Hierarchical Models for Viscoelastic Kelvin-Voigt Prismatic Shells with Voids

George Jaiani*

I. Javakhishvili Tbilisi State University

I. Vekua Institute of Applied Mathematics & Department of Mathematics,
Chair of Mechanics

2 University St., 0186, Tbilisi, Georgia

(Received January 16, 2017; Revised April 19, 2017; Accepted June 1, 2017)

The present paper is devoted to construction of hierarchical models for porous elastic and viscoelastic Kelvin-Voigt prismatic shells on the basis of linear theories. Using I. Vekua’s [1], [2] dimension reduction method, governing systems are derived and in the Nth approximation boundary value problems are set.

Keywords: Hierarchical models, Viscoelastic prismatic shells, Porous elastic prismatic shells, Materials with voids.

AMS Subject Classification: 74K20, 74K25, 74F99, 74D05.

1. Introduction

The present paper is devoted to construction of hierarchical models for porous elastic and viscoelastic Kelvin-Voigt prismatic shells on the basis of linear theories. Using I. Vekua’s [1], [2] dimension reduction method, governing systems are derived and in the Nth approximation boundary value problems are set. In the N = 0 approximation, considering plates of a constant thickness, governing systems mathematically coincide with the governing systems of the plane strain corresponding to the basic three-dimensional (3D) linear theories [3]-[6] up to a separate equation for the out of plane component of the displacement vector in our cases. The ways of investigation of boundary value problems (BVPs) and initial boundary value problems (IBVPs), including the case of cusped prismatic shells [2], are indicated and some preliminary results are presented.

2. Field equations for Kelvin-Voigt materials

The field equations have the following form [4], [5]:

\[ \begin{align*}
X_{ji,j} + \Phi_i &= \rho \ddot{u}_i(x_1, x_2, x_3, t), \\
(x_1, x_2, x_3) &\in \Omega \subset \mathbb{R}^3, \quad t > t_0, \quad i,j = 1, 2, 3;
\end{align*} \]  

(2.1)

*Corresponding author. Email: george.jaiani@gmail.com
\[ H_{j,j} + H_0 = \rho_0 \ddot{\varphi} - \mathcal{F}, \]  

(2.2)

where \( X_{ij} \in C^1(\Omega) \) is the stress tensor; \( \Phi_i \) are the volume force components; \( \rho_0 := \rho k' \) \( (k' \) is equilibrated inertia), \( \rho \) is the reference mass density; \( u_i \in C^2(\Omega) \) are the displacements; \( H_j \in C^1(\Omega) \) is the component of the equilibrated stress vector, \( H_0 \) and \( \mathcal{F} \) are the intrinsic and extrinsic equilibrated volume forces; the points as superscripts mean differentiation with respect to the time, and Einstein’s summation convention is used; indices after comma mean differentiation with respect to the corresponding variables of the Cartesian frame \( O x_1 x_2 x_3 \) (throughout the paper we assume existence of the indicated (continuous) derivatives); dots as superscripts of the symbols mean derivatives with respect to time \( t \);

**Constitutive Equations (isotropic case)**

\[
X_{ij} = \lambda e_{kk} \delta_{ij} + 2\mu e_{ij} + \lambda^* e_{kk} \delta_{ij} + 2\mu^* \dot{e}_{ij} + b \varphi \delta_{ij} + b^* \dot{\varphi} \delta_{ij}, \quad i,j = 1,2,3, 
\]

(2.3)

\[
H_j = \tilde{\alpha} \varphi,_{j} + \alpha^* \dot{\varphi}, j = 1,2,3, 
\]

(2.4)

\[
H_0 = -b e_{kk} - \xi \varphi - \nu^* \dot{e}_{kk} - \xi^* \dot{\varphi},
\]

(2.5)

where \( e_{ij} \in C^1(\Omega) \) is the strain tensor; \( \varphi := \nu_0 - \nu \in C^2(\Omega) \) is the change in the volume fraction from the matrix reference volume fraction \( \nu \) (clearly, the bulk reference density \( \rho = \nu \gamma \), \( 0 < \nu \leq 1 \), here \( \gamma \) is the matrix reference density); \( \lambda, \lambda^*, \mu, \mu^*, b, b^*, \tilde{\alpha}, \alpha^*, \nu^*, \xi, \xi^* \) are the constitutive coefficients;

