CRITICAL GROUPS OF CRITICAL POINTS PRODUCED
BY LOCAL LINKING WITH APPLICATIONS

KANISHKA PERERA

Abstract. We prove the existence of nontrivial critical points with nontrivial critical groups for functionals with a local linking at 0. Applications to elliptic boundary value problems are given.

1. Introduction

Let $F$ be a real $C^1$ function defined on a Banach space $X$. We say that $F$ has a local linking near the origin if $X$ has a direct sum decomposition $X = X_1 \oplus X_2$ with $\dim X_1 < \infty$, $F(0) = 0$, and, for some $r > 0$,

$$
\begin{cases}
F(u) \leq 0 & \text{for } u \in X_1, \|u\| \leq r, \\
F(u) > 0 & \text{for } u \in X_2, 0 < \|u\| \leq r.
\end{cases}
$$

(1)

Then it is clear that 0 is a critical point of $F$.

The notion of local linking was introduced by Li and Liu [7], [8], who proved the existence of nontrivial critical points under various assumptions on the behavior of $F$ at infinity. These results were recently generalized by Brézis and Nirenberg [3], Li and Willem [9], and several other authors.

In infinite dimensional Morse theory (see Chang [5] or Mawhin and Willem [11]), the local behavior of $F$ near an isolated critical point $u_0$, $F(u_0) = c$, is described by the sequence of critical groups

$$
C_q(F, u_0) = H_q(F_c \cap U, (F_c \cap U) \setminus \{u_0\}) \quad q \in \mathbb{Z}
$$

where $F_c$ is the sublevel set $\{u \in X : F(u) \leq c\}$, $U$ is a neighborhood of $u_0$ such that $u_0$ is the only critical point of $F$ in $F_c \cap U$, and $H_\ast(\cdot, \cdot)$ denote the singular relative homology groups.

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It was proved in Liu [10] that if $F$ has a local linking near the origin, $\dim X_1 = j$, and 0 is an isolated critical point of $F$, then $C_j(F,0) \neq 0$.

In the present paper we use this fact to obtain a nontrivial critical point $u$ with either $C_{j+1}(F,u) \neq 0$ or $C_{j-1}(F,u) \neq 0$. When $X$ is a Hilbert space and $F$ is $C^2$, this yields Morse index estimates for $u$ via the Shifting theorem.

When $X$ is a Hilbert space and $dF$ is Lipschitz in a neighborhood of the origin, we extend the result of Liu [10] to the case where $F$ satisfies the “relaxed” local linking condition

$$\begin{cases} 
F(u) \leq 0 \text{ for } u \in X_1, \|u\| \leq r, \\
F(u) \geq 0 \text{ for } u \in X_2, \|u\| \leq r
\end{cases}$$

(see Brézis and Nirenberg [3] and Li and Willem [9]), and thus obtain a nontrivial critical point with a nontrivial critical group in this case also.

We apply our abstract result to elliptic boundary value problems, including an equation asymptotically linear at $-\infty$ and superlinear at $+\infty$, and prove new multiplicity results.

2. Abstract Result

Throughout this section we assume that $F$ satisfies the Palais-Smale compactness condition (PS) and has only isolated critical values, with each critical value corresponding to a finite number of critical points.

**Theorem 2.1.** Suppose that there is a critical point $u_0$ of $F$, $F(u_0) = c$, with $C_j(F,u_0) \neq 0$ for some $j \geq 0$ and regular values $a, b$ of $F$, $a < c < b$, such that $H_j(F_a, F_b) = 0$. Then $F$ has a critical point $u$ with either

$c < F(u) < b$ and $C_{j+1}(F,u) \neq 0$, or

$a < F(u) < c$ and $C_{j-1}(F,u) \neq 0$.

Proof of Theorem 2.1 makes use of the following topological lemma:

**Lemma 2.2.** If $B' \subset B \subset A \subset A'$ are topological spaces such that $H_j(A,B) \neq 0$ and $H_j(A',B') = 0$, then either

$H_{j+1}(A', A) \neq 0$ or $H_{j-1}(B, B') \neq 0$.

