homotopy type of \(X\). In particular, one has an isomorphism of graded algebras \(H^*(\wedge Z, d) \cong H^*(X; \mathbb{Q})\). The model is called minimal if \(dZ \subset \wedge^{\geq 2} Z\).

In this case the vector spaces \(Z^n\) and \(\text{Hom}(\pi_n(X), \mathbb{Q})\) are isomorphic. If \(X\) has the rational homotopy type of a finite CW-complex, we say that \(X\) is elliptic if \(Z\) is finite dimensional, otherwise \(X\) is called hyperbolic.

2. Models of classifying spaces

Henceforth \(X\) will denote a simply connected finite CW-complex and \(L_X\) its homotopy Lie algebra. Let \(\text{aut} X\) denote the space of free self homotopy equivalences of \(X\), \(\text{aut}_1(X)\) the path component of \(\text{aut} X\) containing the identity map of \(X\). The space \(\text{Baut}_1(X)\) classifies fibrations with fibre \(X\) over simply connected base spaces [4].

The Schlessinger-Stasheff model for \(\text{Baut}_1(X)\) is defined as follows [12]. If \((\mathbb{L}(V), \delta)\) is a Quillen model of \(X\), we define a differential Lie algebra \(\text{Der} \mathbb{L}(V) = \bigoplus_{k \geq 1} \text{Der}_k \mathbb{L}(V)\) where \(\text{Der}_k \mathbb{L}(V)\) is the vector space of derivations of \(\mathbb{L}(V)\) which increase the degree by \(k\); with the restriction that \(\text{Der}_1 \mathbb{L}(V)\) is the vector space of derivations of degree 1 that commute with the differential \(\delta\).

Define the differential Lie algebra \((s\mathbb{L}(V) \oplus \text{Der} \mathbb{L}(V), D)\) as follows:

- The graded vector space \(s\mathbb{L}(V) \oplus \text{Der} \mathbb{L}(V)\) is isomorphic to \(s\mathbb{L}(V) \oplus \text{Der} \mathbb{L}(V)\),
- If \(\theta, \gamma \in \text{Der} \mathbb{L}(V)\) and \(sx, sy \in s\mathbb{L}(V)\), then \([\theta, \gamma] = \theta \gamma - (-1)^{|\theta||\gamma|} \gamma \theta\), \([\theta, sx] = (-1)^{|\theta|} \theta(x)\) and \([sx, sy] = 0\).
- The differential \(D\) is defined by \(D\theta = [\delta, \theta]\), \(D(sx) = -s\delta x + ad x\), where \(ad x\) is the inner derivation determined by \(x\).

From the Sullivan minimal model \((\wedge Z, d)\), Sullivan defines the graded differential Lie algebra \((\text{Der} \wedge Z, D)\) as follows [13]. For \(k > 1\), the vector space \((\text{Der} \wedge Z)_k\) consists of the derivations on \(\wedge Z\) that decrease the degree by \(k\) and \((\text{Der} \wedge Z)_1\) is the vector space of derivations of degree 1 verifying \(d\theta + \delta d = 0\). For \(\theta, \gamma \in \text{Der} \wedge V\), the Lie bracket is defined by \([\theta, \gamma] = \theta \gamma - (-1)^{|\theta||\gamma|} \gamma \theta\) and the differential \(D\) is defined by \(D\theta = [d, \theta]\).

We have the following result:

**Theorem 1.** [13, 12, 14] The differential Lie algebras \((\text{Der} \wedge Z, D)\) and \((s\mathbb{L}(V) \oplus \text{Der} \mathbb{L}(V), D)\) are models of the classifying space \(\text{Baut}_1(X)\).

An indirect proof of the Schlessinger-Stasheff model is given in [8, Theorem 2].

3. The classifying space spectral sequence

Recall that if \((L, \delta)\) is a graded differential Lie algebra, then \(L\) becomes an \(UL\) module by the adjoint representation \(ad : L \to \text{Hom}(L, L)\). In the sequel all Lie
algebras are endowed with the above module structure.