**Kinematic Relations**

\[
e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}), \quad i,j = 1,2,3.
\]

(2.6)


Let us consider prismatic shells (see, e.g., Fig. 1 and [2], [7]) occupying 3D domain \( \Omega \) with the projection \( \omega \) (on the plane \( x_3 = 0 \)) and the face surfaces

\[
x_3 = (\pm) h (x_1, x_2) \in C^2(\omega) \quad \text{and} \quad x_3 = (\mp) h (x_1, x_2) \in C^2(\omega), \quad (x_1, x_2) \in \omega.
\]

![Prismatic shell of a constant thickness. \( \partial \Omega \) is a Lipschitz boundary](image)
th order moments of the following quantities are defined as

\[
\left( u_{ir}, X_{ijr}, e_{ijr}, \Phi_{jr}, H_{ir}, H_{0r}, \varphi_r, F_r \right)(x_1, x_2, t)
\]

\[
\left( + \right) \int \left( u_i, X_{ij}, e_{ij}, \Phi_j, H_i, H_0, \varphi, F \right)(x_1, x_2, x_3, t) P_r(ax_3 - b) \, dx_3,
\]

\[
\left( - \right) \frac{h}{h}(x_1, x_2)
\]

\[i, j = 1, 2, 3, \quad (3.1)\]  

where

\[P_r(ax_3 - b) \left( a(x_1, x_2) := \frac{2}{(+) h - (-) h}, \quad b(x_1, x_2) := \frac{(+) h + (-) h}{(+)(-) h - (-) h} \right), \quad r = 0, 1, \ldots, \]

are the \(r\)th order Legendre polynomials.

\[2h(x_1, x_2) := \left( + \right) h \left( x_1, x_2 \right) - \left( - \right) h \left( x_1, x_2 \right), \quad (x_1, x_2) \in \omega,\]

is the thickness of the prismatic shell.

Under the well-know restrictions (see, e.g., [1]) the following Fourier-Legendre series

\[
\left( u_i, X_{ij}, e_{ij}, H_i, H_0, \varphi, F \right)(x_1, x_2, x_3, t)
\]

\[= \sum_{r=0}^{\infty} a \left( r + \frac{1}{2} \right) \left( u_{ir}, X_{ijr}, e_{ijr}, H_{ir}, H_{0r}, \varphi_r, F_r \right)(x_1, x_2, t) P_r(ax_3 - b), (3.2)\]

are convergent.

Therefore on the upper and lower face surfaces of the prismatic shell under consideration

\[
\left( + \right) u_i := u_i(x_1, x_2, h(x_1, x_2), t) = \sum_{s=0}^{\infty} a \left( s + \frac{1}{2} \right) u_{is}(\pm 1)^s = \sum_{s=0}^{\infty} \frac{(\pm 1)^s(2s + 1)}{2h} u_{is},
\]

\[i = 1, 2, 3, \]

whence

\[
\left( + \right) u_i - (-1)^r \left( - \right) u_i = - \sum_{s=0}^{\infty} a_{3s} u_{is}, \quad i = 1, 2, 3,
\]

\[
\left( + \right) u_i h, \alpha - (-1)^r \left( - \right) u_i h, \alpha = \sum_{s=0}^{\infty} a_{3s} u_{is}, \quad \alpha = 1, 2,
\]
where

\[
\begin{align*}
  r_a &= r, \quad r_s = (2r + 1) \frac{h_{a\alpha}}{h}, \\
  r_{\alpha s} &= r_{\alpha s}, \quad s \neq r, \\
  r_{\alpha r}^* &= (2r + 1) \frac{h_{\alpha} - (-1)^r s h_{a\alpha}}{2h}, \quad s \neq r.
\end{align*}
\]