**Proof.** Suppose that $H_{j+1}(A', A) = 0$. Since $H_j(A', B')$ is also trivial, it follows from the following portion of the exact sequence of the triple $(A', A, B')$ that $H_j(A, B') = 0$:

$$H_{j+1}(A', A) \xrightarrow{\partial_*} H_j(A, B') \xrightarrow{i_*} H_j(A', B')$$

Since $H_j(A, B) \neq 0$, now it follows from the following portion of the exact sequence of the triple $(A, B, B')$ that $H_{j-1}(B, B') \neq 0$:

$$H_j(A, B') \xrightarrow{j_*} H_j(A, B) \xrightarrow{\partial_*} H_{j-1}(B, B')$$
Proof of Theorem 2.1. Take \( \epsilon, 0 < \epsilon < \min\{c - a, b - c\} \) such that \( c \) is the only critical value of \( F \) in \([c - \epsilon, c + \epsilon] \). Then, since \( C_j(F, u_0) \neq 0 \), it follows from Chapter I, Theorem 4.2 of Chang [5] that \( H_j(F_{c+\epsilon}, F_{c-\epsilon}) \neq 0 \). Since \( H_j(F_b, F_a) = 0 \), by Lemma 2.2, either \( H_{j+1}(F_b, F_{c+\epsilon}) \neq 0 \) or \( H_{j-1}(F_{c-\epsilon}, F_a) \neq 0 \), and the conclusion follows from Chapter I, Theorem 4.3 and Corollary 4.1 of Chang [5].

As mentioned before, if \( F \) has a local linking near the origin, \( \dim X_1 = j \), then \( C_j(F, 0) \neq 0 \) (see Liu [10]), and hence the following corollary is immediate from Theorem 2.1:

**Corollary 2.3.** Suppose \( F \) has a local linking near the origin, \( \dim X_1 = j \). Assume also that there are regular values \( a, b \) of \( F \), \( a < 0 < b \), such that \( H_j(F_b, F_a) = 0 \). Then \( F \) has a critical point \( u \) with either

\[
0 < F(u) < b \quad \text{and} \quad C_{j+1}(F, u) \neq 0, \quad \text{or} \\
a < F(u) < 0 \quad \text{and} \quad C_{j-1}(F, u) \neq 0.
\]

If \( X \) is a Hilbert space, \( F \) is \( C^2 \), and \( u \) is a critical point of \( F \), we denote by \( m(u) \) the Morse index of \( u \) and by \( m^*(u) = m(u) + \dim \ker d^2F(u) \) the large Morse index of \( u \). We recall that if \( u \) is nondegenerate and \( C_q(F, u) \neq 0 \), then \( m(u) = q \) (see Chapter I, Theorem 4.1 of Chang [5]). Let us also recall that it follows from the Shifting theorem (Chapter I, Theorem 5.4 of Chang [5]) that if \( u \) is degenerate, 0 is an isolated point of the spectrum of \( d^2F(u) \), and \( C_q(F, u) \neq 0 \), then \( m(u) \leq q \leq m^*(u) \). Hence we have the following corollary:

**Corollary 2.4.** Let \( X \) be a Hilbert space and \( F \) be \( C^2 \) in Theorem 2.1. Assume that for every degenerate critical point \( u \) of \( F \), 0 is an isolated point of the spectrum of \( d^2F(u) \). Then \( F \) has a critical point \( u \) with either

\[
c < F(u) < b \quad \text{and} \quad m(u) \leq j + 1 \leq m^*(u), \quad \text{or} \\
a < F(u) < c \quad \text{and} \quad m(u) \leq j - 1 \leq m^*(u).
\]

**Remark 2.5.** In particular, Corollary 2.4 yields a critical point \( u \neq u_0 \) with \( m(u) \leq j + 1 \) and \( j - 1 \leq m^*(u) \). Benci and Fortunato [2] have proved this fact for the special case where \( u_0 \) is a nondegenerate critical point with Morse index \( j \), but without assuming that the critical points of \( F \) are isolated. Their proof is based on a generalized Morse theory due to Benci and Giannoni [1]. However, Corollary 2.4 says, in addition, that \( u \) is at a level different from \( F(u_0) \).

