Nonself-Adjoint Degenerate Differential-Operator Equations of Higher Order

Liparit Tepoyan*

Yerevan State University, A. Manoogian str. 1, 0025, Yerevan, Armenia
(Received September 30, 2012; Revised October 23, 2013; Accepted December 12, 2013)

This article deals with the Dirichlet problem for a degenerate nonself-adjoint differential-operator equation of higher order. We prove existence and uniqueness of the generalized solution as well as establish some analogue of the Keldysh theorem for the corresponding one-dimensional equation.

Keywords: Differential equations in abstract spaces, Degenerate equations, Weighted Sobolev spaces, Spectral theory of linear operators.

AMS Subject Classification: 34G10, 34L05, 35J70, 46E35, 47E05.

1. Introduction

The main object of the present paper is the degenerate differential-operator equation

$$ Lu \equiv (-1)^m (t^\alpha u^{(m)})^{(m)} + A(t^{\alpha-1}u^{(m)})^{(m-1)} + Pt^\beta u = f(t), $$

(1)

where $m \in \mathbb{N}$, $t$ belongs to the finite interval $(0,b)$, $\alpha \geq 0, \alpha \neq 1, 3, \ldots, 2m-1$, $\beta \geq \alpha - 2m$, $A$ and $P$ are linear operators (in general unbounded) in the separable Hilbert space $H$, $f \in L_{2,\beta}((0,b),H)$, i.e.,

$$ \|f\|_{L_{2,\beta}((0,b),H)}^2 = \int_0^b t^{-\beta} \|f(t)\|_H^2 dt < \infty. $$

We suppose that the operators $A$ and $P$ have common complete system of eigenfunctions $\{\varphi_k\}_{k=1}^\infty$, $A\varphi_k = a_k \varphi_k$, $P\varphi_k = p_k \varphi_k$, $k \in \mathbb{N}$, which form a Riesz basis in $H$, i.e., for any $x \in H$ there is a unique representation

$$ x = \sum_{k=1}^\infty x_k \varphi_k $$

* Email: tepoyan@yahoo.com

ISSN: 1512-0082 print
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and there are constants \( c_1, c_2 > 0 \) such that

\[
c_1 \sum_{k=1}^{\infty} |x_k|^2 \leq ||x||^2 \leq c_2 \sum_{k=1}^{\infty} |x_k|^2.
\]

If \( m = 1 \), the operator \( A \) is a multiplication operator, \( Au = au, a \in \mathbb{R}, a \neq 0 \) and \( Pu = -u_{xx}, x \in (0, c) \) then we obtain the degenerate elliptic operator in the rectangle \((0, b) \times (0, c)\). The dependence of the character of the boundary conditions with respect to \( t \) for \( t = 0 \) on the sign of the number \( a \) was first observed by M.V. Keldish in [5] and next generalized by G. Jaiani in [4] (thus the statement of the boundary value problem depends on the “lower order” terms). The case \( m = 1, \beta = 0, 0 \leq \alpha < 2 \) was considered in [2], [6] (here \( A = 0 \)) and the case \( m = 2, \beta = 0, 0 \leq \alpha \leq 4 \) in [8]. In [9] the self-adjoint case of higher order degenerate differential-operator equations for arbitrary \( \alpha \geq 0, \alpha \neq 1, 3, \ldots, 2m - 1 \) has been considered.

Our approach is based on the consideration of the one-dimensional equation (1), when the operators \( A \) and \( P \) are multiplication operators by numbers \( a \) and \( p \) respectively, \( Au = au, Pu = pu, a, p \in \mathbb{C} \) (see [3]).

Observe that this method suggested by A.A. Dezin (see [3]) has been used for the degenerate self-adjoint operator equation on the infinite interval \((1, +\infty)\) in [12] and with arbitrary weight function on the finite interval in [11].