Using

\[
\begin{align*}
  &h_{(+)}(x_1, x_2) \int_{h(x_1, x_2)}^{} P_r(ax_3 - b)f_{x_3} \, dx_3 = f_{r, \alpha} + \sum_{s=0}^{r} r_{\alpha s} f_s - f_{h_{,\alpha}} + (-1)^r f_{h_{,\alpha}}, \quad \alpha = 1, 2, \\
  &h_{(-)}(x_1, x_2) \int_{h(x_1, x_2)}^{} P_r(ax_3 - b)f_{x_3} \, dx_3 = \sum_{s=0}^{r} r_{\alpha s} f_s + f - (-1)^r f,
\end{align*}
\]

from (2.1)-(2.6), after multiplying them by \( P_r(ax_3 - b) \) and then integrating within the limits \( h(x_1, x_2) \) and \( h(x_1, x_2) \) with respect to the thickness variable \( x_3 \), we obtain the following formulas in \( \omega \):

\[
\begin{align*}
  X_{\alpha i r, \alpha} + \sum_{s=0}^{r} r_{\alpha s} X_{j is} + X_i &= \rho \frac{\partial^2 u_{i r}}{\partial t^2}, \quad i = 1, 3, \quad r = 0, 1, \ldots, \\
  H_{\alpha r, \alpha} + \sum_{s=0}^{r} r_{\alpha s} H_{is} + H_0 + \ddot{H} &= \rho_0 \ddot{\varphi}_r - \mathcal{F}_r, \quad r = 0, 1, \ldots, \\
  r \ddot{H} &= h_{(+)} - h_{(-)}(+),
\end{align*}
\]

\[
= H \sqrt{1 + \left( h_{1,1}^2 + h_{2,2}^2 \right) + (-1)^r H \sqrt{1 + \left( h_{1,1}^2 + h_{2,2}^2 \right)}}, \quad r = 0, 1, \ldots,
\]
\[X_{ijr} = \lambda e_{kkr} \delta_{ij} + 2\mu e_{ijr} + \lambda^* e_{kkr} \delta_{ij} + 2\mu^* e_{ijr} + b \varphi_r \delta_{ij} + b^* \varphi_r \delta_{ij},\]
\[i, j = \overline{1, 3}, \quad r = 0, 1, \ldots, \quad (3.5)\]

\[H_{ar} = \hat{\alpha} \left[ \varphi_{r,a} + \sum_{s=0}^{r} a_{as} \varphi_s - \left( (+) \varphi \right)_{h,a} + (-1)^r \varphi_{h,a} \right] + \alpha^* \left[ \varphi_{r,a} + \sum_{s=0}^{r} a_{as} \varphi_s - \left( (-) \varphi \right)_{h,a} + (-1)^r \varphi_{h,a} \right], \quad \alpha = 1, 2, \quad (3.6)\]

\[H_{3r} = \hat{\alpha} \left[ \sum_{s=0}^{r} a_{3s} \varphi_s + \varphi - (-1)^r \varphi \right] + \alpha^* \left[ \sum_{s=0}^{r} a_{3s} \varphi_s + \varphi - (-1)^r \varphi \right], \quad (3.7)\]

i.e.,
\[H_{jr} = \hat{\alpha} \left( \varphi_{r,j} + \sum_{s=r}^{\infty} b_{js} \varphi_s \right) + \alpha^* \left( \dot{\varphi}_{r,j} + \sum_{s=r}^{\infty} b_{js} \dot{\varphi}_s \right), \quad j = \overline{1, 3}, \quad r = 0, 1, \ldots, \quad (3.8)\]

evidently,
\[H_{jr} = \hat{\alpha} \left( h^{r+1} \psi_{r,j} + \sum_{s=r+1}^{\infty} h^{s+1} b_{js} \psi_s \right) + \alpha^* \left( h^{r+1} \dot{\psi}_{r,j} + \sum_{s=r+1}^{\infty} h^{s+1} b_{js} \dot{\psi}_s \right), \quad j = \overline{1, 3}, \quad r = 0, 1, \ldots, \quad (3.9)\]

\[\psi_r := \frac{\varphi_r}{h^{r+1}}.\]

\[H_{0r} = -b e_{kkr} - \xi \varphi_r - \nu^* e_{kkr} - \xi^* \dot{\varphi}_r, \quad r = 0, 1, \ldots.\]

