Reductive – Holistic Cycle: A Model for the Study of the Didactic Procedure

Dimitris Gavalas, Athens (Greece)

Abstract: The concept of *reality level* may be useful as a catalyst among several systems in the area of knowledge. This concept is leading us to ask, if we can make a reduction from a reality level to another, that is to the problem of *reductionism.* Relative to it is the problem of *holism.* At the end these concepts are connected to the category theory and adjoint functors. Within the framework of this aspect we set up a model for the study of the didactic procedure. This model is a feedback system between two reality levels or categories, these of the teacher and of the student. So, the article seeks to enhance and improve the teaching of mathematics by its attempt to understand both student's and teacher's knowledge in the same terms.

Kurzreferat: *Reduktiv-holistischer Zyklus: ein Modell für die Untersuchung didaktischer Prozesse.* Der Begriff des Realitäts-Niveaus kann als Katalysator zwischen verschiedenen Systemen im Bereich des Wissens nützlich sein. Dieser Begriff führt uns zu der Frage, ob wir eine Reduktion von einem Realitäts-Niveau zu einem anderen vornehmen können – ein Reduktionismus-Problem. Relevant ist hier das Problem des Holismus. Am Ende werden diese Begriffe mit der Kategorie-Theorie und adjungierten Funktoren in Verbindung gebracht. Im Rahmen dieses Aspektes erarbeiteten wir ein Modell für die Untersuchung von Lehrprozessen. Dieses Modell ist ein Feedback-System zwischen zwei Realitäts-Niveaus oder Kategorien, dem des Lehrers und dem des Schülers. Der Artikel versucht, das Mathematiklehren zu erklären und zu verbessern indem versucht wird, das Wissen beider, Schüler und Lehrer, mit den gleichen Begriffen zu erfassen.

ZDM-Classification: C70, D20, D40, H70, M90

1. Introduction

A careful examination of some research papers leads to the following: In Simon (Simon 1995) we find that the analysis of the data of his research led to a *model* of teacher decision making with respect to mathematical tasks. This model is called "Mathematics Teaching Cycle" and describes the relationship among various domains of teacher knowledge, the hypothetical learning trajectory and the interactions with students. Central to this model is the creative tension between the teacher's goals with regard to student learning and his responsibility to the sensitive and responsive to the mathematical thinking of the students. The model is a feedback system. According to the same author (Simon 1994), there is the need for the re-conceptualization of mathematics education both for teachers and students. In this paper Simon (1994) uses the model of Karplus (Karplus et al. 1977) "Learning Cycle", which consists of six learning cycles. We move from the first of them, that is the cycle of "Learning Mathematics", to the last one, that is that of "Teaching", through "Developing Knowledge about Mathematics Learning", "Developing Theories of Mathematics Learning", "Understanding Students' Learning" and "Instructional Planning". Every cycle and the whole system is a feedback system. The model is proper only for teachers. Simon in these two papers deals with mathematics learning from the constructivist view. Finally, in Pirie & Kieran (Pirie & Kieran 1994) we find a theory of the growth of mathematical understanding which is based on the consideration of understanding as a whole, dynamic, leveled but non-linear process of growth. This theory demonstrates understanding to be a constant, consistent organization of ones knowledge structures: a dynamic process, not an acquisition of categories of knowing. Its leveled nature has been illustrated through another model of eight embedded cycles, each of which represents a level of understanding activity potentially attainable for any particular topic by any specific person. These levels/cycles move outwards from "Primitive Knowing" to "Inventising" through "Image Making", "Image Having", "Property Noticing", "Formalizing", "Observing" and "Structuring". Next we present a new model for the study of the whole didactic procedure, simpler than the three previous models, which is based on the concept of "reality level"/"category" and that of "reductionism" and on the mathematical category theory.

The concept of reality level/category as well as the relevant concepts in this context reductionism, holism and category theory are fundamental and they contribute to the attempt of creating a multi-leveled hierarchical system of human knowledge. An observer is, however, demanded in order to realize this hierarchical system. Here we need to stress an important point: The observer ought to complete his knowledge about the parts of the unified world, with as much information as necessary for the study of the collective behavior of the parts; because these parts exist within this unified world that acts as a system. By performing this upward procedure, the observer reevaluates the content of knowledge and information that he had lost during the descending process of the gradual analysis of this unified system in parts. In this neutral explanation we can find the reconciliation between reductionism and holism. The fan of reductionism begins from the top of the hierarchy and goes down, winning in this descending process preciseness of information with respect to the parts, but loosing in quality of information with respect to the higher levels which he left behind. The fan of holism proceeds conversely, from bottom to top, trying to gain the lost information through rebuilding the parts. This point has already been discussed in Koestler & Smythies (Koestler & Smythies 1969) and in Koestler (Koestler 1978).