Let \((L(V), \delta)\) be a Quillen model of a finite CW-complex and \((TV, d)\) its enveloping algebra. On the \(TV\)-module \(TV \otimes (Q \oplus sV)\), define a \(Q\)-linear map

\[ S : TV \otimes (Q \oplus sV) \to TV \otimes (Q \oplus sV) \]

as follows:

- \(S(1 \otimes x) = 0\) for all \(x \in Q \oplus sV\),
- \(S(v \otimes 1) = 1 \otimes sv\) for all \(v \in V\),
- If \(a \in TV\) and \(x \in TV \otimes (Q \oplus sV)\) with \(|x| > 0\), then \(S(ax) = (-1)^{|a|}a.S(x)\).

The differential on the \(TV\)-module \(TV \otimes (Q \oplus sV)\) is defined by

\[ D(1 \otimes sv) = v \otimes 1 - S(dv \otimes 1)\] for \(v \in V\) and \(D(1 \otimes 1) = 0\).

It follows from \([1]\) that \((TV \otimes (Q \oplus sV), D)\) is acyclic, hence it is a semifree resolution of \(Q\) as a \((TV, d)\)-module [6, 86].

Using the Schlessinger-Stasheff model of the classifying space, the author proved the following:

**Theorem 2.** [8] The differential graded vector spaces \(Hom_{TV}(TV \otimes (Q \oplus sV), L(V))\) and \(sL(V) \oplus \text{Der} L(V)\) are isomorphic. Moreover, for \(n \geq 0\), the \(Q\)-vector spaces \(\text{Ext}^n_{TV}(Q, L(V))\) and \(\pi_n(\Omega B aut_1 X) \otimes Q\) are isomorphic.

In particular if \(X\) is a coformal space, one has an isomorphism \(\pi_n(B aut_1 X) \otimes Q \cong \text{Ext}^n_{UL_X}(Q, L_X)\). Therefore \(\pi_n(B aut_1 X) \otimes Q\) can be computed by the means of a projective resolution of \(Q\) as an \(UL_X\)-module.

Consider the complex \(P = Hom_{TV}(TV \otimes (Q \oplus sV), L(V))\). Filter \(V\) as follows

\[ F_0V = 0, \quad F_{p+1}V = \{ x \in V : dx \in \mathbb{L}(F_p V) \} \]

We will denote \(V_p = F_p V / F_{p-1} V\). If \(F_{n-1} V \neq F_n V = V\), following Lemaire [10] we say that \(V\) is of length \(n\). We will restrict to spaces with a Quillen model of length \(n\).

Define a filtration on \(P = TV \otimes (Q \oplus sV)\) as follows:

\[ P_0 = TV \otimes Q, \quad P_1 = TV \otimes (Q \oplus sV), \ldots, P_n = TV \otimes (Q \oplus sV) \]

We filter the complex

\[ Hom_{TV}(TV \otimes (Q \oplus sV), L(V)) \]

by

\[ F_k = \{ f : f(P_{k-1}) = 0 \} \]

This yields a spectral sequence \(E_r\) such that \(E_1^{p,q} = \text{Hom}_Q(sV_p, L_X)\) for \(p > 1\), \(E_1^{0,q} = \text{Hom}_Q(Q, L_X)\) and that converges to \(\text{Ext}^*_TV(Q, L(V))\). This sequence will be called the **classifying space spectral sequence** of \(X\).

Now assume that \(X\) is coformal and let \(A = UL_X\). If \(L(V)/I\) is a minimal presentation of \(L_X\), then there is a quasi-isomorphism \((L(V_1 \oplus V_2 \oplus \cdots \oplus V_n), \delta) \to L_X\) which extends to \(p : (TV, d) \xrightarrow{\cong} (A, 0)\). The \((E_1, d)\) term provides a resolution

\[ \cdots \to A \otimes sV_n \to A \otimes sV_{n-1} \to \cdots A \otimes sV_1 \to A \to Q \]
of \( Q \) as an \( A \)-module. Here the differential is given by the composition

\[
sV_n \xrightarrow{D} TV \otimes (Q \oplus sV_{n-1}) \xrightarrow{\rho \otimes id} A \otimes (Q \oplus sV_{n-1}).
\]