\[H_{0r} = -b \left( h^{r+1} \psi_{r,\gamma} + \sum_{s=r+1}^{\infty} h^{s+1} b_{is} \psi_s \right) - \xi h^{r+1} \psi_r\]

\[-\nu^* \left( h^{r+1} \dot{\psi}_{r,\gamma} + \sum_{s=r+1}^{\infty} h^{s+1} b_{is} \dot{\psi}_s \right) - \xi^* h^{r+1} \dot{\psi}_r, \quad r = 0, 1, \ldots.\]
where

\[ \theta_r := e_{ir} = u_{\gamma r, \gamma} + \sum_{s=r}^{\infty} b_{ksu}u_{ks}, \quad r \geq 0, \]

\[ r b_{js} := \begin{cases} 0, & s < r, \\ -(2s + 1) \frac{h_{-a}}{h}, & j = \alpha, s > r, \\ -(r + 1) \frac{h_{-a}}{h}, & j = 3, s > r, \end{cases} \]

\[ \alpha = 1, 2, j = 1, 3, \quad r, s = 0, 1, 2, \cdots; \]

\[ r X_j := X_{3j} - X_{\alpha j} h_{-a} + (-1)^r \left[ -X_{3j} + X_{\alpha j} h_{-a} \right] + \Phi_{jr} \]

\[ = Q_{(+) j}^{(+)} \sqrt{1 + \left( \frac{h_{-a}}{h_{+1}} \right)^2 + \left( \frac{h_{-a}}{h_{+2}} \right)^2} + (-1)^r Q_{(-) j}^{(-)} \left[ \sqrt{1 + \left( \frac{h_{-a}}{h_{-1}} \right)^2 + \left( \frac{h_{-a}}{h_{-2}} \right)^2} + \Phi_{jr}, \right] \]

\[ j = 1, 3, r = 0, 1, 2, \cdots; \]

\[ Q_{(+) j}^{(+)} \quad \text{and} \quad Q_{(-) j}^{(-)} \] are components of the stress vectors acting on the upper and lower face surfaces with normals \( n_h \) and \( n_n \), respectively. So, we get the equivalent \(^1\) to (2.2)-(3.6), infinite system (3.3)-(3.5), (3.8)-(3.10) with respect to the so called \( r \)-th order moments \( X_{ijr}, e_{ijr}, u_{ir}, H_{jr}, H_{0r}, \phi_r \). Then, substituting (3.10) into (3.5) and the obtained into (3.3), and (3.8) into (3.4) we construct an equivalent infinite system with respect to the \( r \)-th order moments \( u_{ir}, \phi_r \). After this, if we suppose that the moments whose subscripts, indicating moments’ order, are greater than \( N \) equal zero and consider only the first \( N + 1 \) equations \( r = 0, N \) in the obtained infinite system of equations with respect to the \( r \)-th order moments \( u_{ir}, \phi_r \), we obtain the \( N \)-th order approximation (hierarchical model) governing system consisting

\(^1\)in the following sense: if \( X_{ij}, e_{ij}, u_i, H_i, H_0, \phi \) satisfy the relations (3.2)-(3.6), then constructed by (3.1) functions \( X_{ijr}, e_{ijr}, u_{ir}, H_{jr}, H_{0r}, \phi_r \) will satisfy the infinite relations (3.3)-(3.5), (3.8)-(3.10) and, vice versa, if \( X_{ijr}, e_{ijr}, u_{ir}, H_{jr}, H_{0r}, \phi_r \) satisfy the infinite relations (3.3)-(3.5), (3.8)-(3.10), then constructed by means of (3.2) functions \( X_{ij}, e_{ij}, u_i, H_i, H_0, \phi \) will satisfy the relations (3.2)-(3.5).
of $4N + 4$ equations with respect to $4N + 4$ unknown functions $\overset{N}{u}_{ir}$, $\overset{N}{\varphi}_r$ (roughly speaking $\overset{N}{u}_{ir}$, $\overset{N}{\varphi}_r$ is an “approximate value” of $u_{ir}$, $\varphi_r$, since $\overset{N}{u}_{ir}$, $\overset{N}{\varphi}_r$ are solutions of the derived finite system), $i = 1, 3$, $r = 0, N$. Because of

