COMMON FIXED POINTS OF ONE-PARAMETER NONEXPANSIVE SEMIGROUPS IN STRICTLY CONVEX BANACH SPACES

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One of our main results is the following convergence theorem for one-parameter nonexpansive semigroups: let \( C \) be a bounded closed convex subset of a Hilbert space \( E \), and let \( \{ T(t) : t \in \mathbb{R}_+ \} \) be a strongly continuous semigroup of nonexpansive mappings on \( C \). Fix \( u \in C \) and \( t_1, t_2 \in \mathbb{R}_+ \) with \( t_1 < t_2 \). Define a sequence \( \{ x_n \} \) in \( C \) by \( x_n = (1 - \alpha_n)/(t_2 - t_1) \int_{t_1}^{t_2} T(s)x_ds + \alpha_n u \) for \( n \in \mathbb{N} \), where \( \{ \alpha_n \} \) is a sequence in \((0, 1)\) converging to 0. Then \( \{ x_n \} \) converges strongly to a common fixed point of \( \{ T(t) : t \in \mathbb{R}_+ \} \).

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1. Introduction

Let \( C \) be a closed convex subset of a Banach space \( E \), and let \( T \) be a nonexpansive mapping on \( C \), that is, \( \|Tx - Ty\| \leq \|x - y\| \) for all \( x, y \in C \). We know that \( T \) has a fixed point in the case that \( E \) is uniformly convex and \( C \) is bounded; see Browder [4], Göhde [10], and Kirk [15]. We denote by \( F(T) \) the set of fixed points of \( T \).

Let \( \{ T(t) : t \in \mathbb{R}_+ \} \) be a strongly continuous semigroup of nonexpansive mappings (nonexpansive semigroup, in short) on a closed convex subset \( C \) of a Banach space \( E \), that is,

(i) for each \( t \in \mathbb{R}_+ \), \( T(t) \) is a nonexpansive mapping on \( C \);

(ii) \( T(s + t) = T(s) \circ T(t) \) for all \( s, t \in \mathbb{R}_+ \);

(iii) for each \( x \in C \), the mapping \( t \mapsto T(t)x \) from \( \mathbb{R}_+ \) into \( C \) is strongly continuous.

We also know that \( \{ T(t) : t \in \mathbb{R}_+ \} \) has a common fixed point in the case that \( E \) is uniformly convex and \( C \) is bounded; see Browder [4]. Bruck [7] prove the following theorem.

**Theorem 1.1** (Bruck [7]). Suppose a closed convex subset \( C \) of a Banach space has the fixed point property for nonexpansive mappings, and \( C \) is either weakly compact, or bounded and separable. Then for any commuting family \( S \) of nonexpansive mappings on \( C \), the set of common fixed points of \( S \) is a nonempty nonexpansive retract of \( C \).
This theorem yields that \( \{ T(t) : t \in \mathbb{R}_+ \} \) has a common fixed point in the case that \( C \) has the fixed point property, and that \( C \) is weakly compact, or bounded and separable.

Several authors have studied about convergence theorems for nonexpansive semigroups; see [1, 2, 13, 16, 19, 21, 22] and others. For example, the following theorem is a corollary of Theorem 8 in [19].

**Theorem 1.2** (Shioji and Takahashi [19]). Let \( C \) be a bounded closed convex subset of a Hilbert space \( E \). Let \( \{ T(t) : t \in \mathbb{R}_+ \} \) be a strongly continuous semigroup of nonexpansive mappings on \( C \). Let \( \{ \alpha_n \} \) and \( \{ t_n \} \) be sequences of real numbers satisfying \( 0 < \alpha_n < 1, \lim_n \alpha_n = 0, t_n > 0 \) and \( \lim_n t_n = \infty \). Fix \( u \in C \) and define a sequence \( \{ x_n \} \) in \( C \) by

\[
x_n = \frac{1 - \alpha_n}{t_n} \int_0^{t_n} T(s)x_n \, ds + \alpha_n u
\]

for \( n \in \mathbb{N} \). Then \( \{ x_n \} \) converges strongly to a common fixed point of \( \{ T(t) : t \in \mathbb{R}_+ \} \).