If \( X \) is a Hilbert space and \( dF \) is Lipschitz in a neighborhood of the origin, we can relax the local linking condition as in (2). This follows from the following extension of the result of Liu [10] (see also Theorem 5.6 of Kryszewski and Szulkin [6]):

**Theorem 2.6.** Let \( X \) be a Hilbert space and \( dF \) be Lipschitz in a neighborhood of the origin. Suppose that \( F \) satisfies the local linking condition (2), \( \dim X_1 = j \). Then \( C_j(F, 0) \neq 0 \).
Our proof of Theorem 2.6 uses the following “deformation” lemma:

**Lemma 2.7.** Under the assumptions of Theorem 2.6 there exist a closed ball $B$ centered at the origin and a homeomorphism $h$ of $X$ onto $X$ such that

1. $0$ is the only critical point of $F$ in $h(B)$,
2. $h|_{B \cap X_1} = id_{B \cap X_1}$,
3. $F(u) > 0$ for $u \in h(B \cap X_2 \setminus \{0\})$.

**Proof.** Take open balls $B'$, $B''$ centered at the origin, with $\overline{B'} \subset B''$, such that $0$ is the only critical point of $F$ in $B'$ and $dF$ is Lipschitz in $B''$, and let $B \subset B'$ be a closed ball centered at the origin with radius $\leq r$ (in (2)). Since $B$ and $(B')^c$ are disjoint closed sets there is a locally Lipschitz nonnegative function $g \leq 1$ satisfying

$$g = \begin{cases} 1 & \text{on } B \\ 0 & \text{outside } B' \end{cases}.$$

Consider the vector field

$$V(u) = g(u) \|P_u\| dF(u)$$

where $P$ is the orthogonal projection onto $X_2$. Clearly $V$ is locally Lipschitz and bounded on $X$. Consider the flow $\eta(t) = \eta(t, u)$ defined by

$$\frac{d\eta}{dt} = V(\eta), \quad \eta|_{t=0} = u.$$

Clearly, $\eta$ is defined for $t \in [0, 1]$. Let $h = \eta(1, \cdot)$. Since $h|(B')^c = id_{(B')^c}$ and $h$ is one-to-one, $h(B) \subset B'$ and 1 follows. For $u \in B \cap X_2 \setminus \{0\},$

$$F(h(u)) = F(u) + \int_0^1 g(\eta(t)) \|P\eta(t)\| \|dF(\eta(t))\|^2 dt > 0$$

since $F(u) \geq 0$ and $g(u) \|P(u)\| \|dF(u)\|^2 > 0$. \[Q.E.D.\]

**Proof of Theorem 2.6.** By 1 of Lemma 2.7, $C_j(F, 0) = H_j(F_0 \cap h(B), F_0 \cap h(B) \setminus \{0\})$.

By the local linking condition (2) and 2 and 3 of Lemma 2.7, $\partial B \cap X_1 \subset F_0 \cap h(B) \setminus \{0\} \subset h(B \cap X_2)$ and $B \cap X_1 \subset F_0 \cap h(B)$. Since $h|_{\partial B \cap X_1} = id_{\partial B \cap X_1}$, the inclusion $\partial B \cap X_1 \hookrightarrow h(B \cap X_2)$ can also be written as the composition of the inclusion $\partial B \cap X_1 \hookrightarrow B \cap X_2$ and the restriction of $h$ to $B \cap X_2$. Hence we have the following commutative diagram induced by inclusions and $h$:

$$\begin{array}{ccc}
H_{j-1}(B \cap X_2) & \xrightarrow{i''_*} & H_{j-1}(\partial B \cap X_1) \\
| & u_* & | \\
H_{j-1}(h(B \cap X_2)) & \xrightarrow{i''_*} & H_{j-1}(F_0 \cap h(B) \setminus \{0\}) \\
| & i_* & | \\
H_{j-1}(h(B \cap X_2)) & \xrightarrow{i_*} & H_{j-1}(F_0 \cap h(B))
\end{array}$$

Since $\partial B \cap X_1$ is a strong deformation retract of $B \cap X_2$ and $h$ is a homeomorphism, $i''_*$ and $u_*$ are isomorphisms and hence $i'_*$ is a monomorphism.
Since \( \text{rank } H_{j-1}(B \cap X_1) < \text{rank } H_{j-1}(\partial B \cap X_1) \), then it follows that \( i_* \) is not a monomorphism.

