Splitting Polytopes
Sven Herrmann (joint work with Michael Joswig)
62ème Séminaire Lotharingien de Combinatoire
Subdivisions and Splits of Convex Polytopes

Regular Subdivisions and Secondary Polytopes

Properties of Splits

Application: Tropical Geometry

Generalizations/Outlook
Definition
A subdivision of $P$ is a collection $\Sigma$ of polytopes (faces) such that

- $\bigcup_{F \in \Sigma} = P$,
- $F \in \Sigma \implies$ all faces of $F$ are in $\Sigma$,
- $F_1, F_2 \in \Sigma \implies F_1 \cap F_2$ is a face of both,
- $F$ 0-dimensional $\implies$ $F$ is a vertex of $P$. 
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Refinements of Subdivisions

Definition

- $\Sigma'$ is a refinement of $\Sigma$ if each face of $\Sigma'$ is contained in a face of $\Sigma$.
- The common refinement of two subdivisions $\Sigma, \Sigma'$ of $P$ is the subdivision $\{S \cap S' \mid S \in \Sigma, S' \in \Sigma'\}$.

- The refinement defines a partial order on the set of all subdivisions of $P$.
- A finest subdivision (minimal element) is a triangulation.
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**Splits of Convex Polytopes**

**Definition**
A **split** $S$ of a polytope $P$ is a subdivision of $P$ with exactly two maximal faces.

- A split $S$ is defined by a hyperplane $H_S$.
- A hyperplane $H$ (that meets the interior of $P$) defines a split if and only if $H$ does not cut any edge of $P$.
- The splits of $P$ only depend on the combinatorics (oriented matroid) of $P$, not on the realization.
- Example: $v$ a vertex of $P$ such that all neighbors of $v$ lie in a common Hyperplane $H_v$: vertex split for $v$. 
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Example: Hypersimplices

\[ \Delta(k, n) := \text{conv} \left\{ \sum_{i \in I} e_i \ \bigg| \ I \in \binom{\{1, \ldots, n\}}{k} \right\} \subset \mathbb{R}^n, \]

- \( n \)-dimensional unit cube cut with the hyperplane \( \sum_i x_i = k \),
- For a partition \((A, B)\) of \( \{1, \ldots, n\} \) define the \((A, B; \mu)\)-hyperplane by
  \[ \sum_{i \in A} x_i = \mu. \]

Satz (Joswig, H. 08)

The splits of \( \Delta(k, n) \) correspond to the \((A, B; \mu)\)-hyperplanes with \( k - \mu + 1 \leq |A| \leq n - \mu - 1 \) and \( 1 \leq \mu \leq k - 1 \).

Theorem (Joswig, H. 08)

The number of splits of \( \Delta(k, n) \) equals \( (k - 1) \left(2^n - (n - 1)\right) - \sum_{i=2}^{k-1} (k - i) \binom{n}{i} \).
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Generalizations/Outlook
Regular Subdivisions

- \( w : \text{vert } P \rightarrow \mathbb{R} \) weight function,
- consider \( \text{conv}\{(v, w(v)) | v \in \text{vert } P\} \),
- project the lower convex hull down to \( P \),
- the resulting subdivision \( \Sigma_w(P) \) is called regular.
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Lemma

Splits are regular.
The Secondary Polytope

- A d-dimensional polytope in $\mathbb{R}^d$ with n vertices $v_1, \ldots, v_n$.

**Theorem (Gel′fand, Kapranov, Zelevinsky 90)**

There exists an $(n - d - 1)$-dimensional polytope $\text{SecPoly}(P)$ (secondary polytope of $P$) whose face lattice is isomorphic to the poset of all regular subdivisions of $P$.

- Vertices of $\text{SecPoly}(P)$ correspond to triangulations $\Sigma$:
  $$x_i^\Sigma = \sum_{v_i \in S \in \Sigma} \text{vol}(S).$$

- Facets of $\text{SecPoly}(P)$ correspond to coarsest regular subdivisions.

- The intersection of two faces corresponds to the common refinement of the subdivisions corresponding to the faces.
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Splits and Secondary Polytopes

- Splits are facets of \( \text{SecPoly}(P) \), they define an approximation \( \text{SplitPoly}(P) \supset \text{SecPoly}(P) \).
- This is a common approximation for all polytopes with the same oriented matroid.

Theorem (Joswig, H. 09)
\( \text{SecPoly}(P) = \text{SplitPoly}(P) \) if and only if \( P \) is a simplex, polygon, cross polytope, prism over a simplex, or a (possible multiple) join of these polytopes.
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(Weakly) Compatible Split Systems

Definition
Let $S$ be a set of splits (split system) of a polytope $P$.

- We call $S$ weakly compatible if the subdivisions $S \in S$ have a common refinement (without new vertices).
- We call $S$ compatible if none of the split defining hyperplanes meet in the interior of $P$.