2. Categories, presentation levels or reality levels

In order to talk about the world, we need a language, a way to present it. This language includes laws or axioms and undefined or defined concepts, which we can divide in several categories. The latter are exactly called presentation levels or reality levels. *Categories* are fundamental divisions of some subject-matter; categorization is the classification of information, objects, properties or relations into categories. Categories have corresponding concepts or may even be said to be concepts, depending on the extent to which one takes categories to be mind-dependent (i.e. having no existence apart from mind). Apart from the ontology of categories, category membership was traditionally thought to be determined by a set of necessary and sufficient conditions, however it is now evident that category membership is in some cases a matter of degree. Members of some categories display overlapping features. Rather than a set of necessary and sufficient conditions, a prototype, or set of properties judged to be most commonly exhibited by members of the category, provides the standard for category membership. Members are determined by their degree of resemblance to the prototype. Categories constitute the conceptual dynamic schemata which organize knowledge.

According to science, there are four basic levels of presenting the world:

1) *The natural,* containing concepts such as space, time, energy and so on.
2) *The biological* which includes concepts about life, evolution, ecosystem and so on.
3) *The psychological,* incorporating concepts such as self-consciousness, consciousness, intention, feeling and so on.
4) *The social,* with concepts such as order, culture, society and so on.

Before we go further to our discussion, we must talk about the terms "definition" and "interpretation". *Definition* is a label covering different names and presentations of the same concept. *Interpretation,* on the other hand, provides us many or even all the possible data in a reference domain L_1 so that the receiver can conceive or understand well a concept, which belongs to a reference domain L_2, where generally $L_1 \cap L_2 = \emptyset$. An interpretation, which is a mere semantic matter, cannot give us immediate realistic results. At best, the interpretation offers us a better insight of the way things are.

We must next make some necessary remarks about communication and language. Man's consciousness communicates with the external and the internal world through processes in order to correspond and represent stimuli and symbols. Consequently, man is literally being bombarded with external and internal stimuli and symbols. This communication is problematic for two main reasons:

a) It is done under conditions of obscurity, complicity and competition.
b) The stimuli and the symbols which man receives every moment are too many.

As a result, the person has to decide on which stimuli and symbols of the environment he will assign his cognitive system and how he will respond to them. Decision making is the procedure of assigning a cognitive attractor to a particular stimulus. The tool for this decision is the symbolic language, which is the partition and categorification of the external and internal world. The more precise and detailed this language is, the more detailed and precise is the division of the world into parts. The symbolic language allows the simulation of the world and the creation of alternative hypotheses, since the brain, which is the biological basis of this procedure, has the ability of a potential prediction and the ability to run an algorithm/program faster than reality. So, there is the possibility for compression, generalization and abstraction of reality to an algorithm/program of smaller length than reality which holds its general characteristics. This situation precisely leads to the creation of a cognitive attractor.

This symbolic language is arranged in three levels:

a) *The syntactic level,* in which the language symbols and their interrelating dynamics are created. Out of the latter, rules are formulated regarding the sequence of personal and collective symbols, that are independent of meaning and correlation or correspondence with declared objects.
b) *The semantic level,* in which the relations, the correspondences and the representations of language symbols to structures and functions of the world are examined. In this level the meaning and the notion for the subject emerges and becomes reinstated.
c) *The pragmatic level,* where the interpersonal relations are under observation, that is the correlation of symbols to the psychical and cognitive structures and patterns of persons that communicate. (Nicolis 1987).

Let us return to the reality levels. We observe that an upper level has as background lower ones, but also includes other concepts strange/disjoint to them. The criterion, which diversifies the one level from the other, is that, while we climb up from level to level there is a new fundamental concept, which is the category. The rest of the concepts of the new level are defined in this category, which additionally has the following property: It can be interpreted but cannot be defined with concepts of lower levels. Anyway, we can distinguish two sorts of demarcation: The "horizontal" and the "perpendicular". The horizontal is the one which is done within the framework of one presentation level, and the perpendicular is the one getting from level to level.