2. One-dimensional case

2.1. Weighted Sobolev spaces \( \tilde{W}_\alpha^m(0, b) \)

Let \( \tilde{C}^m[0, b] \) denote the functions \( u \in C^m[0, b] \), which satisfy the conditions

\[
u^{(k)}(0) = u^{(k)}(b) = 0, k = 0, 1, \ldots, m - 1.
\]

Define \( \tilde{W}_\alpha^m(0, b) \) as the completion of \( \tilde{C}^m[0, b] \) in the norm

\[
||u||^2_{\tilde{W}_\alpha^m(0, b)} = \int_0^b t^\alpha |u^{(m)}(t)|^2 dt.
\]

Denote the corresponding scalar product in \( \tilde{W}_\alpha^m(0, b) \) by \( \{u, v\}_\alpha = (t^\alpha u^{(m)}, v^{(m)}) \), where \( (\cdot, \cdot) \) stands for the scalar product in \( L_2(0, b) \).

Note that the functions \( u \in \tilde{W}_\alpha^m(0, b) \) for every \( t_0 \in (\varepsilon, b] \), \( \varepsilon > 0 \) have the finite values \( u^{(k)}(t_0), k = 0, 1, \ldots, m - 1 \) and \( u^{(k)}(b) = 0, k = 0, 1, \ldots, m - 1 \) (see [1]).

For the proof of the following propositions we refer to [9] and [10].

**Proposition 2.1:** For the functions \( u \in \tilde{W}_\alpha^m(0, b), \alpha \neq 1, 3, \ldots, 2m - 1 \) we have the following estimates

\[
|u^{(k)}(t)|^2 \leq C t^{2m-2k-1-\alpha} ||u||^2_{\tilde{W}_\alpha^m(0, b)}, k = 0, 1, \ldots, m - 1.
\]

It follows from Proposition 2.1 that in the case \( \alpha < 1 \) (weak degeneracy) \( u^{(j)}(0) = 0 \) for all \( j = 0, 1, \ldots, m - 1 \), while for \( \alpha > 1 \) (strong degeneracy) not all \( u^{(j)}(0) = 0 \).
More precisely, for $1 < \alpha < 2m - 1$ the derivatives at zero $u^{(j)}(0) = 0$ only for $j = 0, 1, \ldots, s_\alpha$, where $s_\alpha = m - 1 - \lfloor \frac{\alpha + 1}{2} \rfloor$ (here $[a]$ is the integral part of the $a$) and for $\alpha > 2m - 1$ all $u^{(j)}(0)$, $j = 0, 1, \ldots, m - 1$ in general may be infinite.

Denote $L_{2,\beta}(0,b) = \left\{ f, \int_0^b t^\beta |f(t)|^2 \, dt < +\infty \right\}$. Observe that for $\alpha \leq \beta$ we have $L_{2,\alpha}(0,b) \subset L_{2,\beta}(0,b)$.

**Proposition 2.2:** For $\beta \geq \alpha - 2m$ we have a continuous embedding

$$\hat{W}_\alpha^m(0,b) \subset L_{2,\beta}(0,b),$$

which is compact for $\beta > \alpha - 2m$.

Note that the embedding (4) in the case of $\beta = \alpha - 2m$ is not compact while for $\beta < \alpha - 2m$ it fails.

Denote $d(m,\alpha) = 4^{-m}(\alpha - 1)^2(\alpha - 3)^2 \cdots (\alpha - (2m - 1))^2$. In Proposition 2.2 using Hardy inequality (see [7]) it was proved that

$$\int_0^b t^\alpha |u^{(m)}(t)|^2 \, dt \geq d(m,\alpha) \int_0^b t^{\alpha - 2m} |u(t)|^2 \, dt. \tag{5}$$

Note that here $d(m,\alpha)$ is the exact number. Now it is easy to check that for $\beta \geq \alpha - 2m$

$$\|u\|_{\hat{W}_\alpha^m(0,b)}^2 \geq b^{\alpha - 2m - \beta} d(m,\alpha) \|u\|_{L_{2,\beta}(0,b)}^2. \tag{6}$$

### 2.2. Nonself-adjoint degenerate equations

In this subsection we consider one-dimensional version of equation (1)

$$Su \equiv (-1)^m (t^\alpha u^{(m)})^{(m)} + a(t^{\alpha - 1} u^{(m)})^{(m-1)} + pt^\beta u = f(t), \tag{7}$$

where $\alpha \geq 0, \alpha \neq 1, 3, \ldots, 2m - 1$, $\beta \geq \alpha - 2m$, $f \in L_{2,-\beta}(0,b)$, $a \neq 0$ and $p$ are real constants.