Also, Suzuki [21] proved the following theorem.

**Theorem 1.3** (Suzuki [21]). Let \( E, C, \{ T(t) : t \in \mathbb{R}_+ \} \) be as in Theorem 1.2. Let \( \{ \alpha_n \} \) and \( \{ t_n \} \) be sequences of real numbers satisfying \( 0 < \alpha_n < 1, t_n > 0 \) and \( \lim_n t_n = \lim_n \alpha_n/t_n = 0 \). Fix \( u \in C \) and define a sequence \( \{ x_n \} \) in \( C \) by

\[
x_n = (1 - \alpha_n) T(t_n)x_n + \alpha_n u
\]

for \( n \in \mathbb{N} \). Then \( \{ x_n \} \) converges strongly to a common fixed point of \( \{ T(t) : t \in \mathbb{R}_+ \} \).

We note that in these theorems, real sequences \( \{ t_n \} \) converge to 0 and \( \infty \). So, it is natural to study convergence theorems under the assumption that \( \{ t_n \} \) is a constant sequence. In this paper, motivated by Theorems 1.2 and 1.3, we consider such type of convergence theorems to a common fixed point of \( \{ T(t) : t \in \mathbb{R}_+ \} \).

### 2. Preliminaries

Throughout this paper we denote by \( \mathbb{R} \) the set of real numbers, by \( \mathbb{R}_+ \) the set of nonnegative real numbers, and by \( \mathbb{N} \) the set of positive integers. For a Banach space \( E \), we also denote by \( E^* \) the dual space of \( E \).

We recall that a Banach space \( E \) is called strictly convex if \( \|x + y\|/2 < 1 \) for all \( x, y \in E \) with \( \|x\| = \|y\| = 1 \) and \( x \neq y \). We know the following lemma.

**Lemma 2.1.** Let \( E \) be a Banach space. Then the following are equivalent:

(i) \( E \) is strictly convex;

(ii) \( \|\lambda x + (1 - \lambda) y\| < 1 \) for all \( \lambda \in (0, 1) \) and \( x, y \in E \) with \( \|x\| = \|y\| = 1 \) and \( x \neq y \);

(iii) if \( \|x\| = \|y\| = \|\lambda x + (1 - \lambda) y\| \) for some \( \lambda \in (0, 1) \), then \( x = y \).

A Banach space \( E \) is called uniformly convex if for each \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( \|x + y\|/2 < 1 - \delta \) for all \( x, y \in E \) with \( \|x\| = \|y\| = 1 \) and \( \|x - y\| \geq \varepsilon \). It is clear that a uniformly convex Banach space is strictly convex. The norm of \( E \) is called Fréchet differentiable if for each \( x \in E \) with \( \|x\| = 1 \), \( \lim_{t \to 0} (\|x + ty\| - \|x\|)/t \) exists and is attained uniformly in \( y \in E \) with \( \|y\| = 1 \). A Banach space \( E \) is said to have the Opial property [17]
if for each weakly convergent sequence \( \{x_n\} \) in \( E \) with weak limit \( z \), \( \liminf_n \|x_n - z\| < \liminf_n \|x_n - y\| \) for all \( y \in E \) with \( y \neq z \). All Hilbert spaces, all finite dimensional Banach spaces and \( \ell^p (1 \leq p < \infty) \) have the Opial property. Gossez and Lami Dozo\[11\] prove that every weakly compact convex subset of a Banach space with the Opial property has normal structure. We also know that every separable Banach space can be equivalently renormed so that it has the Opial property; see [23].

3. Common fixed points

In this section, we give our main results. The following proposition plays an important role in this paper.