The spectral sequence will therefore collapse at \( E_2 \) level. Moreover \( \text{Ext}^s_A(Q, \mathcal{L}_X) \) is endowed with a Lie algebra structure verifying

\[
[\text{Ext}^p, s, \text{Ext}^q] \subset \text{Ext}^{p+q-1}.
\]

The Lie bracket can be defined using the bijection between the Koszul complex \( C^*(\mathcal{L}_X, \mathcal{L}_X) \) and derivations on the Sullivan model \( C^*(\mathcal{L}_X, Q) \) of \( X \) \([9, \text{Proposition} 4]\) (see also \([7]\) for a direct definition of the Lie bracket on \( C^*(\mathcal{L}_X, \mathcal{L}_X) \)). Alternatively one may use the bijection

\[
\text{Hom}_{TV}(TV \otimes (Q \oplus sV), \mathbb{L}(V)) \cong s\mathbb{L}(V) \oplus \text{Der} \mathbb{L}(V)
\]

to transfer a Lie algebra structure on \( \text{Hom}_{TV}(TV \otimes (Q \oplus sV), \mathbb{L}(V)) \) from \( s\mathbb{L}(V) \oplus \text{Der} \mathbb{L}(V) \).

**Definition 3.** Let \( L \) be a Lie algebra. An element \( x \in L \) is called locally nilpotent if for every \( y \in L \), there is a positive integer \( k \) such that \((ad x)^k(y) = 0\). A subset \( K \subset L \) is called locally nilpotent if each element of \( K \) is locally nilpotent.

We deduce from Equation (2) the following

**Proposition 4.** Let \( X \) be a coformal space of homotopy Lie algebra denoted \( \mathcal{L}_X \). If \( X \) has a Quillen model \((\mathcal{L}(V), \delta)\), of length \( n \), one has:

1. For \( k \neq 1 \), \( \text{Ext}^k_A(Q, \mathcal{L}_X) \) is locally nilpotent,
2. \( \text{Ext}^1_A(Q, \mathcal{L}_X) \) is a subalgebra of \( \text{Ext}_A(Q, \mathcal{L}_X) \),
3. If \( \text{Ext}^0_A(Q, \mathcal{L}_X) = 0 \), then \( \oplus_{i \geq i_0} \text{Ext}^i_A(Q, \mathcal{L}_X) \) is an ideal of \( \text{Ext}_A(Q, \mathcal{L}_X) \), for \( i_0 \geq 1 \).

We will now assume that \( X \) is a coformal 2-cone. Recall that \( X \) has a Quillen minimal model of the form \((\mathbb{L}(V_1 \oplus V_2), \delta)\), with \( \delta V_1 = 0 \) and \( \delta V_2 \subset \mathbb{L}(V_1) \). Moreover \( \pi_* (\Omega X) \otimes Q = H_*(\mathbb{L}(V_1 \oplus V_2), \delta) = \mathbb{L}(V_1)/J \), where \( J \) is the ideal of \( \mathbb{L}(V_1) \) generated by \( \delta V_2 \).

**Definition 5.** Let \( \mathbb{L}(V) \) be a free Lie algebra where \( \{a, b, c, \ldots \} \) is a basis of \( V \). Denote \( \mathbb{L}^n(V) \) the subspace of \( \mathbb{L}(V) \) consisting of Lie brackets of length \( n \). Consider a basis \( \{u_1, u_2, \ldots \} \) of \( \mathbb{L}^n(V) \) where each \( u_i \) is a Lie monomial. If \( x \in \{a, b, c, \ldots \} \), we define the length of \( u_i \) in the variable \( x \), \( l_x(u_i) \), as the number of occurrences of the letter \( x \) in \( u_i \). If \( u = \sum r_i u_i \in \mathbb{L}^n(V) \), define \( l_x(u) = \min\{l_x(u_i)\} \) and if \( v = \sum v_i \) where \( v_i \in \mathbb{L}^n(V) \), \( l_x(v) = \min\{l_x(v_i)\} \).

