Chaos Indicators


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Introduction

- Major question in celestial mechanics:
  -> Stability of (multi) planetary systems

  - Stability issues analysed by
    - Laplace, Lagrange,
    - Gauss, Poincaré,
    - Kolmogorov, Arnold, Moser, …

- Multi-planetary systems:
  - generally N-body problems

- N-Body problem:
  - No general, analytical solution -> approximations -> perturbation theory

Credit: http://www.cosmos.esa.int/web/plato
Introduction

- 2-Body problem:
  - exactly solvable
  - Kepler’s laws, Kepler equation, Kepler orbits
  - completely solved by Johann Bernoulli (1734)
  - elliptic, parabolic, hyperbolic solutions (conic sections)

\[
\ddot{x}_1 = -G m_2 \frac{x_1 - x_2}{|x_1 - x_2|^3}
\]
\[
\ddot{x}_2 = -G m_1 \frac{x_2 - x_1}{|x_1 - x_2|^3}
\]

- Jacobi coordinates

\[
\ddot{r} := \ddot{x}_1 - \ddot{x}_2
\]
\[
\ddot{R} := \frac{m_1 \ddot{x}_1 + m_2 \ddot{x}_2}{m_1 + m_2}
\]

- Integrals of motion: Energy:
  \[
  E = \frac{\mu \dot{r}^2}{2} + \frac{L^2}{2\mu r^2} - \frac{GM\mu}{r}
  \]
  Angular momentum: \( L = \mu r^2 \dot{\phi} \)
Introduction

- 3-Body problem:
  - No general, analytic solution
  - However, special solutions do exist:
    - Euler solution (1767): collinear aligned masses (3 solution families)
    - Lagrange solution (1772): masses form equilateral angle (2 solution families)
    - => these solutions become Lagrange points in the Restricted 3-Body problem
    - Sundman solution (Karl Frithiof Sundman, 1909): (extremely slowly) convergent infinite power series, practically one needs $10^{8,000,000}$ (!!) terms (Beloriszky, 1930)
  - Restricted 3-Body problem ($m_P \sim 0$):
    - MacMillan / Sitnikov problem

- N-Body problem:
  - No general, analytical solution
Chaos theory

- Studies behavior of dynamical systems
- Small differences in initial conditions
  -> widely diverging outcomes
- => long term prediction nearly impossible in general
- Systems are deterministic -> ‘Deterministic Chaos’
- Future fully determined by initial conditions -> no random elements
- “When the present determines the future, but the approximate present does not approximately determine the future”
- Applications: meteorology, sociology, physics, computer science, engineering, economics, biology, ecology, philosophy, …

Credit: https://en.wikipedia.org/wiki/Chaos_theory
Definition of Chaos:

- Common usage (non-scientific): “chaos” means “a state of disorder”
- Definition in Poincaré sense: dynamical behavior is not quasi-periodic
  - does not necessarily mean that system will disintegrate during any limited period of time (→ solar system)
- Stability in Poisson sense: stability is related to the preservation of a certain neighborhood relative to the initial position of the trajectory
  - in conservative systems, quasi periodic orbits remain always confined within certain limits, in this sense they are stable
Definition of Chaos:

- no universally accepted mathematical definition of chaos exists

Mathematical definition by Robert L. Devaney (1989):

- to classify a dynamical system, i.e. map $f: X \rightarrow X$, as chaotic, it must fulfil these properties:
  
  1. $f$ must be sensitive to initial conditions
  2. $f$ must have topological mixing
  3. the set of periodic orbits of $f$ is dense in $X$
Definition of Chaos:

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  3. the set of periodic orbits of $f$ is dense in $X$

Six iterations of map $x_{k+1} = 4x_k (1 - x_k)$
Chaos Theory - Systems

- Logistic Map

\[ x_{n+1} = F(x) = rx_n(1 - x_n) \]

- Fix point (order k):

\[ x_0 = F(F(F(\ldots F(x_0) \ldots)))) \quad \text{k-times} \]

- Map becomes chaotic for \( r \geq r_\infty = 3.5699\ldots \)

Credit: https://en.wikipedia.org/wiki/Logistic_map

\[ r_\infty = 3.5699\ldots \]
Chaos Theory - Systems

- Standard Map
  \[ I_{n+1} = I_n + K \sin \theta_n \]
  \[ \theta_{n+1} = \theta_n + I_{n+1} \]