**Definition 2.3:** A function $u \in \hat{W}_\alpha^m(0,b)$ is called a generalized solution of equation (7), if for arbitrary $v \in \hat{W}_\alpha^m(0,b)$ we have

$$\{u,v\}_\alpha + a(-1)^{m-1}(t^{\alpha - 1} u^{(m)}, v^{(m-1)}) + p(t^\beta u, v) = (f,v). \tag{8}$$

**Theorem 2.4:** Let the following condition be fulfilled

$$a(\alpha - 1)(-1)^m > 0, \quad \gamma = b^{\alpha - 2m - \beta} (d(m,\alpha) + a(\alpha - 1)(-1)^m d(m - 1, \alpha - 2)) + p > 0. \tag{9}$$

Then the generalized solution of equation (7) exists and is unique for every $f \in L_{2,-\beta}(0,b)$.

**Proof:** Uniqueness. To prove the uniqueness of the solution we set in equality (8) $f = 0$ and $v = u$. Let $\alpha > 1$ (in the case $\alpha < 1$ the proof is similar and we use
Hence using inequality (6) we obtain
\[
\left(t^{\alpha-1}u^{(m-1)}(t)\right)_{t=0} = 0, \text{ which follows from Proposition 2.1}. \text{ Then integrating by parts we obtain}
\[
\left(t^{\alpha-1}u^{(m)}, u^{(m-1)}\right) = -\frac{1}{2} \left(t^{\alpha-1}|u^{(m-1)}(t)|^2\right)_{t=0} - \frac{\alpha - 1}{2} \int_0^b t^{\alpha-2}|u^{(m-1)}(t)|^2 \, dt.
\]
It follows from the inequality (3) for \(k = m-1\) that the value \(\left(t^{\alpha-1}|u^{(m-1)}(t)|^2\right)_{t=0}\) is finite. On the other hand, using inequality (5) we get
\[
\int_0^b t^{\alpha-2}|u^{(m-1)}(t)|^2 \, dt \geq d(m-1, \alpha - 2) \int_0^b t^{\alpha-2m}|u(t)|^2 \, dt.
\]
Hence using inequality (6) we obtain
\[
0 = \{u, u\}_\alpha + a(-1)^{m-1}\left(t^{\alpha-1}u^{(m)}, u^{(m-1)}\right) + p(t^\beta u, u)
\leq \frac{a}{2}(-1)^m\left(t^{\alpha-1}|u^{(m-1)}(t)|^2\right)_{t=0} + \gamma \int_0^b t^\beta|u(t)|^2 \, dt.
\]
Now uniqueness of the generalized solution follows from condition (9).