$$
\overset{r}{b}_{3r} = 0, \quad h^{r+1}(h^{-r-1})_{,\alpha} = \overset{r}{b}_{\alpha}, \quad \alpha = 1, 2,
$$

we can rewrite (3.3) for

$$
v_{ir} := h^{-r-1}u_{ir}
$$

as follows

$$
e_{ijr} = \frac{1}{2} h^{r+1}(v_{ir,j} + v_{jr,i}) + \frac{1}{2} \sum_{s=r+1}^{\infty} h^{s+1}(\overset{r}{b}_{is}v_{js} + \overset{r}{b}_{js}v_{is}),
$$

$$
i, j = 1, 3, \quad r = 0, 1, \cdots.
$$

(3.11)

$$
\theta_r := e_{ir} = h^{r+1}v_{\gamma r, \gamma} + \sum_{s=r+1}^{\infty} h^{s+1}b_{ks}v_{ks}, \quad r = 0, 1, \cdots.
$$

Multiplying equality (3.3) by $h^r$ and, taking into account that $\overset{r}{a}_{ir} = rh^{-1}h_{,a}$, we get

$$
(h^r X_{\alpha fr})_{,\alpha} + h^r \sum_{s=0}^{r-1} \overset{r}{a}_{is} X_{ij} + h^r X_j = \rho h^r \frac{\partial^2 h^{r+1}v_{jr}}{\partial t^2},
$$

$$
\quad j = 1, 3, \quad r = 0, 1, \cdots.
$$

(3.12)

Substituting (3.11) into (3.5) we have

$$
X_{ijr} = \lambda \delta_{ij} h^{r+1}v_{\gamma r, \gamma} + \mu h^{r+1}(v_{ir,j} + v_{jr,i}) + \sum_{s=r+1}^{\infty} \overset{r}{B}_{ijks} h^{s+1}v_{ks}
$$

$$
+ \lambda^* \delta_{ij} h^{r+1}\overset{r}{v}_{\gamma r, \gamma} + \mu^* h^{r+1}(\overset{r}{v}_{ir,j} + \overset{r}{v}_{jr,i}) + \sum_{s=r+1}^{\infty} \overset{r}{B}_{ijks} h^{s+1}\overset{r}{v}_{ks}
$$

$$
+ bh^{r+1}\psi_r \delta_{ij} + b^* h^{r+1}\overset{r}{\psi}_r \delta_{ij}, \quad i, j = 1, 3, \quad r = 0, 1, \cdots,
$$

where

$$
\overset{r}{B}_{ijks} := \lambda \delta_{ij} b_{ks} + \mu \delta_{kj} b_{is} + \mu \delta_{ik} b_{js}.
$$
\[ B_{ijks} := \lambda^r \delta_{ij} b_{ks} + \mu^r \delta_{kj} b_{is} + \mu^r \delta_{ik} b_{js}, \]

Multiplying (3.4) by \( h^r \) we obtain

\[ (h^r H_{\alpha r})_{, \alpha} + h^r \sum_{s=0}^{r-1} r_{is} H_{is} + h^r H_{0r} + h^r H = \rho_0 h^r \varphi_r - h^r \mathcal{F}_r. \] (3.13)