- Surface of Section

Chaos sets in at \( K \sim 0.9716535\ldots \)
(Golden KAM-Torus)

Phase Space \((\theta, I)\) for \( K = 0.6 \)

Phase Space \((\theta, I)\) for \( K = 0.971635 \)

Credit: https://en.wikipedia.org/wiki/Standard_map
Chaos Theory – Lyapunov Exponent

- Lyapunov Characteristic Exponent (LCE)
  - is a quantity of a dynamical system that characterizes the rate of separation of infinitesimally close trajectories
  - Two trajectories in phase space with initial separation $\delta Z_0$ diverge at a rate given by
    $$ |\delta Z(t)| \approx e^{\lambda t} |\delta Z_0| $$
    where $\lambda$ is the Lyapunov exponent
  - generally, rate of separation is different for different orientations of initial separation vector
  - there is a spectrum of Lyapunov exponents $\lambda_1, \lambda_2, \ldots, \lambda_N$, equal in number to dim of phase space -> largest $\lambda_k$ … Maximal Lyapunov exponent (MLE)
  - Positive MLE is an indication that the system is chaotic

Credit: http://www.slideshare.net/dvidby0/lyapunov-exponent-of-time-series-data
Chaos Theory – Lyapunov Exponent

Lyapunov Exponent

- Maximal Lyapunov Exponent

\[
\lambda = \lim_{t \to \infty} \lim_{\delta Z_0 \to 0} \frac{1}{t} \ln \frac{|\delta Z(t)|}{|\delta Z_0|}
\]

Credit: http://www.slideshare.net/dvidby0/lyapunov-exponent-of-time-series-data
Chaos Theory – Lyapunov Exponent

- Lyapunov Exponent for Logistic Map

Credit: http://math.arizona.edu/~ura-reports/001/huang.pojen/2000_Report.html
Stability of Planet orbit

- Stability of planetary orbits in binary systems (Z. E. Musielak, et al., 2005)

Stability of Planet orbit

Stability of Solar system


- Hipparchus, Ptolemy: epicycles
- Copernicus, Kepler (laws: 1609-1618)
- Newton’s gravitation law 1687
- Laplace-Lagrange
  - Correctly formulated equations of motion
  - Perturbation theory
- Hamilton, Jacobi, Poincaré: 1892-1899 not possible to integrate equations of motion of 3-body problem
- Kolmogorov, Arnold, Moser (KAM theorem), 1950-60: if masses, eccentricities, inclinations of planets are small enough -> many initial conditions lead to quasiperiodic trajectories, actual masses of planets are much too large to apply directly to solar system (Michel Hénon computed that masses needs to be smaller than $10^{-320}$)
Stability of Solar system

- Chaos in the Solar system:
  - Integration over 200 Mio years (Laskar, 1989) showed that solar system is in principle chaotic, with Lyapunov time of about 50 Mio years
  - Marginal stability:
    - Solar system is full
    - Resonances important (MMR, Secular)
    - 3-5 billion years to allow collision
    - Solar system is in principal unstable
    - But catastrophe time-scale is 5 billion year
Chaos Indicators - Overview

- Chaos Indicators are techniques to detect chaos (not to prove chaos)
- Indicators versus Order Parameter (e.g. magnetization M in ferromagnetism)

- “Slow” techniques:
  - Poincaré Surfaces of Section
    - Poincaré, Birkhoff, Hénon & Heiles
    - 2 degrees of freedom
    - costly numerical integrations
    - in some cases it is impossible to obtain a transverse section for the whole flow
  - Maximum Lyapunov Exponent (MLE)
    - \( \lambda = \lim_{t \to \infty} \lim_{\delta Z_0 \to 0} \frac{1}{t} \ln \frac{|\delta Z(t)|}{|\delta Z_0|} \)
    - long time integration
Chaos Indicators – Fast Techniques


- OFLI / OFLI2: “Orthogonal Fast Lyapunov Indicator” (M. Fouchard, C. Froeschlé, E.Lega, 2002; R. Barrio, P.M. Cincotta)

- MEGNO: “Mean Exponential Growth of Nearby Orbits” (P.M. Cincotta, C. Simó, 2000; C.M. Giordano, N. Maffione)


Fast Lyapunov Indicator (FLI)