Now it follows from the following portion of the exact sequence of the pair \((F_0 \cap h(B), F_0 \cap h(B) \setminus \{0\})\) that \( C_j(F,0) = H_j(F_0 \cap h(B), F_0 \cap h(B) \setminus \{0\}) \neq 0: \)
\[
\begin{align*}
C_j(F,0) &\xrightarrow{\partial_*} H_{j-1}(F_0 \cap h(B) \setminus \{0\}) \\
&\xrightarrow{i_*} H_{j-1}(F_0 \cap h(B))
\end{align*}
\]

3. **Elliptic Boundary Value Problems**

Consider the problem
\[
\begin{align*}
\{ -\Delta u &= g(u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega
\end{align*}
\]
where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \) and \( g \in C^1(\mathbb{R}, \mathbb{R}) \) satisfies
\[
\begin{align*}
(g_1): |g(u)| &\leq C (1 + |u|^{p-1}) \quad \text{with } 2 < p < \frac{2n}{n-2}, \text{ for some } C > 0, \\
(g_2): g(0) = 0 &= g(a) \quad \text{for some } a > 0, \\
(g_3): \text{there are constants } \mu > 2 \text{ and } A > 0 \text{ such that } \quad 0 < \mu G(u) &\leq u g(u) \quad \text{for } |u| \geq A,
\end{align*}
\]
where \( G(u) := \int_0^u g(t) \, dt. \)

Let \( \lambda = g'(0) \) and let \( 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \) be the eigenvalues of \(-\Delta\) with Dirichlet boundary condition.

**Theorem 3.1.** Assume that \( g \) satisfies \((g_1) - (g_3)\) and one of the following conditions:
\begin{enumerate}
\item \( \lambda_j < \lambda < \lambda_{j+1} \),
\item \( \lambda_j = \lambda < \lambda_{j+1} \) and, for some \( \delta > 0, \)
\[
G(u) \geq \frac{1}{2} \lambda u^2 \quad \text{for } |u| \leq \delta,
\]
\item \( \lambda_j < \lambda = \lambda_{j+1} \) and, for some \( \delta > 0, \)
\[
G(u) \leq \frac{1}{2} \lambda u^2 \quad \text{for } |u| \leq \delta.
\]
\end{enumerate}

If \( j \geq 3 \), problem (3) has at least four nontrivial solutions.

**Proof.** Solutions of (3) are the critical points of the \( C^2 \) functional
\[
F(u) = \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 - G(u) \right)
\]
defined on \( X = H_0^1(\Omega) \). It is well known that \( F \) satisfies (PS).

By a standard argument involving a cut-off technique and the strong maximum principle, \( F \) has a local minimizer \( u_0 \) with \( 0 < u_0 < a, \)
\[
\text{rank } C_q(F, u_0) = \delta_{q0}.
\]
Since \( \lim_{t \to \infty} F(\pm t \phi_1) = -\infty \), where \( \phi_1 > 0 \) is the first Dirichlet eigenfunction of \(-\Delta\), then \( F \) also has two mountain pass points \( u_1^\pm \) with \( u_1^- < u_0 < u_1^+ \),

\[
\text{rank } C_q(F, u_1^\pm) = \delta_{q1}
\]
(see the proof of Theorem B in Chang, Li, and Liu [4]).