In the N-th Approximation the governing system has the following form

\[
\begin{align*}
\tilde{a}[\left(h^{2r+1} \psi_{r, \alpha}\right)_{, \alpha} + \sum_{s=r+1}^{N} \left(h^{r+s+1} \psi_{s, \alpha}\right)_{, \alpha}] \\
+ \alpha^r \left(h^{2r+1} \psi_{r, \alpha}\right)_{, \alpha} + \sum_{s=r+1}^{N} \left(h^{r+s+1} \psi_{s, \alpha}\right)_{, \alpha} \\
+ \tilde{a} \sum_{s=0}^{r-1} a_{is} \left(h^{r+s+1} \psi_{s, i}\right) + \sum_{l=s+1}^{N} \left(h^{r+l+1} b_{il} \psi_{l}\right) \\
+ \alpha^r \sum_{s=0}^{r-1} a_{is} \left(h^{r+s+1} \psi_{s, i}\right) + \sum_{l=s+1}^{N} \left(h^{r+l+1} b_{il} \psi_{l}\right) \\
- b \left(h^{2r+1} v_{\gamma r, \gamma} + \sum_{s=r+1}^{N} h^{r+s+1} b_{is} v_{is} \right) - \xi h^{2r+1} \psi_r \\
- \nu^r \left(h^{2r+1} v_{\gamma r, \gamma} + \sum_{s=r+1}^{N} h^{r+s+1} b_{is} v_{is} \right) \\
- \xi^r h^{2r+1} \psi_r + h^r H = \rho_0 h^r \varphi_r - h^r \mathcal{F}_r, \quad r = 0, N. \] (3.14)
\]

\[
\mu \left[ \left(h^{2r+1} v_{\alpha r, i}\right)_{, \alpha} + \left(h^{2r+1} v_{ir, \alpha}\right)_{, \alpha} \right] + \lambda \delta_{\alpha i} \left(h^{2r+1} v_{\gamma r, \gamma}\right)_{, \alpha} \\
+ \sum_{s=r+1}^{N} \left( B_{ai ks} h^{r+s+1} n_{v, ks} \right)_{, \alpha} + \sum_{l=0}^{r-1} \lambda \delta_{ji} \left[h^{r+i+1} v_{\gamma l, \gamma} + \mu h^{r+i+1} \left(n_{v, j l, i} + n_{v, d, j}\right) \right] \\
+ \sum_{s=l+1}^{N} B_{jiks} \left[h^{r+s+1} n_{v, ks}\right] \\
+ \mu^r \left[\left(h^{2r+1} v_{\alpha r, i}\right)_{, \alpha} + \left(h^{2r+1} v_{ir, \alpha}\right)_{, \alpha} \right] + \lambda^r \delta_{\alpha i} \left(h^{2r+1} v_{\gamma r, \gamma}\right)_{, \alpha}
\]
\[ + \sum_{s=r+1}^{N} \left( B_{\alpha ks}^r h^{r+s+1} N \frac{v_{ks}}{h} \right) + \sum_{i=0}^{r-1} a_{ijl} \lambda_i \delta_{jil} h^{r+s+1} N \frac{v_{il,j}}{h} + \mu h^{r+i+1} \left( N \frac{v_{il,i}}{h} + N \frac{v_{il,j}}{h} \right) \]

\[ + \sum_{s=l+1}^{N} B_{jiks}^r h^{r+s+1} N \frac{v_{ks}}{h} \]

\[ + b\left( h^{2r+1} N \frac{v_{ir}}{h} \right) + \sum_{s=0}^{r-1} a_{ils} h^{r+s+1} N \frac{\psi_s}{h} \]

\[ + b\left( h^{2r+1} N \frac{v_{ir}}{h} \right) + \sum_{s=0}^{r-1} a_{ils} h^{r+s+1} N \frac{\psi_s}{h} \]

\[ + h^r \hat{X}_i = \rho h^r \frac{\partial^2 h^{r+1} N \frac{v_{ir}}{h}}{\partial t^2}, \quad r = 0, N, \quad i = 1, 3, \quad \sum_{q=0}^{q-1} \cdots \equiv 0, \quad (3.15) \]

where

\[ \frac{N}{v_{kr}} := \frac{N}{u_{kr}} \frac{h_{r+1}}{h_{r+1}}, \quad k = 1, 3, \quad r = 0, N. \quad (3.16) \]

Now, we consider the following two mixed 3D BCs:

- on the face surfaces of the prismatic shell under consideration the stress vectors

\[ Q_{\alpha j}^+, j = 1, 2, 3, \quad Q_{\alpha j}^-, j = 1, 2, 3, \quad \text{and equilibrated stress vectors} \quad H_j, j = 1, 2, 3, \]

\[ H_j, j = 1, 2, 3, \quad \text{are prescribed}; \]

- on the lateral boundary

\[ \Gamma := \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \partial \omega, \quad h_j(x_1, x_2) < x_3 < h_j(x_1, x_2) \} \]

of the prismatic shell

either BCs:

\[ u_i \equiv \bar{u}_i, \quad \varphi = \bar{\varphi}, \]

or BCs:

\[ X_{ji} n_j = f_i, \quad H_j n_j = g, \]

where \( \bar{u}_i, f_i, \bar{\varphi}, g \) are given continuous functions, are prescribed.