- FLI introduced by C. Froeschlé, et al. (1997)
- Based on Lyapunov Characteristic Exponent (LCE)
  - Exponential-like divergence of originally nearby trajectories
- Easy (to implement) and sensitive tool for detection of chaos ("Arnold web")

\[
\begin{align*}
\mathbf{w} &= \frac{\partial \mathbf{y}}{\partial s} \bigg|_{s=0} \\
\lim_{t \to \infty} \frac{1}{t} \ln \frac{|| D \phi^t_x \mathbf{w} ||}{|| \mathbf{w} ||} &= \chi(x, w)
\end{align*}
\]

Credit: C. Froeschlé, 1984
Fast Lyapunov Indicator (FLI)

- **Definition of Fast Lyapunov Indicator**
  
  - Set of differential equations:
    \[
    \frac{dx}{dt} = F(x), \quad x = (x_1, x_2, \ldots, x_n)
    \]

  - Equations of motion:
    \[
    \frac{dx}{dt} = F(x) \\
    \frac{dv}{dt} = \frac{\partial F}{\partial x} v.
    \]

  - Evolution of tangent vector \( v \):
    \[
    x(t + 1) = \psi(x(t)) \\
    v(t + 1) = \frac{\partial \psi}{\partial x}(x(t))v(t).
    \]

  \[
  \Rightarrow \text{Fast Lyapunov indicator:} \quad \text{FLI}_t(x(0), v(0)) = \log \frac{||v(t)||}{||v(0)||}
  \]

  improved version (reduce fluctuations):
  \[
  \text{FLI}(x(0), v(0), t) = \sup_{0 \leq k \leq t} \log ||v(k)||
  \]

  Credit: C. Froeschlé, 1984

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**Chaos Indicators**

November 17th 2016  Seite 21
Fast Lyapunov Indicator (FLI)

Properties of FLI

- Quantity  $\text{FLI}(x(0), v(0))/t$ tends to the largest Lyapunov exponent (of spectrum of LCEs) as $t$ goes to infinity.

- If differential equations are Hamiltonian and if motion is regular, the largest Lyapunov exponent is zero, otherwise it is positive. This property is largely used to discriminate between chaotic and ordered motions.

- However, among regular motions the ordinary Lyapunov exponent does not distinguish between circulation and libration orbits.

- In contrast, the FLI distinguish between them.

- How to choose $v(0)$ for practical implementation? Special choices of $v(0)$ have to be avoided.

- $\Rightarrow$ compute average (or alternatively the maximum) of the FLIs obtained for an orthonormal basis of tangent vectors.
Chaos Indicators

Fast Lyapunov Indicator (FLI)

- FLI applied on Standard Map

\[(I(t + 1), \varphi(t + 1)) = \psi(I(t), \varphi(t))\]

\[\psi(I, \varphi) = (I + \epsilon \sin(\varphi + I), \varphi + I)\]

- 3 orbits
  - Libration orbit
  - Circulation orbit
  - Chaotic orbit

Credit: E. Lega, M. Guzzo, C. Froeschlé, 2015
Fast Lyapunov Indicator (FLI)

- FLI applied on Standard Map

\[ \text{FLI}_t(x(0), v(0)) = \log \frac{\|v(t)\|}{\|v(0)\|} \]

Time evolution of FLI

Credit: E. Lega, M. Guzzo, C. Froeschlé, 2015
Fast Lyapunov Indicator (FLI)

- FLI applied on Standard Map

\[
FLI_t(x(0), v(0)) = \log \frac{\|v(t)\|}{\|v(0)\|}
\]

\[
FLI(x(0), v(0), t) = \sup_{0 \leq k \leq t} \log \|v(k)\|
\]

Credit: E. Lega, M. Guzzo, C. Froeschlé, 2015
Fast Lyapunov Indicator (FLI)

- FLI applied on Standard Map
  - FLI for $t = 1000$
  - Grid of 900 x 900 initial conditions
  - 2 orthogonal initial vectors
    - $v(0) = (1,0)$, $w(0) = (0,1)$
    - Largest FLI is plotted

$$\text{FLI}(x(0), v(0), t) = \sup_{0 \leq k \leq t} \log ||v(k)||$$

Credit: E. Lega, M. Guzzo, C. Froeschlé, 2015
Fast Lyapunov Indicator (FLI)