Let \( X_1 \) be the \( j \)-dimensional space spanned by the eigenfunctions corresponding to \( \lambda_1, \ldots, \lambda_j \) and let \( X_2 \) be its orthogonal complement in \( X \). Then \( F \) has a local linking near the origin with respect to the decomposition \( X = X_1 \oplus X_2 \) (see the proof of Theorem 4 in Li and Willem [9]) and hence

\[
C_j(F, 0) \neq 0.
\]

Also, for \( \alpha < 0 \) and \( |\alpha| \) sufficiently large,

\[
H_q(X, F_\alpha) = 0 \quad \forall q \in \mathbb{Z}
\]
(see Lemma 3.2 of Wang [13]). Therefore, by Theorem 2.1, \( F \) has a nontrivial critical point \( u_j \) with either

\[
C_{j+1}(F, u_j) \neq 0 \text{ or } C_{j-1}(F, u_j) \neq 0.
\]

Since \( j \geq 3 \), a comparison of the critical groups shows that \( u_0, u_1^\pm, u_j \) are distinct nontrivial critical points of \( F \).

Next we consider the following asymmetric problem of the Ambrosetti-Prodi type

\[
\begin{cases}
-\Delta u + a(x) u = g(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega
\end{cases}
\]
where \( a \in L^\infty(\Omega) \) and \( g \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R}) \) satisfies

\begin{enumerate}
\item[(g1):] \(|g(x, u)| \leq C (1 + |u|^{p-1})\) with \( 2 < p < \frac{2n}{n-2} \), for some \( C > 0 \),
\item[(g2):] \( g(x, 0) = g_u(x, 0) = 0 \),
\item[(g3):] \( \lim_{u \to -\infty} \frac{g(x, u)}{u} < \lambda_1 \), uniformly in \( \overline{\Omega} \),
\item[(g4):] \( \lim_{u \to -\infty} \left( G(x, u) - \frac{1}{2} u g(x, u) \right) < +\infty \), uniformly in \( \overline{\Omega} \),
\item[(g5):] there are \( \mu > 2 \) and \( A > 0 \) such that
\[
0 < \mu G(x, u) \leq u g(x, u) \quad \text{for } u \geq A,
\]
\end{enumerate}

where \( G(x, u) := \int_0^u g(x, t) \, dt \).

Here \( \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \) denote the eigenvalues of \(-\Delta + a\) with Dirichlet boundary condition.

\textbf{Theorem 3.2.} Assume that \( g \) satisfies \((g_1) - (g_5)\) and one of the following conditions:

1. \( \lambda_j < 0 < \lambda_{j+1} \),
2. \( \lambda_j = 0 < \lambda_{j+1} \) and, for some \( \delta > 0 \),
\[
G(x, u) \geq 0 \quad \text{for } |u| \leq \delta,
\]
3. \( \lambda_j < 0 = \lambda_{j+1} \) and, for some \( \delta > 0 \),
\[
G(x, u) \leq 0 \quad \text{for } |u| \leq \delta.
\]

If \( j \geq 3 \), problem (4) has at least three nontrivial solutions.
We seek critical points of

\[ F(u) = \int_\Omega \frac{1}{2} (|\nabla u|^2 + a(x) u^2) - G(x, u) \]

on \( X = H^1_0(\Omega) \).

Lemma 3.3. If \( g \) satisfies \((g_1), (g_3) - (g_5)\), then, for \( \alpha < 0 \) and \( |\alpha| \) sufficiently large,

\[ H_q(X, F_\alpha) = 0 \ \forall q \in \mathbb{Z}. \]

Proof. Let \( \tilde{X} = C^1_0(\overline{\Omega}) \) and \( \tilde{F} = F|_{\tilde{X}} \). By elliptic regularity, \( F \) and \( \tilde{F} \) have the same critical set. If \( F \) does not have any critical values in \((\alpha, \alpha')\), then \( F_\alpha \) (respectively \( \tilde{F}_\alpha \)) is a strong deformation retract of \( \{u \in X : F(u) < \alpha'\} \) (respectively \( \{u \in \tilde{X} : \tilde{F}(u) < \alpha'\} \)) (see Chapter I, Theorem 3.2 and Chapter III, Theorem 1.1 of Chang [5]). Since \( \tilde{X} \) is dense in \( X \), by a theorem of Palais [12],

\[ H_q(X, \{F < \alpha'\}) \cong H_q(\tilde{X}, \{\tilde{F} < \alpha'\}). \]