**Existence.** To prove the existence of the generalized solution define a linear functional \(l_f(v) = (f, v), v \in \dot{W}^m_\alpha(0, b)\). From the continuity of the embedding (4) it follows that
\[
|l_f(v)| \leq \|f\|_{L_2, \alpha(0, b)} \|v\|_{L_2, \alpha(0, b)} \leq c\|f\|_{L_2, \alpha(0, b)} \|v\|_{\dot{W}^m_\alpha(0, b)},
\]
therefore the linear functional \(l_f(v)\) is bounded on \(\dot{W}^m_\alpha(0, b)\). Hence it can be represented in the form \(l_f(v) = (f, v) = \{u^*, v\}, u^* \in \dot{W}^m_\alpha(0, b)\) (this follows from the Riesz theorem on the representation of the linear continuous functional). The last two terms in the left hand-side of equality (8) also can be regarded as a continuous linear functional relative to \(u\) and represented in the form \(\{u, Kv\}_\alpha, Kv \in W^m_\alpha(0, b)\). In fact, using inequality (5) we may write
\[
|a(-1)^{m-1}\left(t^{\alpha-1}u^{(m)}, v^{(m-1)}\right) + p(t^\beta u, v)|
\leq |a(t^\alpha u^{(m)}, t^\beta v^{(m-1)})| + |p(t^\beta u, t^\beta v)|
\leq c_1\|u\|_{\dot{W}^m_\alpha(0, b)} \left\{ \int_0^b t^{\alpha-2}|v^{(m-1)}(t)|^2 \, dt \right\}^{1/2}
+ c_2\|u\|_{L_2, \alpha-2m(0, b)} \|v\|_{L_2, \alpha-2m(0, b)}
\leq \frac{2c_1}{\alpha - 1}\|u\|_{\dot{W}^m_\alpha(0, b)} \|v\|_{\dot{W}^m_\alpha(0, b)} + c_3\|u\|_{\dot{W}^m_\alpha(0, b)} \|v\|_{\dot{W}^m_\alpha(0, b)}
= c\|u\|_{\dot{W}^m_\alpha(0, b)} \|v\|_{\dot{W}^m_\alpha(0, b)}.
\]
From equality (8) we deduce that for any \(v \in \dot{W}^m_\alpha(0, b)\) we have
\[
\{u, (I + K)v\}_\alpha = \{u^*, v\}_\alpha.
\]
Observe that the image of the operator $I + K$ is dense in $\tilde{W}_\alpha^m(0, b)$. Indeed, if we have some $u_0 \in \tilde{W}_\alpha^m(0, b)$ such that

$$\{u_0, (I + K)v\}_\alpha = 0$$

for every $v \in \tilde{W}_\alpha^m(0, b)$, we obtain $u_0 = 0$, since we have already proved uniqueness of the generalized solution for equation (7).

Assume that $0 < \sigma d(m, \alpha)b^{\alpha-2m-\beta} \leq \gamma$. Then we can write

$$\{u, (I + K)u\}_\alpha \geq \sigma\{u, u\}_\alpha + \left( b^{\alpha-2m-\beta}((1-\sigma)d(m, \alpha) + p) \int_0^b t^\beta |u(t)|^2 \, dt \right.$$

$$\left. + \frac{a}{2}(\alpha-1)(-1)^m d(m-1, \alpha-2) \right) \int_0^b t^\beta |u(t)|^2 \, dt$$

$$= \sigma\{u, u\}_\alpha + (\gamma - \sigma d(m, \alpha)b^{\alpha-2m-\beta}) \int_0^b t^\beta |u(t)|^2 \, dt$$

$$\geq \sigma\{u, u\}_\alpha.$$

Finally we get

$$\{u, (I + K)u\}_\alpha \geq \sigma\{u, u\}_\alpha. \quad (11)$$

From (11) it follows that $(I + K)^{-1}$ is defined on $\tilde{W}_\alpha^m(0, b)$ and is bounded. Consequently there exist operator $I + K^*$ and $(I + K^*)^{-1} = ((I + K)^{-1})^*$ (here $K^*$ means the adjoint operator). Hence from (10) we obtain

$$u = (I + K^*)^{-1}u^*.$$

Define an operator $S : D(S) \subset \tilde{W}_\alpha^m(0, b) \subset L_{2,\beta}(0, b) \to L_{2,-\beta}(0, b)$.

**Definition 2.5:** We say that $u \in \tilde{W}_\alpha^m(0, b)$ belongs to $D(S)$ if there exists $f \in L_{2,-\beta}(0, b)$ such that equality (8) is fulfilled for every $v \in \tilde{W}_\alpha^m(0, b)$. In this case we write $Su = f$.

The operator $S$ acts from the space $L_{2,\beta}(0, b)$ to $L_{2,-\beta}(0, b)$. It is easy to check that $S := t^{-\beta}S, D(S) = D(S), S : L_{2,\beta}(0, b) \to L_{2,\beta}(0, b)$ is an operator in the space $L_{2,\beta}(0, b)$, since if $f \in L_{2,\beta}(0, b)$ then $f_1 := t^{-\beta}f \in L_{2,\beta}(0, b)$ and $\|f\|_{L_{2,\beta}(0, b)} = \|f_1\|_{L_{2,\beta}(0, b)}$.