- FLI applied on Standard Map
  - Choice of integration time:
    - $t = 10$ (top left), $t = 100$ (top right)
    - $t = 1000$ (bottom left), $t = 10000$ (bottom right)
  - $t = 1000$ seems to be appropriate in that example

Credit: E. Lega, M. Guzzo, C. Froeschlé, 2015
Fast Lyapunov Indicator (FLI)

- FLI applied on Standard Map

Detection of regular/chaotic regions via method of set propagation -> rather costly

Credit: E. Lega, M. Guzzo, C. Froeschlé, 2015
Fast Lyapunov Indicator (FLI)

- FLI applied to Continuous System: $H(\rho, z, p_\rho, p_z) = \frac{1}{2}(p_\rho^2 + p_z^2) + \frac{h_z^2}{2c^2} + \frac{1}{2}\log(\rho^2 + z^2) - z$

Credit: E. Lega, M. Guzzo, C. Froeschlé, 2015
Orthogonal Fast Lyapunov Indicator (OFLI)

- OFLI introduced by M. Fouchard, C. Froeschlé, E. Lega, 2002

- in case of OFLI one takes component orthogonal to flow

\[
\text{FLI}(\mathbf{y}(0), \delta \mathbf{y}(0), t_f) = \sup_{0 < t < t_f} \log \| \delta \mathbf{y}(t) \| \\
\text{OFLI}(\mathbf{y}(0), \delta \mathbf{y}(0), t_f) = \sup_{0 < t < t_f} \log \| \delta \mathbf{y}^\perp(t) \|
\]

- with OFLI one can distinguish between periodicity among the regular component
- OFLI tends to a constant value for a periodic orbit
- for quasiperiodic and chaotic motion same behavior as FLI
FLI Example – Hénon-Heiles system

\[ \mathcal{H}(x, y, X, Y) = \frac{1}{2} (X^2 + Y^2 + x^2 + y^2) + x^2 y - \frac{1}{3} y^3 \]
MEGNO

- MEGNO ("Mean Exponential Growth of Nearby Orbits") (P.M. Cincotta, C. Simó, 2000)
  - Suitable fast indicator to separate regular from chaotic motion
  - Provides relevant information of global dynamics and the fine structure of phase space
  - Yields good estimate of the LCN with a comparatively small computational effort
  - Provides clear picture of resonance structures, location of stable and unstable periodic orbits, as well as measure of rate of divergence of unstable orbits
  - Feasible to investigate nature of orbits that have small, positive Lyapunov number
  - Converges to null value of LCN faster than classical algorithm to compute the LCN
Definition of MEGNO Indicator

- N-dim Hamiltonian $H(p, q)$ with $p, q \in \mathbb{R}^N$
  \[ x = (p, q) \in \mathbb{R}^{2N}, \quad v = \left(-\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}\right) \in \mathbb{R}^{2N} \]

- Equations of motion: $\dot{x} = v(x)$

- $\gamma(x_0; t) : \text{arc of an orbit of the flow over compact energy surface}$
  \[ M_h \subset \mathbb{R}^{2N}, \quad M_h = \{ x : H(p, q) = h \} \quad h = \text{constant} \]
  \[ \gamma(x_0; t) = \{ x(t'; x_0) : x_0 \in M_h, \quad 0 \leq t' < t \} \]

- Largest Lyapunov Characteristic Number (LCN)

\[
\sigma(\gamma) = \lim_{t \to \infty} \sigma_1(\gamma(x_0; t)), \quad \sigma_1(\gamma(x_0; t)) = \frac{1}{t} \ln \left[ \frac{\|\delta \gamma(x_0; t)\|}{\|\delta \gamma(x_0; t)\|} \right] \\
\sigma(\gamma) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \frac{\delta \gamma(x_0; t')}{\delta \gamma(x_0; t')} \, dt' = \left( \frac{\delta}{\delta} \right)
\]
Definition of MEGNO Indicator

\[
\sigma(\gamma) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \frac{\dot{\delta} \gamma(x_0; t')}{\delta \gamma(x_0; t')} \, dt' = \left( \frac{\delta}{\delta x_0} \right) \\
\delta \equiv \| \vec{\delta} \|, \quad \dot{\delta} \equiv \frac{d\delta}{dt} = \vec{\delta} \cdot \frac{\vec{\delta}}{\| \vec{\delta} \|} \\
\vec{\delta} = \Lambda(\gamma(x_0; t)) \cdot \vec{\delta},
\]