**Proposition 3.1.** Let \( C \) be a closed convex subset of a strictly convex Banach space \( E \). Let \( \tau_\infty > 0 \) and let \( \{T(t) : t \in [0, \tau_\infty)\} \) be a family of mappings on \( C \) satisfying the following:

(i) for each \( t \in [0, \tau_\infty) \), \( T(t) \) is nonexpansive;

(ii) there exists a strictly increasing sequence \( \{\tau_n\} \) in \( [0, \tau_\infty) \) such that \( \tau_1 = 0 \), \( \{\tau_n\} \) converges to \( \tau_\infty \), and mappings \( t \mapsto T(t)x \) are weakly continuous on \( [\tau_n, \tau_{n+1}) \) for all \( x \in C \) and \( n \in \mathbb{N} \).

Suppose that

\[
\bigcap_{t \in [0, \tau_\infty)} F(T(t)) \neq \emptyset. \tag{3.1}
\]

Then

\[
\bigcap_{t \in [0, \tau_\infty)} F(T(t)) = F(S), \tag{3.2}
\]

where \( S \) is a nonexpansive mapping on \( C \) defined by

\[
Sx = \frac{1}{\tau_\infty} \int_0^{\tau_\infty} T(s)x \, ds \tag{3.3}
\]

for all \( x \in C \).

**Remark 3.2.** We do not assume \( \{T(\cdot)\} \) is a nonexpansive semigroup.

**Proof.** Fix \( f \in E^* \). Then the functions \( t \mapsto f(T(t)x) \) from \( [\tau_n, \tau_{n+1}) \) into \( \mathbb{R} \) are continuous on \( [\tau_n, \tau_{n+1}) \) for \( x \in C \) and \( n \in \mathbb{N} \). So, the functions \( t \mapsto f(T(t)x) \) from \( [0, \tau_\infty) \) into \( \mathbb{R} \) are measurable for \( x \in C \). We also have \( \{T(t)x : t \in [0, \tau_\infty)\} \) is separable for each \( x \in C \). Fix \( w \in \bigcap_{t \in [0, \tau_\infty)} F(T(t)) \). Since

\[
\|T(t)x\| = \|T(t)x\| - \|T(t)w\| + \|w\| \leq \|T(t)x - T(t)w\| + \|w\| \\
\leq \|x - w\| + \|w\|, \tag{3.4}
\]

for \( x \in C \) and \( t \in [0, \tau_\infty) \), we have that the mappings \( t \mapsto T(t)x \) are Bochner integrable for all \( x \in C \) and hence \( S \) is well-defined. Using the separation theorem, we can easily prove
that $S$ is a mapping on $C$. Since

$$\|Sx - Sy\| = \left\| \frac{1}{\tau_\infty} \int_0^{\tau_\infty} (T(s)x - T(s)y) \, ds \right\|$$

$$\leq \frac{1}{\tau_\infty} \int_0^{\tau_\infty} \|T(s)x - T(s)y\| \, ds$$

$$\leq \frac{1}{\tau_\infty} \int_0^{\tau_\infty} \|x - y\| \, ds = \|x - y\|$$

for $x, y \in C$, $S$ is nonexpansive. Therefore $S$ is a nonexpansive mapping on $C$. It is obvious that $\bigcap_{t \in [0, \tau_\infty)} F(T(t)) \subset F(S)$. We assume that $z \in F(S) \setminus \bigcap_{t \in [0, \tau_\infty)} F(T(t))$. Then there exists $t_1 \in [0, \tau_\infty)$ such that $T(t_1)z \neq z$. Fix $g \in E^*$ with

$$\|g\| = 1, \quad g(T(t_1)z - z) = \|T(t_1)z - z\|. \quad (3.6)$$

For some $m \in \mathbb{N}$, $t_1$ belongs to $[\tau_m, \tau_{m+1})$. From the assumption (ii), there exists $t_2 \in (t_1, \tau_{m+1})$ such that