It is straightforward that the above definition extends to the enveloping algebra \( T(V) \).

**Theorem 6.** Let \( X \) be a coformal 2-cone and \((\mathbb{L}(V_1 \oplus V_2), \delta)\) be its Quillen minimal model. Choose a basis \( \{x_1, x_2, \ldots \} \) for \( V_1 \) and a basis \( \{y_1, y_2, \ldots \} \) for \( V_2 \). If for some \( x_i \in \{x_1, x_2, \ldots \} \), \( l_{x_i}(\delta y_j) \geq 2 \) for all \( y_j \in \{y_1, y_2, \ldots \} \), then \( \text{Ext}^2_A(Q, \mathcal{L}_X) \) is infinite dimensional.
Proof. Note that for \( i \neq k \) the element \((ad x_i)^n(x_k)\) is a nonzero homology class in 
\( H_*([L(V_1 \oplus V_2), \delta]) \) as it contains only one occurrence of \( x_k \). Take \( y_t \in \{y_1, y_2, \ldots \} \) 
and \( x_m \in \{x_1, x_2, \ldots \} \) with \( m \neq k \). For each \( n \geq 1 \), define \( f_n \in \text{Hom}_A(A \otimes sV_2, L_X) \) by 
\( f_n(sy_j) = (ad x_m)^n(x_k) \) and \( f_n(sy_j) = 0 \) for \( j \neq t \). Obviously \( f_n \in \text{Hom}_A(A \otimes sV_2, L_X) \) is a cocycle. Suppose that \( f_n \) is a coboundary. There exists 
\( g_n \in \text{Hom}_A(A \otimes sV_1, L_X) \) such that \( f_n(sy_j) = g_n(ds y_t) \). From the definition of 
the differential \( d \), one has \( ds y_t = \sum p_i s x_i \), where the \( p_i \)'s are polynomials in the 
variables \( x_1, x_2, \ldots \). From the hypothesis on the differential \( ds y_t \), one knows that 
\( l_{x_i}(p_i) \geq 2 \) for \( i \neq k \) and \( l_{x_i}(p_k) \geq 1 \). By using the number of occurrences of the 
variable \( x_k \), one deduces from the previous equalities that \((ad x_m)^n(x_k)\) equals the 
component of length 1 in \( x_k \) of \( p_k g_n(s x k) \). Therefore, in the monomial decomposition 
of \( g_n(s x k) \) (resp. \( p_k \)) there must exist \((ad x_m)^{n-k}(x_k)\) (resp. \( x_m^k \)). We obtain a 
contradiction with \( l_{x_i}(p_k) \geq 1 \).

The cocycles \( f_n \) create an infinite number of non zero classes (of distinct degrees) 
and the space \( \text{Ext}^2_A(Q, L_X) \) is infinite dimensional.

\[ \text{Corollary 7.} \] If hypotheses of the above theorem are satisfied, then \( \text{cat}(B \text{aut}_1(X)) = \infty \).

Proof. If \( sx \in \text{Ext}^{0,*} \subset L(V_1)/I \) and \( f \in \text{Ext}^{2,*} \) then \([f, sx] = \pm sf(x)\). As elements 
of \( \text{Ext}^{2,*} \) vanish on \( V_1 \), we deduce that \([\text{Ext}^{2,*}, \text{Ext}^{0,*}] = 0\). It follows from Theorem \(6 \) that \( J = \text{Ext}^{2,L_X}_I(Q, L_X) \) is an infinite dimensional ideal of \( \pi_1(\Omega B \text{aut}_1(X)) \). 
Moreover it follows from Equation (2) that \( J \) is abelian. We deduce that the category 
of \( B \text{aut}_1(X) \) is infinite [5, Theorem 12.2].