To the last BCs on the lateral boundary \( \Gamma \) correspond the following BCs in the \( N \)th approximation:

\[ v_{ir} = \bar{v}_{ir} \frac{h_{r+1}}{h_{r+1}}, \quad i = 1, 2, 3, \quad \psi_r = \bar{\psi}_r \frac{h_{r+1}}{h_{r+1}}, \quad r = 0, N, \quad \text{on} \ \partial \omega, \]
and
\[ X_{ji}n_j = f_{ir}, \quad i = 1, 2, 3, \quad H_{jr}n_j = g_r, \quad r = 0, N, \quad \text{on} \ \partial \omega, \]
respectively, where \( \tilde{u}_{ir}, \tilde{\varphi}_r, f_{ir}, g_r, \quad i = 1, 2, 3, \) are \( r \)-th order moments of \( \tilde{u}, \tilde{\varphi}, f, g, \quad i = 1, 2, 3, \) correspondingly.

Note that, if we take \( \lambda^* = 0, \mu^* = 0, b^* = 0, \alpha^* = 0, \nu^* = 0, \xi^* = 0, \)
from the above obtained governing system, we get hierarchical models for porous elastic prismatic shells.

4. \( \text{N}=0 \) approximation for viscoelastic Kelvin-Voigt prismatic shells

The governing system has the following form (see (3.14), (3.15)):
\[
\mu \left[ (hv_{\alpha_0, \beta}, \alpha + (hv_{30, \alpha}, \alpha) \right] + \lambda (hv_{\gamma_0, \gamma}, \beta + b(h\psi_0), \beta + \mu^* (hv_{\alpha_0, \beta}, \alpha + (hv_{30, \alpha}, \alpha) \right]
+ \lambda^* (hv_{\gamma_0, \gamma}, \beta + b^* (h\psi_0), \beta + \frac{0}{X_3} = \rho h \ddot{v}_{30}, \quad \beta = 1, 2; \quad (4.1)
\]
\[
\frac{0}{X_3} = \rho h \ddot{v}_{30}; \quad (4.2)
\]
\[
\ddot{\alpha}(h\psi_0, \alpha) - bh h_{\gamma_0, \gamma} - \xi h\psi_0 + \alpha^* (h\dot{\psi}_0, \alpha) - \nu^* h_{\gamma_0, \gamma} - \xi^* h\dot{\psi}_0 + \frac{0}{H} = \rho \ddot{\psi}_0 - F_0. \quad (5.1)
\]

5. \( \text{N}=0 \) approximation for porous elastic prismatic shells

From (4.1)-(4.3), taking into account (3.17), we get the following governing system
\[
\mu \left[ (hv_{\alpha_0, \beta}, \alpha + (hv_{30, \alpha}, \alpha) \right] + \lambda (hv_{\gamma_0, \gamma}, \beta + b(h\psi_0), \beta + \frac{0}{X_3} = \rho h \ddot{v}_{30}, \quad \beta = 1, 2; \quad (5.1)
\]
\[
\frac{0}{X_3} = \rho h \ddot{v}_{30}; \quad (5.2)
\]
\[
\ddot{\alpha}(h\psi_0, \alpha) - bh h_{\gamma_0, \gamma} - \xi h\psi_0 + \frac{0}{H} = \rho \ddot{\psi}_0 - F_0. \quad (5.3)
\]

BCs for the weighted displacements and the weighted volume fraction are non-classical in the case of cusped prismatic shells (see Figures 2, 3). Namely, we are not always able to prescribe them at cusped edges.

