On the dynamical stability of the solar system

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Introduction

• chaotic behavior of solar system (Laskar 1989)
• 4 terrestrial planets: 5 mio. yrs Lyapunov-time (Laskar 1989)
• mass ratios of the planets to the sun are much larger than those required from KAM

◊ Is the solar system dynamically stable?
Introduction

• long-term numerical integration of the full solar system over 20 Gyrs
• dynamically allowed evolutions in which the planetary orbits become unstable
• effects of general relativity on the dynamical stability
• dynamical lifetime of Uranus
chaotic motion - Lyapunov time

\[ \gamma = \lim_{N \to \infty} \Sigma_{k=1}^{N} \ln \left( \frac{s_k}{s_0} \right)/(N\Delta t) \]

\( s_0 \) ... separation vector (150 m, radially outward)
\( \Delta t \) ... 10,000 yr
\( N \) ... 100
1/\( \gamma \) ... Lyapunov time

\( s/s_0 = e^{(\gamma \Delta t)} \)
Table 1: Lyapunov exponents and times for the Solar System

<table>
<thead>
<tr>
<th>Planet</th>
<th>Lyapunov exp. (yrs $10^{-1}$)</th>
<th>Lyapunov time (yrs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>$7.32029 \times 10^{-7}$</td>
<td>$1.36607 \times 10^6$</td>
</tr>
<tr>
<td>Venus</td>
<td>$1.38561 \times 10^{-7}$</td>
<td>$7.21703 \times 10^6$</td>
</tr>
<tr>
<td>Earth</td>
<td>$2.07484 \times 10^{-7}$</td>
<td>$4.81964 \times 10^6$</td>
</tr>
<tr>
<td>Mars</td>
<td>$2.22353 \times 10^{-7}$</td>
<td>$4.49736 \times 10^6$</td>
</tr>
<tr>
<td>Jupiter</td>
<td>$1.19528 \times 10^{-7}$</td>
<td>$8.36623 \times 10^6$</td>
</tr>
<tr>
<td>Saturn</td>
<td>$1.56875 \times 10^{-7}$</td>
<td>$6.37452 \times 10^6$</td>
</tr>
<tr>
<td>Uranus</td>
<td>$1.33793 \times 10^{-7}$</td>
<td>$7.47423 \times 10^6$</td>
</tr>
<tr>
<td>Neptune</td>
<td>$1.49602 \times 10^{-7}$</td>
<td>$6.68440 \times 10^6$</td>
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chaotic motion - Lyapunov time

• Lyapunov time - some $10^6$ yrs

◊ Why long-term integrations?

They give probabilistic evaluations of the solar system’s future behavior.
direct long-term integration

• start: current configuration
• timespan: over 20 Gyrs
• timestep: 8 days
• conservation of total energy:
  \( \Delta E/E < 10^{-7} \)
• conservation of total angular momentum:
  \( \Delta L/L < 10^{-9} \)
Eccentricity of earth as function of time
Eccentricity of mercury as function of time
direct long-term integration

◊ solar system seems to be stable over its lifetime?

but: its only one possible trajectory! (Laskar 1994)

◊ better: any timescale for occurring instabilities might be long
The Laskar - method

A shift of 150m of earth’s position 500 Myrs in the future correspond to an initial error of $10^{-42}$m
The Laskar - method

numerical induced chaos - limiting factor:
accurately resolving Mercury’s orbit (Yoshida, 1993)

◊ conservation of energy and ang. momentum
◊ criteria for this work:
$\Delta E/E \sim 10^{-8}$ and $\Delta L/L \sim 10^{-10}$

◊ variation of timestep form 3 to 1.2 days to correct the violation of these criterias
The Laskar - experiment

- 2 experiments: 150m and 15m

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<tr>
<td>1</td>
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<td>0.2907</td>
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<tr>
<td>2</td>
<td>500-797</td>
<td>0.4391</td>
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<tr>
<td>3</td>
<td>797-862</td>
<td>0.8257</td>
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<td>0.4139</td>
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<tr>
<td>3</td>
<td>994-1207</td>
<td>0.4874</td>
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<tr>
<td>4</td>
<td>1207-1261</td>
<td>0.9751</td>
</tr>
</tbody>
</table>

◊ Collision with Venus ◊ Collision with Sun
Architecture of solar system

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Fig. 10. The minimal distance of approach during a series of close encounters between Mercury and Venus as a function of time. The collision takes place at $t \sim 861.455\text{Myr}$, when $d_{\text{min}} = 5.5561 \times 10^{-5} \text{AU} < r_{\text{venus}} + r_{\text{mercury}} = 5.6762 \times 10^{-5} \text{AU}$.
Evolution of Mars’ eccentricity in the 150 m experiment at $t = 822$ Myrs - also Mars’ semimajor axis increased and after it reached a distance $> 100$ AU, Mars was assumed to be ejected from the solar system.
The Laskar - experiment