If \( X \) is an elliptic space of Sullivan minimal model \((\wedge Z, d)\) then \( \text{Der} \wedge Z \) is a 
finiti-dimensional \( Q \)-vector space. Hence the homotopy Lie algebra of \( B \text{aut}_1(X) \) is 
finite dimensional, therefore \( \pi_1(\Omega B \text{aut}_1(X)) \otimes Q \) is nilpotent. In [11], P. Salvatore 
proved that if \( X = S^{2n+1} \vee S^{2n+1} \), then \( \pi_1(\Omega B \text{aut}_1(X)) \otimes Q \) contains an element 
\( \alpha \) that is not locally nilpotent. The proof consists in the construction of two outer 
derivations \( \alpha \) and \( \beta \) of the free Lie algebra \( L(a, b) \), where \( |a| = |b| = 2n \), such that 
\( (ad \alpha)^i(\beta) \neq 0 \), for every integer \( i > 0 \). The technique can be applied to any free 
Lie algebra with at least two generators. Therefore \( \pi_1(\Omega B \text{aut}_1(X)) \otimes Q \) contains 
an element \( \alpha \) that is not locally nilpotent if \( X \) is a wedge of two spheres or more.

P. Salvatore asked if \( \pi_1(\Omega B \text{aut}_1(X)) \otimes Q \) has always such a property for ev-
ery hyperbolic space \( X \). A positive answer to this question would provide another 
characterization of the elliptic-hyperbolic dichotomy [5].

For a product space we have the following

\[ \text{Proposition 8.} \] If \( X = Y \times Z \) is a product space such that the Lie algebra 
\( \pi_1(\Omega B \text{aut}_1(Y)) \otimes Q \) is not nilpotent, then \( \pi_1(\Omega B \text{aut}_1(X)) \otimes Q \) is not nilpotent.

Proof. Let \((\wedge V, d)\) and \((\wedge W, d')\) be Sullivan models of \( Y \) and \( Z \) respectively. Therefore 
\((\wedge V \otimes \wedge W, d \otimes d')\) is a Sullivan model of \( X \). It follows from [12] that 
\( H_*(\text{Der}(\wedge V \otimes \wedge W)) \cong H_*(\text{Der} \wedge V) \otimes H^*(\wedge W) \oplus H^*(\wedge V) \otimes H_*(\text{Der} \wedge W) \).

Therefore \( \pi_1(\Omega B \text{aut}_1(Y)) \otimes Q \) is a subalgebra of \( \pi_1(\Omega B \text{aut}_1(X)) \otimes Q \).
In particular if $Y$ is a wedge of at least two spheres, then the Lie algebra $\pi_*(\Omega \text{Aut}_1(Y)) \otimes \mathbb{Q}$ is not nilpotent and so is $\pi_*(\Omega \text{Aut}_1(X)) \otimes \mathbb{Q}$.

We can extend Salvatore’s result to some certain types of coformal hyperbolic 2-cones.

**Theorem 9.** Under the hypotheses of Theorem 6, the rational homotopy Lie algebra of $\text{Aut}_1(X)$ is not nilpotent.

**Proof.** For $i \neq k$, let $(ad x_k)^n(x_i)$ be a nonzero element of $\mathcal{L}_X$. Define $\alpha_n \in Ext^1_A(\mathbb{Q}, \mathcal{L}_X)$ by $\alpha_n(sx_i) = (ad x_k)^n(x_i)$ and zero on the other generators of $\mathcal{L}_X$. Take $w \in V_2$ and define $\beta_m \in Ext^2_A(\mathbb{Q}, \mathcal{L}_X)$ by $\beta_m(sw) = (ad x_k)^m(x_i)$ and zero elsewhere. A short computation shows that $[\alpha_n, \beta_m] = \pm \beta_{m+n}$. Hence $(ad \alpha_n)^l(\beta_m) \neq 0$ for all $l \geq 1$. Therefore $\pi_*(\Omega \text{Aut}_1(X)) \otimes \mathbb{Q}$ is not nilpotent.

**Example 10.** Consider the space $X$ for which the Quillen minimal model is $(L(a, b, c, d))$ with $da = db = 0$ and $dc = [b, [b, a]]$. The space $X$ satisfies the hypothesis of Theorem 6. Therefore $\text{cat}(\text{Aut}_1(X))$ is infinite. Moreover the homotopy Lie algebra of $\text{Aut}_1(X) \otimes \mathbb{Q}$ is not nilpotent.

**References**


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