- Example: Vertex splits are (weakly) compatible if and only if the corresponding vertices are not connected by an edge.
- Stable set of the edge graph of a polytope yields a compatible split system.
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The Split Complex

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The split complex of a polytope $P$ is the simplicial complex

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- These complexes can be seen as (kind of) subcomplexes of the dual complex of the secondary polytope of $P$. 
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The dual graph of a compatible split system is a tree.
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Compatibility for Hypersimplices

Satz (Joswig, H. 08)

Two splits \((A, B; \mu)\) and \((C, D; \nu)\) of \(\Delta(k, n)\) are compatible if and only if one of the following holds:

\[
\begin{align*}
|A \cap C| & \leq k - \mu - \nu, \\
|B \cap C| & \leq \mu - \nu, \\
|A \cap D| & \leq \nu - \mu, \\
|B \cap D| & \leq \mu + \nu - k.
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This allows an explicit computation of the split complex of \(\Delta(k, n)\).
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This allows an explicit computation of the split complex of \(\Delta(k, n)\).
A decomposition $w + w'$ of weight functions is called coherent if $\Sigma_w(P)$ and $\Sigma_{w'}(P)$ have a common refinement ($\Sigma_{w+w'}(P)$).

A weight function $w$ is called split prime if $\Sigma_w(P)$ does not refine any split.

**Theorem (Bandelt, Dress 92; Hirai 06; Joswig, H. 08)**

Each weight function $w$ for a polytope $P$ has a coherent decomposition

$$w = w_0 + \sum_{S \in S} \alpha^w_{w_S} w_S,$$

where $S$ is some weakly compatible set of splits and $w_0$ is split prime. This decomposition is unique.
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The Second Hypersimplex and Metric Spaces

- \( \Delta(2, n) = \text{conv} \{e_i + e_j \mid 1 \leq i < j \leq n \} \).
- Lifting functions of \( \Delta(2, n) \) correspond to (pseudo-)metrics on \( n \) points.
- Splits of \( \Delta(2, n) \) are in bijection with partitions \((A, B)\) of \( \{1, \ldots, n\} \) where each part has at least two elements.
- Originally, these were the splits of finite metric spaces defined by Bandelt and Dress (92) for applications in biology.
- Two splits \((A, B)\) and \((C, D)\) of \( \Delta(2, n) \) are compatible if and only if one of the four sets \( A \cap C \), \( A \cap D \), \( B \cap C \), and \( B \cap D \) is empty.
- This is the original definition of compatibility for splits of finite metric spaces.
- There is also a combinatorial condition for weak compatibility.
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The Second Hypersimplex and Metric Spaces

- $\Delta(2, n) = \text{conv} \{e_i + e_j \mid 1 \leq i < j \leq n\}$.
- Lifting functions of $\Delta(2, n)$ correspond to (pseudo-)metrics on $n$ points.
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Subdivisions and Splits of Convex Polytopes

Regular Subdivisions and Secondary Polytopes

Properties of Splits

Application: Tropical Geometry

Generalizations/Outlook
A subdivision $\Sigma$ of $\Delta(k, n)$ is called a **matroid subdivision** if all edges of $\Sigma$ are edges of $\Delta(k, n)$.

(Equivalently: Each face of $\Sigma$ is a matroid polytope $P_M$, i.e. each vertex of $P_M$ corresponds to a basis of $\mathcal{M}$.)

The **Dressian** is the polyhedral complex

$$Dr(k, n) := \left\{ w \in \mathbb{R}^{{{n}\choose{k}}} \mid \Sigma_w(\Delta(k, n)) \text{ is a matroid subdivision} \right\} \cap {n\choose{k}}^{-1}.$$

Elements of $Dr(k, n)$ are the tropical Plücker vectors (Speyer 08).

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The Split Complex and the Dressian

\( k = 2: \)

- Lifting functions of \( \Delta(2, n) \) correspond to (pseudo-)metrics on \( n \) points.
- \( Gr(2, n) = Dr(2, n) \cong \text{Split}(\Delta(2, n)) \) is the space of metric trees (Bunemann 74; Billera, Holmes & Vogtmann 01).

**Theorem (Joswig, H. 08)**

\( \text{Split}(\Delta(k, n)) \) is a subcomplex of \( Dr(k, n) \).

- Proof idea:
  - Splits are matroid subdivisions.
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*The dimension of the Dressian $\Delta(3, n)$ is of order $\Theta(n^2)$.*

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Further Coarsest Subdivisions

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- Three maximal faces: one combinatorial type, regular.
- Four maximal faces: three combinatorial types.
- More than four: gets complicated...

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A subdivision of $\Sigma$ of $P$ is called $k$-split, if the tight span of $\Sigma$ is a $(k - 1)$-dimensional simplex.
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Thanks for your attention!