3. The problem of reductionism

We can describe the problem of reductionism by using the concepts of reality levels and of their perpendicular demarcation. This problem is tantamount to conclude whether it is possible to reduce a presentation level to another lower one. This reduction should be understood as an expression of concepts and laws of a level in respect to the concepts and laws of a lower level. Instead of talking about presentation levels we can equivalently talk about categories that are ranked in a hierarchical way, having thus a presentation of the reductionism of a category to another.

Talking about concepts, relevant terms like definition and interpretation come up. Already, from the definition of the presentation levels, it is obvious that a concept can be only interpreted by concepts of lower levels. As for the laws, though, we use the term "explanation" corresponding to the term interpretation. We can define explanation as follows:

The *explanation* of a law l, which belongs to the reference domain L_2, in terms of a set of laws L, which belong to the reference domain L_1, is the interpretation of all the "relevant concepts" that belong to L_2, in terms of the set of all the relevant concepts that belong to L_1. This interpretation is such that l can be understood as logic extension of laws L from L_1 to L_2. "Relevant concepts" are the concepts, in terms of which a law is expressed.

In addition, *category theory* is a general mathematical theory of structures and systems of structures. It allows us to see, among other things, how structures of different kinds are related to one another as well as the universal components of a family of structures of a given kind. The theory is philosophically relevant in more than one way. For one thing, it is considered by many as being an alternative to set theory as a foundation for mathematics. Furthermore, it can be thought of as constituting a theory of concepts. Finally, it sheds a new light on many traditional philosophical questions, for instance on the nature of reference and truth. The present author, in this paper, uses category theory as an answer to the problem of reductionism.

Finally we have to notice that a functor is a morphism between categories which preserves the structure. For the word functor, according to Lawvere (Lawvere 1987), we have: Functor = funct + or = representor of functions. The point of the subject is that a functor is a kind of quantity that takes every quantity of one type into a quantity of the other type.

We can see in "Fig. 1" how the various terms that we use are expressed and interconnected. We consider that the reference domains correspond to presentation levels:

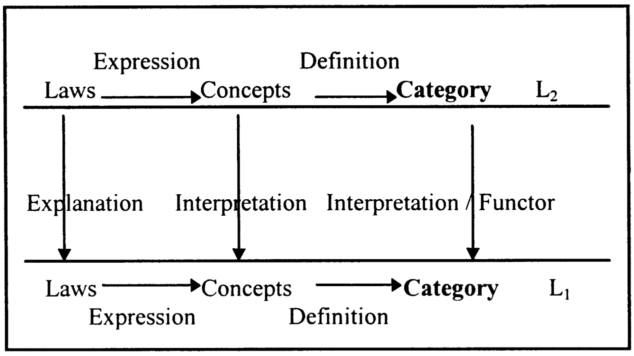

Figure 1: Reduction of the conceptual category L_2 to L_1

We must make a distinction between logical and empirical reductionism:

a) *Logical reductionism* means that a concept or law can be reduced only if it necessarily derives from concepts or laws of a lower level, where "it necessarily derives from" means that it can be invented or discovered from the knowledge of the concepts and laws of the lower level.

b) *Empirical reductionism* means that a concept or law can be reduced when it can be interpreted or explained by means of concepts or laws of a lower level, considering both presentation levels known.

Logical reductionism is impossible and this is directly concluded by comparing the definition of logical reductionism with that of category: Since a category cannot be defined in terms of concepts of a lower level, but can only be interpreted through them, then it cannot necessarily be derived by them. Besides, a law cannot be expressed without the use of the relevant concepts. On the contrary, empirical reductionism of concepts is possible. This is Carnap's view (Carnap 1934, 1970). The unity of science is secured by the fact that all the propositions in science can finally be expressed in a physical language, i.e. in the form of sentences that assign quantity values in certain places of a space-time system of reference. From this point of view, all the apparently non physical concepts are expressed on the basis of some sensory criteria. This possibility is based on the fact that our understanding of the real world is the understanding of the natural phenomena, which we eventually integrate in order to formulate the non-natural concepts. So, reductionism is considered to be the opposite process. The natural language is therefore the universal language of science. Finally, to the question of whether the empirical reductionism of laws is possible, the answer is *possibly yes*, because we cannot know for sure whether this reductionism is possible, but we can try to perform it.