Therefore it suffices to prove that, for \( \alpha < 0 \) and \( |\alpha| \) large,

\[ H_q(\tilde{X}, \tilde{F}_\alpha) = 0 \ \forall q \in \mathbb{Z}. \]

Let \( S^\infty = \{u \in \tilde{X} : \|u\|_X = 1\} \) be the unit sphere in \( \tilde{X} \) and let \( S^\infty_+ = \{u \in S^\infty : u > 0 \text{ somewhere}\} \), which is a relatively open subset of \( S^\infty \), contractible to \( \{\phi_1\} \) via \((t, u) \mapsto \frac{(1-t) u + t \phi_1}{\|(1-t) u + t \phi_1\|} \ t \in [0, 1] \). We shall show that \( \tilde{F}_\alpha \) is homotopy equivalent to \( S^\infty_+ \) for \( \alpha < 0 \) and \( |\alpha| \) large.

By \((g_3)\) and \((g_5)\),

\[ -C (1 + u^2) \leq G(x, u) \leq \frac{1}{2} \lambda_1 u^2 + C \ \text{for} \ u \leq A, \]

\[ G(x, u) \geq C u^\mu \ \text{for} \ u \geq A, \]

where \( C \) denotes (possibly different) positive constants. Thus for \( u \in S^\infty_+ \),

\[ \tilde{F}(tu) = \frac{1}{2} \left( 1 + \int_\Omega a u^2 \right) t^2 - \int_\Omega G(x, tu) \]

\[ \leq C \left( 1 + t^2 - t^\mu \int_{tu \geq A} u^\mu \right) \]

and it follows that

\[ \lim_{t \to \infty} \tilde{F}(tu) = -\infty. \]

On the other hand, in \( N = \{u \in \tilde{X} : u \leq 0 \text{ everywhere}\} \), the nonpositive cone in \( \tilde{X} \),

\[ \tilde{F}(u) \leq \frac{1}{2} \int_\Omega \left( |\nabla u|^2 + a(x) u^2 - \lambda_1 u^2 \right) - C \geq -C. \]

By \((g_4)\) and \((g_5)\),

\[ \gamma := \sup_{\Omega \times \mathbb{R}} \left( G(x, u) - \frac{1}{2} u g(x, u) \right) < +\infty. \]
Thus for \( u \in S_+^\infty \) and \( t > 0 \),
\[
\frac{d}{dt} \tilde{F}(tu) = \left( 1 + \int_{\Omega} au^2 \right) t - \int_{\Omega} ug(x, tu) \\
= \frac{2}{t} \left\{ \tilde{F}(tu) + \int_{\Omega} G(x, tu) - \frac{1}{2} tu g(x, tu) \right\} \\
\leq \frac{2}{t} \left\{ \tilde{F}(tu) + \gamma |\Omega| \right\} < 0
\]
if \( \tilde{F}(tu) < -\gamma |\Omega| \).

Fix \( \alpha < \min \left\{ \inf_N \tilde{F}, -\gamma |\Omega|, \inf_{\|u\|<1} \tilde{F} \right\} \). Then it follows that for each \( u \in S_+^\infty \) there exists a unique \( T(u) \geq 1 \) such that
\[
\tilde{F}(tu) \begin{cases} > \alpha & \text{for } 0 \leq t < T(u) \\
= \alpha & \text{for } t = T(u) \\
< \alpha & \text{for } t > T(u)
\end{cases}
\]
and
\[
\tilde{F}_\alpha = \{ tu : u \in S_+^\infty, t \geq T(u) \}.
\]
By the implicit function theorem, \( T \in C(S_+^\infty, [1, \infty)) \). Hence
\[
\eta(s, tu) = \begin{cases} (1 - s) tu + s T(u) u & \text{if } 1 \leq t < T(u) \\
tu & \text{if } t \geq T(u)
\end{cases}
\]
defines a strong deformation retraction of \( \{ tu : u \in S_+^\infty, t \geq 1 \} \simeq S_+^\infty \) onto \( \tilde{F}_\alpha \).