$$g(T(t)z - z) > \frac{1}{2} \|T(t_1)z - z\| \quad (3.7)$$

for all $t \in [t_1, t_2)$. Define nonexpansive mappings $S_1$ and $S_2$ on $C$ by

$$S_1x = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} T(s)x \, ds,$$

$$S_2x = \frac{1}{\tau_\infty - t_2 + t_1} \left( \int_0^{t_1} T(s)x \, ds + \int_{t_1}^{\tau_\infty} T(s)x \, ds \right) \quad (3.8)$$

for all $x \in C$. We note that

$$Sx = \frac{t_2 - t_1}{\tau_\infty} S_1x + \frac{\tau_\infty - t_2 + t_1}{\tau_\infty} S_2x \quad (3.9)$$

for all $x \in C$. We have

$$g(S_1z - Sz) = g\left(\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} T(s)z \, ds - z\right)$$

$$= g\left(\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} (T(s)z - z) \, ds\right)$$

$$= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} g(T(s)z - z) \, ds$$

$$\geq \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \frac{1}{2} \|T(t_1)z - z\| \, ds$$

$$= \frac{1}{2} \|T(t_1)z - z\| > 0. \quad (3.10)$$

Hence

$$g(S_2z - Sz) = \frac{t_2 - t_1}{\tau_\infty - t_2 + t_1} g(Sz - S_1z) < 0. \quad (3.11)$$
Therefore $S_1z \neq S_2z$. Fix $w \in \bigcap_{t \in [0,\tau_\infty)} F(T(t))$. Then we note that $S_1w = S_2w = w$. We have

$$
\|z - w\| = \|Sz - w\| = \left\| \frac{t_2 - t_1}{\tau_\infty} S_1z + \frac{\tau_\infty - t_2 + t_1}{\tau_\infty} S_2z - w \right\|
\leq \frac{t_2 - t_1}{\tau_\infty} \|S_1z - w\| + \frac{\tau_\infty - t_2 + t_1}{\tau_\infty} \|S_2z - w\|
= \frac{t_2 - t_1}{\tau_\infty} \|S_1z - S_1w\| + \frac{\tau_\infty - t_2 + t_1}{\tau_\infty} \|S_2z - S_2w\|
\leq \frac{t_2 - t_1}{\tau_\infty} \|z - w\| + \frac{\tau_\infty - t_2 + t_1}{\tau_\infty} \|z - w\| = \|z - w\|
$$

and hence

$$
\|Sz - w\| = \|S_1z - w\| = \|S_2z - w\|.
$$

This contradicts the strict convexity of $E$. Therefore, $F(S) \subset \bigcap_{t \in [0,\tau_\infty)} F(T(t))$. This completes the proof. \qed

As a direct consequence of Proposition 3.1, we can prove the following, which was proved by Bruck [6]; see also [20].

**Corollary 3.3 (Bruck [6]).** Let $C$ be a closed convex subset of a strictly convex Banach space $E$. Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on $C$. Suppose $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\{\alpha_n\}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty} \alpha_n = 1$. Define a nonexpansive mapping $S$ on $C$ by

$$
Sx = \sum_{n=1}^{\infty} \alpha_n T_n x
$$

for $x \in C$. Then $F(S) = \bigcap_{n=1}^{\infty} F(T_n)$ holds.

**Proof.** Define a strictly increasing sequence $\{\tau_n\}$ in $[0,1)$ by $\tau_1 = 0$ and

$$
\tau_n = \sum_{k=1}^{n-1} \alpha_k
$$

for $n \in \mathbb{N}$ with $n \geq 2$. We note that $\lim_{n} \tau_n = 1$. Define a family $\{T(t) : t \in [0,1)\}$ of nonexpansive mappings as follows: If $\tau_n \leq t < \tau_{n+1}$, then