**Proposition 2.6:** Under the assumptions of Theorem 2.4 the inverse operator $S^{-1} : L_{2,\beta}(0, b) \to L_{2,\beta}(0, b)$ is continuous for $\beta \geq \alpha - 2m$ and compact for $\beta > \alpha - 2m$.

**Proof:** For the proof first observe that for $u \in D(S)$ we have

$$\|u\|_{L_{2,\beta}(0, b)} \leq c\|f\|_{L_{2,-\beta}(0, b)} = c\|f_1\|_{L_{2,\beta}(0, b)}.$$
considerations of Theorem 2.4, we get
\[
\sigma b^{\alpha - 2m - \beta} d(m, \alpha) \| u \|_{L^2_{\alpha, \beta}(0, b)}^2 \leq \sigma d(m, \alpha) \| u \|_{W^{m, \beta}(0, b)}^2 \\
\leq \{(I + K)u, u\}_\alpha = (f, u) \\
\leq \| f \|_{L^2_{\alpha, \beta}(0, b)} \| u \|_{L^2_{\alpha, \beta}(0, b)} \\
= \| f_1 \|_{L^2_{\alpha, \beta}(0, b)} \| u \|_{L^2_{\alpha, \beta}(0, b)}.
\]
Thus we obtain
\[
\| S^{-1} f_1 \|_{L^2_{\alpha, \beta}(0, b)} \leq c \| f_1 \|_{L^2_{\alpha, \beta}(0, b)},
\]
consequently the continuity of \( S^{-1} \) for \( \beta \geq \alpha - 2m \) is proved. To show the compactness of \( S^{-1} \) for \( \beta < \alpha - 2m \) it is enough to apply the compactness of the embedding (4) for \( \beta < \alpha - 2m \).

Let us consider the following equation
\[
Tv = \begin{pmatrix} -1 \end{pmatrix}^m \left(t^{\alpha} v^{(m)}(m) \right) - a(t^{\alpha - 1} v^{(m - 1)}(m) + pt^\beta v = g(t), \tag{13}
\end{align}
\]
where \( \alpha \geq 0, \alpha \neq 1, 3, \ldots, 2m - 1, \beta \geq \alpha - 2m, g \in L_{2, \beta}(0, b), a \neq 0 \) and \( p \) are real constants.

**Definition 2.7:** We say that \( v \in L^2_{\alpha, \beta}(0, b) \) is a generalized solution of equation (13), if for every \( u \in D(S) \) the following equality holds
\[
(Su, v) = (u, g). \tag{14}
\]

Let \( g_1 := t^{-\beta} g \). Definition 2.7 of the generalized solution as above defines an operator \( T : L^2_{\alpha, \beta}(0, b) \to L^2_{\alpha, \beta}(0, b) \), \( T := t^{-\beta} T \). Actually we have defined the operator \( \overline{T} \) as the adjoint to \( S \) operator in \( L^2_{\alpha, \beta}(0, b) \), i.e.,
\[
\overline{T} = S^*.
\]

**Theorem 2.8:** Under the assumptions of Theorem 2.4 the generalized solution of equation (13) exists and is unique for every \( g \in L_{2, \beta}(0, b) \). Moreover, the inverse operator \( \overline{T}^{-1} : L^2_{\alpha, \beta}(0, b) \to L^2_{\alpha, \beta}(0, b) \) is continuous for \( \beta \geq \alpha - 2m \) and compact for \( \beta > \alpha - 2m \).

**Proof:** Solvability of the equation \( Su = f_1 \) for any \( f_1 \in L^2_{-\beta}(0, b) \) (see Theorem 2.4) implies uniqueness of the solution of equation (13), while existence of the bounded inverse operator \( S^{-1} \) (see Proposition 2.6) implies solvability of (13) for any \( g \in L_{2, \beta}(0, b) \) (see, for instance, [13]). Since we have \((S^*)^{-1} = (S^{-1})^*\), boundedness and compactness of the operator \( S^{-1} \) imply boundedness and compactness of the operator \( \overline{T}^{-1} \) for \( \beta \geq \alpha - 2m \) and \( \beta > \alpha - 2m \) respectively (see Proposition 2.6).