The study of the relation between two categories d_1, d_2 consists in three partial studies :

1) The study of the relation of categories d_1, d_2 as special branches of knowledge: Which this relation is, which the preconditions for the definition of this relation are, whether the effects of this research can be valid or not.
2) The study of the relation between particular laws of d_1 and the corresponding laws of d_2, in other words the explanation of the laws of d_2 by laws of d_1.
3) The study of the relation between concepts of d_1 and concepts of d_2, that is the interpretation of concepts of d_2 by concepts of d_1.

This study is exactly placed into the framework of reductionism, that means whether we can reduce or not the concepts and the laws of category d_2 to the concepts and the laws of category d_1.

Now we can discuss in details why the answer to the question about the empirical reductionism of laws is "possibly yes". A law in level d_1 can be adequate enough to explain the phenomena of d_1, but when we climb to the upper level d_2 the same law may be inadequate or it may lead to contradictions. The solution, in this case, is not to abandon the application of the law of d_1 to d_2, but to reexamine the law of d_1 considering the new experience, that we have gained after having tried to apply it to d_2, and to judge the law's conceptual basis. So, it is possible to enrich it by widening its application field – something that it is not a priori certain – and that's why we originally answer "possibly yes".

In order to interpret the concepts of d_2 by concepts of d_1 we must begin from an epistemology of level d_1 inspired by the epistemology of level d_2. Then, after the clarification of the epistemology that we finally use, we move on to the attempt to explain the laws of d_2 by the laws of d_1. Therefore, we have a perpendicular interaction of the levels d_2 and d_1.

We give an example from science; a good example here is biophysics, that is the attempt to explain the laws of biology (d_2) by laws of physics (d_1). The particular laws can refer to phenomena which belong to irrelevant areas, while the concepts have a close structural relation. Every concept is defined and related, on the horizontal reality level, with other concepts and the basic concept for the category of discipline d_2 is a concept, let say, c – in our example, for the category of biology is the concept of organization. Having as fundamental the concept c, we can define all the other concepts and in this way we construct a conceptual category or a conceptual network. In our example, the basic concept of organization is the one which separates the biological level from its lower one, that is the natural level. The biological level has an upper one, the psychological level. The basic concept of the category of the psychological level is that of self-consciousness.

4. Category theory as an answer to the problem of reductionism

Category theory plays a key role in interpreting those reductive relations. This point is fully discussed in Lawvere (1987, 1994), Lawvere & Schanuel (1993), Mac Lane (1971, 1986, 1996), Mac Lane & Moerdijk (1992), Magnan & Reyes (1994) and in Viswanathan (1989). Sneed (Sneed 1984) represents, as well, the empirical categories as a network of connected "theoretical elements". A *theoretical element* T consists of:

– Some concepts, mental and empirical, let us call them C, which are related within a certain category and which we use in order to say something about a gnostic domain.
– The intended applications of those concepts, let us call them I.

So, a theoretical element is a kind of an ordered pair: $T = \langle C, I(C) \rangle$.

The invariant properties of the categories are significant. Category theory is the most suitable language to represent this kind of properties. Category theory uses functors among categories to describe invariant properties and relations. This view stresses the use of category theory for interpreting categories because it allows to express invariant properties and relations for these categories across transformations and representations in alternative languages. That means that truth does not refer to just one model and one language, but it can be defined in different languages and different structures. Consequently, the concept of invariance can be expressed within the category theory semantics for the mathematical, empirical or conceptual categories (Stefanic, 1996).

According to Nicolis (1987), to understand means to reduce. The reduction of numerous empirical observations to a few basic axioms/laws does not solve every problem completely, because these axioms/laws are not going to be proved compatible between them or that they form a complete group (Goedel). In addition, each one of them is expressed by initial undetermined concepts, which we cannot further reduce. Therefore, we have to accept the fact that the empirical categories do not interpret completely anything; they just go up and down between successive hierarchical levels of reality, since the presentation into only one level is incomplete or even contradictory sometimes.

Finally, we must note that humans think with concepts, which are classified into sets of concepts. These concepts are related to each other and so we can form conceptual categories. These categories are natural categories of the mind. In mathematics also, we form conceptual mathematics (Drossos 1987), (Lawvere 1987, 1994), (Lawvere & Schanuel 1993) and categories of mathematical objects, which are related by morphisms, and by doing that, we are conducted to the mathematical category theory. According to these, we can consider that there is a natural, objective relation/correspondence between, the *human thought and its conceptual categories* on the one hand and *the mathematical category theory* on the other hand.