$$
T(t)x = T_n x
$$

for all $x \in C$. Then we note that

$$
Sx = \sum_{n=1}^{\infty} \alpha_n T_n x = \sum_{n=1}^{\infty} \int_{\tau_n}^{\tau_{n+1}} T(s)x \, ds = \int_{0}^{1} T(s)x \, ds = \frac{1}{\tau_{n+1}} \int_{0}^{1} T(s)x \, ds
$$

for $n \in \mathbb{N}$ with $n \geq 2$. We note that $\lim_{n} \tau_n = 1$. Define a family $\{T(t) : t \in [0,1)\}$ of nonexpansive mappings as follows: If $\tau_n \leq t < \tau_{n+1}$, then

$$
T(t)x = T_n x
$$

for all $x \in C$. Then we note that

$$
Sx = \sum_{n=1}^{\infty} \alpha_n T_n x = \sum_{n=1}^{\infty} \int_{\tau_n}^{\tau_{n+1}} T(s)x \, ds = \int_{0}^{1} T(s)x \, ds = \frac{1}{\tau_{n+1}} \int_{0}^{1} T(s)x \, ds
$$

(3.17)
for \( x \in C \) and
\[
\bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{t \in [0,1)} F(T(t)). \tag{3.18}
\]
So, by Proposition 3.1, we obtain the desired result. \( \square \)

As another direct consequence of Proposition 3.1, we obtain the following proposition.

**Proposition 3.4.** Let \( C \) be a closed convex subset of a strictly convex Banach space \( E \). Let \( \tau > 0 \) and let \( \{T(t) : t \in [0,\tau)\} \) be a family of mappings on \( C \) satisfying the following:

(i) for each \( t \in [0,\tau) \), \( T(t) \) is nonexpansive;
(ii) mappings \( t \mapsto T(t)x \) are weakly continuous on \( [0,\tau) \) for all \( x \in C \).

Suppose that
\[
\bigcap_{t \in [0,\tau)} F(T(t)) \neq \emptyset. \tag{3.19}
\]
Then
\[
\bigcap_{t \in [0,\tau)} F(T(t)) = F(S), \tag{3.20}
\]
where \( S \) is a nonexpansive mapping on \( C \) defined by
\[
Sx = \frac{1}{\tau} \int_{0}^{\tau} T(s)x \, ds \tag{3.21}
\]
for all \( x \in C \).

Now, we prove one of our main results.

**Theorem 3.5.** Let \( C \) be a closed convex subset of a strictly convex Banach space \( E \) and let \( \{T(t) : t \in \mathbb{R}_+\} \) be a strongly continuous semigroup of nonexpansive mappings on \( C \). Suppose that
\[
\bigcap_{t \in \mathbb{R}_+} F(T(t)) \neq \emptyset. \tag{3.22}
\]
Fix \( t_1, t_2 \in \mathbb{R}_+ \) with \( t_1 < t_2 \), and define a nonexpansive mapping \( S \) on \( C \) by
\[
Sx = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} T(s)x \, ds \tag{3.23}
\]
for all \( x \in C \). Then
\[
\bigcap_{t \in \mathbb{R}_+} F(T(t)) = F(S) \tag{3.24}
\]
holds.
Proof. It is clear that \( \bigcap_{t \in \mathbb{R}^+} F(T(t)) \subset F(S) \). Fix \( w \in F(S) \). By Proposition 3.4, we have

\[
\bigcap_{t \in [t_1, t_2]} F(T(t)) = F(S). \tag{3.25}
\]

So, \( T(t)w = w \) for \( t \in [t_1, t_2] \). Hence, for every \( t \in [0, (t_2 - t_1)/2] \), we have

\[
T(t)w = T(t) \circ T(t_1)w = T(t + t_1)w = w. \tag{3.26}
\]

Let \( t \in \mathbb{R}^+ \) be fixed. Then there exist \( m \in \mathbb{N} \cup \{0\} \) and \( u \in [0, (t_2 - t_1)/2) \) such that \( t = u + m(t_2 - t_1)/2 \). We have

\[
T(t)w = T\left(u + m \frac{t_2 - t_1}{2}\right)w = T(u) \circ T\left(\frac{t_2 - t_1}{2}\right)^m w = T(u)w = w, \tag{3.27}
\]

where \( T\left((t_2 - t_1)/2\right)^0 \) is the identity mapping on \( C \). Therefore \( w \) is a common fixed point of \( \{T(t) : t \in \mathbb{R}^+\} \). This completes the proof. \( \square \)

Similarly we can prove the following theorem.