**Remark 1:** For \( \alpha > 1 \) and for every generalized solution \( v \) of equation (13) we
have
\[
\left( t^{\alpha-1} |u^{(m-1)}(t)|^2 \right) \big|_{t=0} = 0.
\] (15)

In fact, replacing \( g \) by \( Tv \) in equality (14), integrating by parts the second term and using equality (8) we obtain (15). Note also that for equation (7) the left-hand side of (15) is only bounded. This is some analogue of the Keldysh theorem (see [5]).

**Remark 2:** Note another interesting phenomenon connected with degenerate equations, namely appearing continuous spectrum. Assume that in equation (7) \( a = p = 0 \) and \( \beta = \alpha - 2m \). In [10] it was proved that the spectrum of the operator \( Bu := (−1)^m t^{2m−\alpha} (t^\alpha u^{(m)})^{(m)} \), \( B : L_{2,\alpha−2m}(0,b) \to L_{2,\alpha−2m}(0,b) \)

is purely continuous and coincides with the ray \([d(m,\alpha), +\infty)\). Note also that the spectrum of the operator \( Qu := (−1)^m t^{−\beta} (t^\alpha u^{(m)})^{(m)} \), \( Q : L_{2,\beta}(0,b) \to L_{2,\beta}(0,b) \)

for \( \beta > \alpha - 2m \) is discrete.

3. **Dirichlet problem for degenerate differential-operator equations**

In this section we consider the operator equation

\[
Lu \equiv (−1)^m (t^\alpha u^{(m)})^{(m)} + A(t^{\alpha-1} u^{(m)})^{(m-1)} + P t^\beta u = f(t),
\] (16)

where \( \alpha \geq 0, \alpha \neq 1, 3, \ldots, 2m - 1, \beta \geq \alpha - 2m \), \( A \) and \( P \) are linear operators in the separable Hilbert space \( H \), \( f \in L_{2,-\beta}((0,b),H) \).

By assumption linear operators \( A \) and \( P \) have common complete system of eigenfunctions \( \{\varphi_k\}_{k=1}^\infty \), \( A\varphi_k = a_k\varphi_k, \ P\varphi_k = p_k\varphi_k, k \in \mathbb{N}, \) which forms a Riesz basis in \( H \), i.e., we can write

\[
u(t) = \sum_{k=1}^\infty u_k(t)\varphi_k, \quad f(t) = \sum_{k=1}^\infty f_k(t)\varphi_k. \] (17)

Hence operator equation (16) can be decomposed into an infinite chain of ordinary differential equations

\[
L_k u_k \equiv (−1)^m (t^\alpha u_k^{(m)})^{(m)} + a_k(t^{\alpha-1} u_k^{(m)})^{(m-1)} + p_k t^\beta u_k = f_k(t), k \in \mathbb{N}. \] (18)

It follows from the condition \( f \in L_{2,-\beta}((0,b),H) \) that \( f_k \in L_{2,-\beta}(0,b), k \in \mathbb{N}. \) For one-dimensional equations (18) we can define the generalized solutions \( u_k(t), k \in \mathbb{N} \) (see Section 2).

**Definition 3.1:** A function \( u \in L_{2,\beta}((0,b),H) \) admitting representation

\[
u(t) = \sum_{k=1}^\infty u_k(t)\varphi_k,
\]
where \( u_k(t), k \in \mathbb{N} \) are the generalized solutions of the one-dimensional equations (18) is called a generalized solution of the operator equation (16).

Actually we have defined the operator \( L \) as the closure of the differential operation \( L(D) \) originally defined on all finite linear combinations of functions \( u_k(t)\varphi_k, k \in \mathbb{N} \), where \( u_k \in D(L_k) \).

The following result is a consequence of the general results of A.A. Dezin (see [3]).