5. The model for the didactic procedure

Let us now study, in sum, the didactic procedure:

– First, the work of the teacher and that of the student have different directions. On the one hand, the teacher has to present knowledge in, appropriate and familiar to student, frameworks and, in a way, he has to personalize/individualize it. On the other hand, the student has to follow the opposite direction. Starting from the specific frameworks and continuing with successive abstractions and generalizations he will conquer the mathematical structure of the subject. This empirical finding constitutes the substantiation of the categorical concept of *adjoint functors.* According to these, we have a functor from the conceptual category of the teacher towards either to the conceptual category of each student individually, or to the average category of the students of a class, and vise versa.
– Second, to understand means to reduce. The problem of didactics is to reduce the conceptual category $\mathbf{C}_{\mathrm{T}}$ of the teacher, to a lower one $\mathbf{C}_{\mathrm{S}}$, that of the student. It is about, of course, an empirical reductionism. If, according to the above, the answer of this problem is a functor between these two categories, then we have to find an appropriate functor F from teacher's conceptual category $\mathbf{C}_{\mathrm{T}}$ to student's one $\mathbf{C}_{\mathrm{S}}$. This problem is related to the traditional didactic method. In modern didactics there is an interaction between the teacher and the student. That means, that there is a functor G from student's conceptual category $\mathbf{C}_{\mathrm{S}}$ to teacher's one $\mathbf{C}_{\mathrm{T}}$, too. This empirical finding leads to the categorical concept of adjoint functors between the two categories $\mathbf{C}_{\mathrm{T}}$ and $\mathbf{C}_{\mathrm{S}}$:

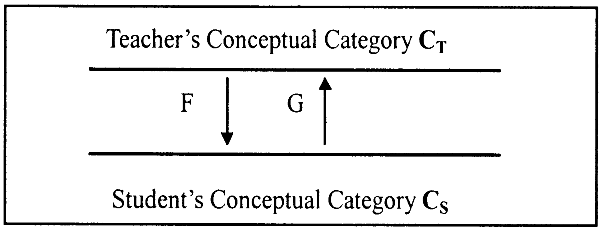

Figure 2 : Adjoint functors (F,G) between the two categories $\mathbf{C}_T$ and $\mathbf{C}_S$

These empirical adjoint functors (F,G) consist the interaction between the two basic elements of the teaching system, which are the teacher and the student.

In order to proceed in our analysis we have to present a few necessary elements about the concept of adjoint functors. In Mac Lane & Moerdijk (1992) we find that, if we consider two categories $\mathbf{A}$ and $\mathbf{X}$ and two functors between them in opposite directions, say

$$F : \mathbf{X} \longrightarrow \mathbf{A}, \qquad G : \mathbf{A} \longrightarrow \mathbf{X},$$

then one says that G is *right adjoint* to F and that F is *left adjoint* to G, when for any two objects X from $\mathbf{X}$ and A from **A** there is a natural bijection between morphisms

$$\frac{X \xrightarrow{f} GA}{FX \xrightarrow{h} A} \qquad (1)$$

in the sense that each morphism f, as displayed, uniquely determines a morphism h, and conversely. This bijection is to be natural in the following sense: Given any morphisms

$$\text{á} : A \longrightarrow A' \text{ in } \mathbf{A} \quad \text{and} \quad \text{î} : X' \longrightarrow X \text{ in } \mathbf{X}$$

and corresponding arrows f and h as in (1) the composites also correspond under the bijection (1):

$$\frac{X' \xrightarrow{\text{î}} X \xrightarrow{f} GA \xrightarrow{G\text{á}} GA'}{FX' \xrightarrow{F\text{î}} FX \xrightarrow{h} A \xrightarrow{\text{á}} A'} \qquad (2).$$

According to all of the above, we can consider two conceptual categories, the one of the teacher $\mathbf{C}_{\mathrm{T}}$ and that of the student $\mathbf{C}_{\mathrm{S}}$, the objects of which are concepts and the arrows are relations/processes between concepts:

Table 1: The didactic prodecure

	$\mathbf{C}_{\mathrm{T}}$	$\mathbf{C}_{\mathrm{T}}$
Objects	X′: The concept, which the teacher wants to introduce, in an initial and informal form	FX′: The new concept, which the student intuitively understands in an initial and informal look.
	X: The concept in its mathematical/scientific type	FX: The new concept in mathematical/scientific type, as the student understands it
	GA: The concept A as it turns back from the student to the teacher	A: The through-out concept in the student's mind, as an isolated mental object
	GA′: The concept A as it turns back from the student to the teacher	A′: The final, assimilated concept after its incorporation and interrelation within a preexistent conceptual network/category
Arrows	î: The transition from the initial and informal form of the concept, which the teacher wants to introduce, to its mathematical/scientific type	Fî: The transition from the intuitive understanding of the new concept to its mathematical/scientific understanding by the student
	f: Control and judgment of the GA, so as the teacher to help the student to reconsider his understanding and to construct his knowledge	h: Reflection in order to become conscious and to understand in depth the concept FX
	Gá: Final control of the concept A′, as it turns back to the teacher, in order to check whether it is reduced correctly to the conceptual network/category of the student	á: Assimilation, reduction, interaction of the new concept within a preexistent conceptual network/ category

We have to note that, if the final control proved positive, then the didactic procedure comes to an end for the concept which the teacher wants to teach to the student. In the opposite case, the didactic procedure is repeated after a point. Also, the functor F represents reduction while the functor G represents holism. This happens because, by the functor F the teacher wants to reduce his conceptual category $\mathbf{C}_{\mathrm{T}}$ to a lower one, that of the student $\mathbf{C}_{\mathrm{S}}$, while the student's attempt is characterized by the reverse direction, that means to reach finally the teacher's conceptual category. Together F and G, as adjoint functors, substantiate the didactic procedure.

The following diagram shows the situation we analyzed in the previous:

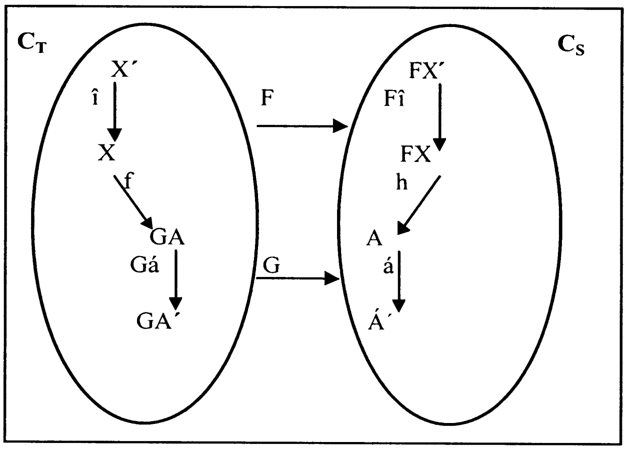

Figure 3: The reductive-holistic cycle

This model is a feedback system which we call RHC (Reductive-Holistic Cycle). The RHC works in two processes which interact through the adjoint functors F and G, that is reduction and holism respectively. The RHC is a first attempt in order to search if and how a modern mathematical theory, such as category theory, can play an important role in the field of mathematics education.

The RHC enables teachers to realize what comprises the didactic procedure and to systematize it. This helps him/her practically to make a lesson plan according to this model, to follow it step by step and to control the procedure, because this is a feedback model that helps the interaction between student and teacher. So, this model can be used practically in classroom. In addition, RHC can be an alternative theoretical proposal and can impact educational theory for further research about the use of category theory in mathematics education.

6. Conclusion

The concept of reality levels and that of the correspondent hierarchical system, which is created by them, is very important for the presentation of the world and the knowledge. In this attempt the question is whether we can reduce a reality level into a lower one, that is to express the concepts and the laws of a reality level in relation to the concepts and laws of a lower reality level in the hierarchical system. That is precisely the problem of reductionism, which is based on the analytical method. In contrast, holism depends on the systemic method. The essential concept here is category, which connects in a unified whole the concepts and their relations in every reality level. Thus, the problem of reductionism becomes a matter of connecting those conceptual categories. Here the solution is given by category theory, which links categories via functors. Therefore, category theory represents a valuable tool in the case of epistemological problems too, as i.e. with reality levels and reductionism and the relations between scientific and especially empirical theories, since this link succeeds by the concept of functor among categories. Finally, category theory can be used in research concerning the conceptual dimension of mathematics teaching. To study the whole didactic procedure in mathematics teaching we set up a model which is based on the previous concepts: Reality level/category, reductionism, holism, category theory and adjoint functors. This model is called RHC and represents the processes in the teacher's and in the student's conceptual category which interact via adjoint functors, namely reduction and holism respectively.

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Author

Gavalas, Dimitris, Dr., Pedagogical Institute, 396 Messogeion Avenue, 153 41 Agia Paraskevi, Athens, Greece.
E-mail: dimgav@aurora.eexi.gr