**Proof of Theorem 3.2.** Since \( F(-t\phi_1) < 0 \) for \( t > 0 \) sufficiently small, by standard arguments, \( F \) has a local minimizer \( u_0 \) with \( u_0 < 0 \),
\[
\text{rank } C_q(F, u_0) = \delta_{q0}.
\]
Since \( \lim_{t \to \infty} F(t\phi_1) = -\infty \), then \( F \) also has a mountain pass point \( u_1 \),
\[
\text{rank } C_q(F, u_1) = \delta_{q1}.
\]

As in the proof of Theorem 3.1,
\[
C_j(F, 0) \neq 0,
\]
so, using Lemma 3.3, \( F \) also has a nontrivial critical point \( u_j \) with either
\[
C_{j+1}(F, u_j) \neq 0 \text{ or } C_{j-1}(F, u_j) \neq 0.
\]
Since \( j \geq 3, u_0, u_1, u_j \) are distinct nontrivial solutions of (4).

Finally we give an application of Theorem 2.1 to the problem
\[
\begin{cases}
-\Delta u + a(x) u = \lambda g(u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\tag{5}
\]
where \( a \in L^\infty(\Omega) \) and \( g \in C^1(\mathbb{R}, \mathbb{R}) \) satisfies
\[
(g_1) \colon \lim_{|u| \to \infty} \frac{g(u)}{u} < 0,
\]
\[
(g_2) \colon g(0) = g'(0) = 0.
\]
**Theorem 3.4.** Assume that $g$ satisfies $(g_1)$, $(g_2)$, and one of the following conditions:

1. $\lambda_j < 0 < \lambda_{j+1}$,  
2. $\lambda_j = 0 < \lambda_{j+1}$ and, for some $\delta > 0$,  
   \[ G(u) \geq 0 \text{ for } |u| \leq \delta, \]
3. $\lambda_j < 0 = \lambda_{j+1}$ and, for some $\delta > 0$,  
   \[ G(u) \leq 0 \text{ for } |u| \leq \delta. \]

If $j \geq 3$, problem (5) has at least four nontrivial solutions for every $\lambda$ sufficiently large.

**Example 3.5.** $g(u) = \pm |u| u - u^3$

**Remark 3.6.** See Brézis and Nirenberg [3] and Li and Willem [9] for at least two nontrivial solutions.

**Proof of Theorem 3.4.** Since, for $\lambda$ sufficiently large, there is an a priori estimate for the solutions of (5) by the maximum principle, we may also assume that $g(u) = bu$ with $b < 0$, for $|u|$ large. Then the functional

\[ F(u) = \int_{\Omega} \frac{1}{2} \left( |\nabla u|^2 + au^2 \right) - \lambda G(u) \]

is well defined on $X = H^1_0(\Omega)$, and bounded below and satisfies (PS) for $\lambda$ large.

Since $F(\pm t\phi_1) < 0$ for $t > 0$ sufficiently small, $F$ has two local minimizers $u_0^\pm$ with $u_0^- < 0 < u_0^+$,  

\[ \text{rank } C_q(F, u_0^\pm) = \delta_{q0}. \]

Then $F$ also has a mountain pass point $u_1$,

\[ \text{rank } C_q(F, u_1) = \delta_{q1}. \]

As before,

\[ C_j(F, 0) \neq 0, \]

and, for $\alpha < \inf F$,

\[ \text{rank } H_q(X, F_\alpha) = \delta_{q0}, \]

so $F$ has a (fourth) nontrivial critical point $u_j$ with either

\[ C_{j+1}(F, u_j) \neq 0 \text{ or } C_{j-1}(F, u_j) \neq 0. \]

**References**


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**DEPARTMENT OF MATHEMATICS**

**UNIVERSITY OF CALIFORNIA IRVINE**

**IRVINE, CA 92697-3875, USA**

E-mail: kperera@math.uci.edu
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Lead Guest Editor

Juan J. Nieto, Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de Compostela, Santiago de Compostela 15782, Spain; juanjose.nieto.roig@usc.es

Guest Editor

Donal O’Regan, Department of Mathematics, National University of Ireland, Galway, Ireland; donal.oregan@nuigalway.ie