- tangent vector \( \vec{\delta} \) satisfies variational equation where \( \Lambda \) is the Jacobian matrix

- Introduce MEGNO \( Y(\gamma(x_0; t)) \)

\[
Y(\gamma(x_0; t)) = \frac{2}{t} \int_0^t \frac{\dot{\delta} \gamma(x_0; t')}{\delta \gamma(x_0; t')} \, t' \, dt' \]

, average of \( \ln \left[ \frac{\delta \gamma(x_0; t)}{\delta \gamma(x_0; t_0)} \right] = \lambda t \)
Properties of MEGNO

- for quasi-periodic orbits:

\[ Y(\gamma_q(x_0; t)) \approx 2 - \frac{\ln(1 + \lambda_q t)^2}{\lambda_q t} + O(\gamma_q(x_0; t)) \]

Introducing time average

\[ \overline{Y}(\gamma_q(x_0; t)) \equiv \frac{1}{t} \int_0^t Y(\gamma_q(x_0; t')) dt' \]

gives

\[ \overline{Y}(\gamma_q) \equiv \lim_{t \to \infty} \overline{Y}(\gamma_q(x_0; t)) = 2 \]
Properties of MEGNO

- Asymptotic behavior:

\[ \overline{Y}[\gamma(x_0; t)] \approx a_\gamma t + b_\gamma \]

- Irregular, chaotic motion: \( a_\gamma = \chi_\gamma / 2 \) and \( b_\gamma \approx 0 \)

- Quasiperiodic motion: \( a_\gamma = 0 \) and \( b_\gamma \approx 2 \)

- Stable periodic orbits (resonant elliptic tori): \( b_\gamma \lesssim 2 \)

- Unstable periodic orbits (hyperbolic tori): \( b_\gamma \gtrsim 2 \)
MEGNO - Example

$V(x, y, z) = -f_0(x, y, z) - f_x(x, y, z) \cdot (x^2 - y^2) - f_z(x, y, z) \cdot (z^2 - y^2)$

$f_n(x, y, z) = \frac{\alpha_n}{[p_n^{\alpha_n} + \delta_n^{\alpha_n}] \frac{\alpha_n}{\alpha_n}}$

$p_n^2 = x^2 + y^2 + z^2 + \epsilon^2$

Credit: N. Maffione, et al., 2011
Smaller/Generalized Alignment Index (SALI / GALI) introduced by H. Skokos (2001)

Definition SALI
- Orbit in n-dim space with initial condition: \( P(0) = (x_1(0), x_2(0),..., x_n(0)) \)
- Deviation vector: \( v(0) = (dx_1(0), dx_2(0),..., dx_n(0)) \)
- Evolution in time of two different deviation vectors \((v_1(0), v_2(0))\)
- Define SALI as:

\[
SALI(t) = \min \left\{ \| \hat{v}_1(t) + \hat{v}_2(t) \|, \| \hat{v}_1(t) - \hat{v}_2(t) \| \right\}
\]

\[
\hat{v}_1(t) = \frac{v_1(t)}{\| v_1(t) \|}
\]
For chaotic orbits the two initially different deviation vectors tend to coincide with the direction defined by the maximal Lyapunov exponent.

\[ \text{SALI}(t) = \min \left\{ \| \hat{v}_1(t) + \hat{v}_2(t) \|, \| \hat{v}_1(t) - \hat{v}_2(t) \| \right\} \]

\[ \text{SALI}(t) \to 0 \]
Properties of SALI

- Behavior for chaotic orbits:

\[ SALI(t) \propto e^{-(\chi_1 - \chi_2) \cdot t} \quad \chi_i \quad \text{two largest LCEs} \]

- SALI \to 0

- Behavior for regular orbits:

- SALI oscillates within the interval (0,2)
SALI / GALI - Examples

- Chaotic Motion of 3 D Hamiltonian

\[ H = \sum_{i=1}^{3} \frac{\omega_i}{2} (q_i^2 + p_i^2) + q_1^2 q_2 + q_1^2 q_3 \]

\[ \omega_1 = 1, \quad \omega_2 = 1.4142, \quad \omega_3 = 1.7321, \quad H = 0.09 \]