**Theorem 3.2:** The operator equation (16) is uniquely solvable for every \( f \in L_{2,-\beta}((0, b), H) \) if and only if the equations (18) are uniquely solvable for every \( f_k \in L_{2,-\beta}(0, b), k \in \mathbb{N} \) and uniformly with respect to \( k \in \mathbb{N} \)

\[
\|u_k\|_{L_{2,\beta}(0, b)} \leq c\|f_k\|_{L_{2,-\beta}(0, b)}.
\]

Theorems 2.4 and 2.8 shows us that a sufficient condition for relations (19) are the conditions

\[
\gamma_k = t^{\alpha-2m-\beta}(d(m, \alpha) + \frac{a_k}{2}(\alpha-1)(-1)^m d(m-1, \alpha-2)) + p_k > \varepsilon > 0, k \in \mathbb{N}.
\]

Here we assume that \( a_k \neq 0, a_k \) and \( p_k \) are real for \( k \in \mathbb{N} \). Thus we get the following result.

**Theorem 3.3:** Let the condition (20) be fulfilled. Then operator equation (16) has a unique generalized solution for every \( f \in L_{2,-\beta}((0, b), H) \).

**Proof:** Since the system \( \{\varphi_k\}_{k=1}^{\infty} \) forms a Riesz basis in \( H \) then according to (19) we can write

\[
\|u\|_{L_{2,\beta}(0, b), H}^2 = \int_0^b t^\beta \|u(t)\|_H^2 \, dt 
\leq c_1 \int_0^b t^\beta \sum_{k=1}^{\infty} |u_k(t)|^2 \, dt 
\leq c_2 \sum_{k=1}^{\infty} \|f_k\|_{L_{2,-\beta}(0, b)}^2 
\leq C \|f\|_{L_{2,-\beta}((0, b), H)}.
\]

(21)

It follows from inequality (21) that the inverse operator \( L^{-1} : L_{2,-\beta}((0, b), H) \to L_{2,\beta}((0, b), H) \) is bounded for \( \beta \geq \alpha - 2m \). In contrast to the one-dimensional case (see Proposition 2.6 and Theorem 2.8) this operator for \( \beta > \alpha - 2m \) will not be compact (it will be a compact operator only in case when the space \( H \) is finite-dimensional). The operator \( L \) acts from the space \( L_{2,\beta}((0, b), H) \) to the space \( L_{2,-\beta}((0, b), H) \). As in one-dimensional case define an operator acting in the same space, which is necessary to explore spectral properties of the operators. Set \( f = t^\beta g \). Then \( \|f\|_{L_{2,-\beta}((0, b), H)} = \|g\|_{L_{2,\beta}((0, b), H)} \). Hence the operator \( L = t^{-\beta}L \) is an operator in the space \( L_{2,\beta}((0, b), H) \). As a consequence of Theorem 3.3 we can state that \( 0 \in \rho(L) \), where \( \rho(L) \) is the resolvent set of the operator \( L \).
Remark 1: The simplest example of the operators described in Introduction consists of the operators on the $n$-dimensional cube $V = [0, 2\pi]^n$, generated by differential expressions of the form

$$L(-iD)u \equiv \sum_{|\alpha| \leq m} a_\alpha D^\alpha u$$

with constant coefficients. Here $\alpha \in \mathbb{Z}_+^n$ is a multi-index. This class of operators is at the same time quite a large class. Let $\mathcal{P}^\infty$ be the set of smooth functions that are periodic in each variable. Let $s \in \mathbb{Z}^n$. To every differential operation $L(-iD)$ we can associate a polynomial $A(s)$ with constant coefficients such that

$$A(-iD) e^{ix} = A(s) e^{ix}, \quad s \cdot x = s_1 x_1 + s_2 x_2 + \ldots + s_n x_n.$$ 

We define the corresponding operator $A : L^2(V) \to L^2(V)$ to be the closure in $L^2(V)$ of the differential operation $A(-iD)$ first defined on $\mathcal{P}^\infty$. Such operators are called $\Pi$-operators and have many interesting properties. The role of the functions $\{\varphi_k\}_{k=1}^\infty$ is played by the functions $e^{ix}$, $s \in \mathbb{Z}^n$. For details see the book of A.A. Dezin [3].

References

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