**Theorem 3.6.** Let \( C \) be a closed convex subset of a strictly convex Banach space \( E \) and let \( \{\{T_n(t) : t \in \mathbb{R}^+\} : n \in \mathbb{N}\} \) be a sequence of strongly continuous semigroups of nonexpansive mappings on \( C \). Let \( \{U_n : n \in \mathbb{N}\} \) be a sequence of nonexpansive mappings on \( C \). Suppose that

\[
\bigcap_{n=1}^{\infty} \bigcap_{t \in \mathbb{R}^+} F(T_n(t)) \cap \bigcap_{n=1}^{\infty} F(U_n) \neq \emptyset. \tag{3.28}
\]

Let \( \{t_n\}, \{u_n\}, \{\alpha_n\} \) and \( \{\beta_n\} \) be real sequences such that \( 0 \leq t_n < u_n, \alpha_n > 0 \) and \( \beta_n > 0 \) for all \( n \in \mathbb{N} \), and \( \sum_{n=1}^{\infty} \alpha_n + \sum_{n=1}^{\infty} \beta_n = 1 \). Define a nonexpansive mapping \( S \) on \( C \) by

\[
Sx = \sum_{n=1}^{\infty} \frac{\alpha_n}{u_n - t_n} \int_{t_n}^{u_n} T_n(s)x \, ds + \sum_{n=1}^{\infty} \beta_n U_n x \tag{3.29}
\]

for all \( x \in C \). Then

\[
\bigcap_{n=1}^{\infty} \bigcap_{t \in \mathbb{R}^+} F(T_n(t)) \cap \bigcap_{n=1}^{\infty} F(U_n) = F(S). \tag{3.30}
\]

holds.

We recall that a closed convex subset \( C \) of a Banach space \( E \) is said to have the fixed point property for nonexpansive mappings (FPP, in short) if for every bounded closed convex subset \( D \) of \( C \), every nonexpansive mapping on \( D \) has a fixed point. So, by the results of Browder [4] and Göhde [10], every uniformly convex Banach space has FPP. Also, by Kirk’s fixed point theorem [15], every weakly compact convex subset with normal structure has FPP.

As a direct consequence of Theorem 3.6, we obtain the following corollary.
Corollary 3.7. Let $E$, $C$, $\{T_n(t) : t \in \mathbb{R}_+ \}$, $\{U_n : n \in \mathbb{N} \}$, $\{t_n\}$, $\{u_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ be as in Theorem 3.6. Assume that $C$ is weakly compact and has FPP, and

$$T_m(s) \circ T_n(t) = T_n(t) \circ T_m(s), \quad U_m \circ U_n = U_n \circ U_m, \quad U_m \circ T_n(t) = T_n(t) \circ U_m$$

(3.31)

for all $s, t \in \mathbb{R}_+$ and $m, n \in \mathbb{N}$. Define a nonexpansive mapping $S$ on $C$ as in Theorem 3.6. Then

$$\bigcap_{n=1}^{\infty} F(T_n(t)) \cap \bigcap_{n=1}^{\infty} F(U_n) = F(S) \neq \emptyset.$$  

(3.32)

4. Convergence theorems

Using Theorem 3.5, we can prove many convergence theorems to a common fixed point of nonexpansive semigroups. In this section, we state some of them.

From the result of Ishikawa [14], we obtain the following theorem see also Edelstein [8].