Credit: H. Skokos, 2010
Application to Hénon-Heiles system:

For $E=1/8$ we consider the orbits with initial conditions:

Ordered orbit, $x=0$, $y=0.55$, $p_x=0.2417$, $p_y=0$

Chaotic orbit, $x=0$, $y=-0.016$, $p_x=0.49974$, $p_y=0$

Chaotic orbit, $x=0$, $y=-0.01344$, $p_x=0.49982$, $p_y=0$
SALI / GALI – Definition GALI

SALI effectively measures the ‘area’ of the parallelogram formed by the two deviation vectors.

\[
\text{Area} = \|\hat{\mathbf{v}}_1 \wedge \hat{\mathbf{v}}_2\| = \frac{\|\hat{\mathbf{v}}_1 - \hat{\mathbf{v}}_2\| \cdot \|\hat{\mathbf{v}}_1 + \hat{\mathbf{v}}_2\|}{2} = \max\left\{\|\hat{\mathbf{v}}_1 - \hat{\mathbf{v}}_2\|, \|\hat{\mathbf{v}}_1 + \hat{\mathbf{v}}_2\|\right\} \times \text{SALI} \cdot \frac{2}{\text{Area} \propto \text{SALI}}
\]

Credit: H. Skokos, 2010
Definition GALI ("Generalized Alignment Index")

- Orbit in n-dim space with initial condition:
- \( k \) deviation vectors \( 2 \leq k \leq 2N \)
- Define GALI as:

\[
\text{GALI}_k (t) = \left\| \hat{v}_1 (t) \wedge \hat{v}_2 (t) \wedge \ldots \wedge \hat{v}_k (t) \right\|
\]

- Wedge product:

\[
\hat{v}_1 \wedge \hat{v}_2 \wedge \ldots \wedge \hat{v}_k = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq 2N} \begin{vmatrix}
V_{1i_1} & V_{1i_2} & \cdots & V_{1i_k} \\
V_{2i_1} & V_{2i_2} & \cdots & V_{2i_k} \\
\vdots & \vdots & \ddots & \vdots \\
V_{ki_1} & V_{ki_2} & \cdots & V_{ki_k}
\end{vmatrix} \hat{e}_{i_1} \wedge \hat{e}_{i_2} \wedge \ldots \wedge \hat{e}_{i_k}
\]
SALI / GALI

Properties of GALI

- Behavior for chaotic orbits:

\[ GALI_k(t) \propto e^{-(\chi_1 - \chi_2) + (\chi_1 - \chi_3) + \ldots + (\chi_1 - \chi_k) \cdot t} \]

- Behavior for regular orbits:

\[ GALI_k(t) \propto \begin{cases} 
\text{constant if } 2 \leq k \leq M \\
\frac{1}{t^{(k-M)}} & \text{if } M < k \leq 2N - M \\
\frac{1}{t^{2(k-N)}} & \text{if } 2N - M < k \leq 2N 
\end{cases} \]
Behaviors of GALI for Chaotic Motion of 2 D Hamiltonian (Henon-Heiles system)
Relative Lyapunov Indicator (RLI)

- RLI introduced by Z. Sandor, et al., 2004

- Definition of RLI:
  
  - LI difference for “base” orbit and its “shadow”

  \[
  \Delta LI(x_0; j) = ||LI(x_0 + \Delta x; j) - LI(x_0; j)||
  \]

  - define RLI as

  \[
  RLI(t) = \langle \Delta LI(x_0) \rangle_t = \frac{1}{t} \sum_{i=1}^{t/\delta t} \Delta LI(x_0, i \cdot \delta t)
  \]

- RLI values for chaotic motion are several orders of magnitude higher than for regular motion
Usage of MEGNO \(<Y>\) for HD 160692 system: 2:1 mean motion resonance

Applications of Fast Chaos Indicators

- Stability map for Earth-like planet in the $a_{\text{Jup}}$, $e_{\text{Jup}}$ for Sun-Jupiter system

Summary

- Classical methods to distinguish between regular and chaotic dynamical states like Lyapunov Characteristic Numbers (LCN), Poincare Surfaces of Section, require costly computations over long evolutionary times.

- Fast Chaos Indicators (FLI, MEGNO, SALI/GALI, …) represent useful and efficient methods to distinguish between regular and chaotic planetary configurations.
References


References