Let \( \omega \) be a domain bounded by a sufficiently smooth arc (\( \partial \omega \setminus \gamma_0 \)) lying in the half-plane \( x_2 > 0 \) and a segment \( \gamma_0 \) of the \( x_1 \)-axis (\( x_2 = 0 \)).
If the thickness looks like

$$2h(x_1, x_2) = h_0x_2^\kappa, \quad h_0, \kappa = \text{const} > 0, \quad (5.4)$$

then the displacements and volume fraction we can prescribe at cusped edge $\gamma_0$ if $\kappa < 1$, while we cannot do it if $\kappa \geq 1$.

Let us show it for the particular case of deformation when

$$v_{\alpha 0} \equiv 0, \quad \alpha = 1, 2; \quad v_{30} \neq 0.$$ 

Then in the static case, taking into account (5.4), from (5.2), (5.3) we get

$$x_2\psi_{0,0} + \kappa\psi_{30,2} = 2(\mu h_0)^{-1}x_2^{1-\kappa}\chi_3, \quad (5.5)$$

$$x_2\psi_{0,0} + \kappa\psi_{2,0} - \xi\psi_{0,2} = -2(\tilde{\alpha}h_0)^{-1}x_2^{1-\kappa}\left(H + F_0\right), \quad (5.6)$$

respectively.

Problem D (Find $v_{30}, \Psi_0 \in C^2(\omega) \cap C(\tilde{\omega})$ by their values prescribed on $\partial \omega$) and Problem E (Find bounded $v_{30}, \Psi_0 \in C^2(\omega) \cap C(\omega \cup (\partial \omega \setminus \gamma_0))$ by their values prescribed only on the arc $\partial \omega \setminus \gamma_0$) are uniquely solvable for equations (5.5), (5.6) by $\kappa_2 < 1$ and $\kappa_2 \geq 1$, correspondingly. It follows from the theorem (see [8]).

**Theorem 5.1:** If the coefficients $a_\alpha, \alpha = 1, 2,$ and $c$ of the equation

$$x_2^{\kappa_\alpha}u_{\alpha \alpha} + a_\alpha(x_1, x_2)u_{\alpha} + c(x_1, x_2)u = 0, \quad c \leq 0, \quad \kappa_\alpha = \text{const} \geq 0, \quad \alpha = 1, 2,$$

are analytic in $\varpi$, then

(i) if either $\kappa_2 < 1$, or $\kappa_2 \geq 1$,

$$a_2(x_1, x_2) < x_2^{\kappa_2 - 1} \quad (5.7)$$

in $\varpi_\delta$ for some $\delta = \text{const} > 0$, where

$$\omega_\delta := \{(x_1, x_2) \in \omega : 0 < x_2 < \delta\},$$

the Dirichlet problem (Problem D) is well-posed;
(ii) if $\kappa_2 \geq 1$,

$$a_2(x_1, x_2) \geq x_2^{\kappa_2 - 1}$$  \hspace{1cm} (5.8)

in $\omega_3$ and $a_1(x_1, x_2) = O(x_2^{\kappa_1})$, $x_2 \to 0+$ ($O$ is the Landau symbol), the Keldysh problem (Problem E) is well-posed.

Indeed, from (5.7) and (5.8), it follows $a_2(x_1, x_2) = \kappa < 1$ for Problem D and $a_2(x_1, x_2) = \kappa \geq 1$ for Problem E, respectively, since $\kappa_1 = \kappa_2 = 1$.

To the general system (5.1)-(5.3) in the static case we apply results obtained for the more general system (see [9]).

**Remark 1:** In the similar way we construct hierarchical models for piezoelectric thermoviscoelastic Kelvin-Voigt prismatic shells with voids as well. It turned out that considering cusped prismatic shells by setting BCs at cusped edges the electric and magnetic potentials expose the same peculiarities which characterize the displacements and volume fraction by setting BCs for the displacements, volume fraction, electric and magnetic potentials on the lateral boundary of the prismatic shells. These results will be published in the forthcoming paper.

**Acknowledgements**

This work is partially supported by Shota Rustaveli National Science Foundation (SRNSF) [217596, Construction and investigation of hierarchical models for thermoelastic piezoelectric structures].

**References**