Theorem 4.1. Let $E$ be a compact convex subset of a strictly convex Banach space $E$. Let $\{T(t) : t \in \mathbb{R}_+ \}$ be a strongly continuous semigroup of nonexpansive mappings on $C$. Fix $t_1, t_2 \in \mathbb{R}_+$ with $t_1 < t_2$. Define a sequence $\{x_n\}$ in $C$ by $x_1 \in C$ and

$$x_{n+1} = \frac{\alpha_n}{t_2 - t_1} \int_{t_1}^{t_2} T(s)x_n \, ds + (1 - \alpha_n)x_n$$

(4.1)

for $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0, 1]$ satisfying $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_n \alpha_n < 1$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T(t) : t \in \mathbb{R}_+ \}$.

From the results of Edelstein and O’Brien [9], and Reich [18], we obtain the following theorem.

Theorem 4.2. Let $E$ be a Banach space. Suppose either of the following holds:

(i) $E$ is strictly convex and has the Opial property; or

(ii) $E$ is uniformly convex and its norm is Fréchet differentiable.

Let $C$ be a weakly compact convex subset of $E$, and let $\{T(t) : t \in \mathbb{R}_+ \}$ be a strongly continuous semigroup of nonexpansive mappings on $C$. Fix $t_1, t_2 \in \mathbb{R}_+$ with $t_1 < t_2$. Define a sequence $\{x_n\}$ in $C$ by $x_1 \in C$ and

$$x_{n+1} = x_n$$

(4.2)

for $n \in \mathbb{N}$, where $\alpha$ is a constant number in $(0, 1)$. Then $\{x_n\}$ converges weakly to a common fixed point of $\{T(t) : t \in \mathbb{R}_+ \}$.

We note that

$$x \mapsto (1 - \alpha)Tx + \alpha u$$

(4.3)
is a contractive mapping if \( T \) is a nonexpansive mapping and \( \alpha \in (0, 1) \). By the Banach contraction principle [3], such mappings have a unique fixed point. From the results of Browder [5], and Wittmann [24], we obtain the following theorem; see also [12]. Compare Theorem 4.3 with Theorems 1.2 and 1.3.

**Theorem 4.3.** Let \( C \) be a bounded closed convex subset of a Hilbert space \( E \), and let \( \{ T(t) : t \in \mathbb{R}_+ \} \) be a strongly continuous semigroup of nonexpansive mappings on \( C \). Fix \( u \in C \) and \( t_1, t_2 \in \mathbb{R}_+ \) with \( t_1 < t_2 \). Define a sequence \( \{ x_n \} \) in \( C \) by

\[
x_n = \frac{1 - \alpha_n}{t_2 - t_1} \int_{t_1}^{t_2} T(s)x_n \, ds + \alpha_n u
\]

for \( n \in \mathbb{N} \), where \( \{ \alpha_n \} \) is a sequence in \( (0, 1) \) converging to 0. Then \( \{ x_n \} \) converges strongly to a common fixed point of \( \{ T(t) : t \in \mathbb{R}_+ \} \).

**Theorem 4.4.** Let \( E, C, \{ T(t) : t \in \mathbb{R}_+ \}, u, t_1 \) and \( t_2 \) be as in Theorem 4.3. Define a sequence \( \{ x_n \} \) in \( C \) by \( x_1 \in C \) and

\[
x_{n+1} = \frac{1 - \alpha_n}{t_2 - t_1} \int_{t_1}^{t_2} T(s)x_n \, ds + \alpha_n u
\]

for \( n \in \mathbb{N} \), where \( \{ \alpha_n \} \) is a sequence in \([0, 1]\] satisfying the following:

\[
\lim_{n \to \infty} \alpha_n = 0; \quad \sum_{n=1}^{\infty} \alpha_n = \infty; \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.
\]

Then \( \{ x_n \} \) converges strongly to a common fixed point of \( \{ T(t) : t \in \mathbb{R}_+ \} \).

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**References**


10 Nonexpansive semigroup in SC


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This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

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