- Mercury shows different ways of evolution
- Mercury tends to switch from regular to irregular motion
  - Exists any association with timesteps or integral algorithm?
  - Reintegration of a 22 Myr time interval where Mercury became unstable
The Laskar - experiment

150 m symplectic algorithm 0.5 days time step  
- 22 Myrs Bulirsch-Stoer algorithm 0.5 days time step

15 m symplectic algorithm 0.5 days time step  
- 20 Myrs Bulirsch-Stoer algorithm 0.5 days time step

almost identical increase of eccentricity as in the primary solution
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The Laskar - experiment

Starting at 778 Myrs for the 150 m:
- adding perturbations of 15 m and 150 m in four directions and integrating over 22 Myrs
  ◊ no changes

Starting at 1190 Myrs for the 15 m:
- adding perturbations of 15 m and 150 m in four directions and integrating over 20 Myrs
  ◊ no changes
The Laskar - experiment

• until 780 Myrs Mercury’s eccentricity varies within a narrow and well-defined range

• shortly thereafter the eccentricity increases and leads to collisions,.....

◊ secular resonances
secular resonances

stable motion versus unstable motion

\[<R_{i}^{\text{sec}}>=n_{i}a_{i}^{2}\{0.5A_{ii}e_{i}^{2}+0.5B_{ii}I_{i}^{2}+\sum[A_{ij}e_{i}e_{j}\cos(\varpi_{i}-\varpi_{j})+B_{ij}I_{i}I_{j}\cos(\Omega_{i}-\Omega_{j})]\}\]

\[R_{i}=\sum[(Gm_{j})/(|r_{j}-r_{i}|) - Gm_{j}(r_{j} \times r_{i})/(r_{j}^{3})]\]
secular resonances

The dominant frequencies of the secular disturbing function can be identified by Fourier-analyzing the numerically computed time-series for Mercury’s full disturbing function.

\[ g_i = \langle \wp_i \rangle \]

stable motion \hspace{1cm} unstable motion

\((g_1 - g_5) \approx 0.9389'' \text{ yr}^{-1} \hspace{1cm} (g_1 - g_5) \approx 0.0538'' \text{ yr}^{-1} \)

(Laskar 1990)
secular resonances - stable motion
secular resonances - unstable motion
secular resonances

Classical Laplace-Lagrange secular solution for Mercury’s eccentricity vector:

\[ h_1 = e_1 \sin \varpi_1 \]
\[ k_1 = e_1 \cos \varpi_1 \]
secular resonances

\[ h_1 = e_{1f} \sin (g_1 t + \beta_1) - \sum [(v_j / (g_1 - g_j)) \sin (g_j t + \beta_j)] \]

\[ k_1 = e_{1f} \cos (g_1 t + \beta_1) - \sum [(v_j / (g_1 - g_j)) \cos (g_j t + \beta_j)] \]

small divisors indicate large influence !!
secular resonances

classical value of Mercury’s proper frequ.:
\( g_1 = 5.4058'' \)
during Mercury’s evolution of ecc.:
\( g_1 = 4.9273'' \)
\( g_j = 4.24354'' \)
◊ linear secular resonances
◊ large variations in the ecc. vector
secular resonances

\( \varpi_1 - \varpi_5 \) librates between +19.8° and -43.56°
secular resonances

\[ \frac{de_{15}}{dt} = A_{15} e_5 (1-e^2)^{0.5} \sin(\varpi_1 - \varpi_5) \]

\[ \cdot \]

\[ g_1 \Rightarrow \langle \varpi_1 \rangle \text{ and } g_5 \Rightarrow \langle \varpi_5 \rangle \]

\[ (g_1 - g_5) \] secular resonance responsible for the climb in Mercury’s eccentricity.
secular resonances

Influence of Venus:

\[ g_2 \Rightarrow \langle \bar{\omega}_2 \rangle \] (Laskar, 1990)

\[ e_{12} \] constructed as before
relativistic influence

• adding 0.43“ yr\(^{-1}\)
• \(g_1 - g_5\) gets about 40 % higher

◊ Mercury enters later in the linear secular resonance with Jupiter
◊ relativistic influence stabilizes Mercury’s orbit
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Uranus dynamical lifetime

- Laskar experiment for Uranus
- Sun (+4 terrestrial planets) and the 4 gasgiants
- Jupiter was repositioned 1500m
- Integration intervals: 5 Gyrs

◊ Uranus ecc. never exceeded 0.078!!
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Merry Christmas
one and all

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