Stabilising relativistic fluids on slowly expanding cosmological spacetimes

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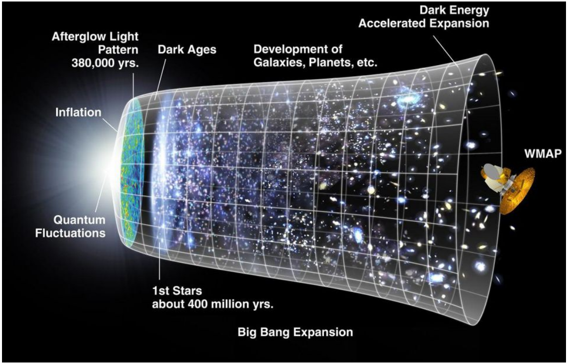
Setting for today



Goal: understand the stability of particular cosmological spacetimes filled with idealised perfect fluids.

- CMP 2021, with Fajman (Vienna) and Oliynyk (Monash)
- arXiv:2107.00457 with Fajman and Ofner (Vienna)

Cosmological spacetimes

Interested in cosmological solutions to the Einstein equations

$$\mathrm{Ric}[g]_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathsf{R}[g] + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1}$$

Cosmological principle:

- fundamental oberservers whose timelines span the spacetime and whose proper time t_c is **cosmic time**
- at large scales, one sees the same distribution of matter regardless of
  - direction (**isotropic**)
  - location (**homogeneous**)

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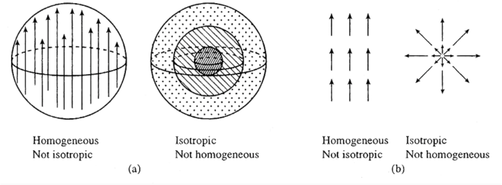
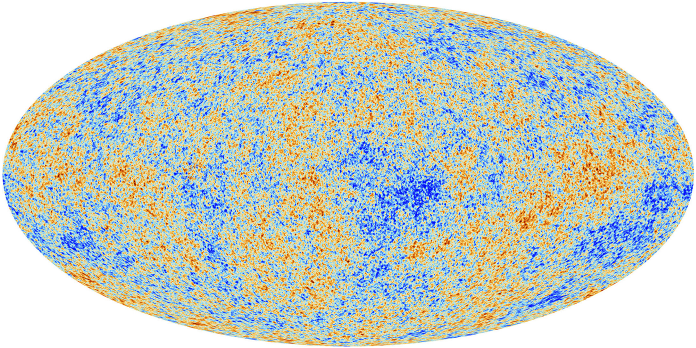


Observations

Expect isotropy about every point



(WMAP, NASA) temperature range $\pm$ 200 μK

Uniform radiation to roughly 1 part in 100,000.

Cosmological models

Spacetime splits as $\bar{M} = I \times M$ with metric

$$g = -dt_c^2 + h \tag{2}$$

where $h = h(t_c)$ is a Riemannian metric on M. Require (M, h) to be isotropic.

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Isotropy at *a* point $x \in M$ implies

- sectional curvature independent of choice of plane
- Riemann tensor takes the form

$$\mathrm{Riem}[g]_{abcd}(x) = \kappa(x)\big(h_{ac}h_{bd} - h_{bc}h_{ad}\big)(x) \tag{3}$$

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Isotropic at *every* point implies

$$\kappa = \kappa_0 \text{ constant} \tag{4}$$

and hence (M, h) is locally isometric to $\{\mathbb{H}^3, \mathbb{R}^3, \mathbb{S}^3\}$

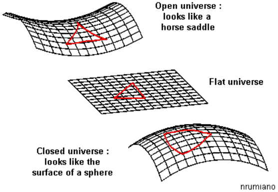
Cosmological models

Can characterise the metric in spherical coordinates as

$$g = -dt_c^2 + a(t_c)^2 \left(\frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right) \quad (5)$$

where $a(t_c)$ is an arbitrary function of time, and

$$\kappa = \begin{cases} -1 & \text{negatively curved} \\ 0 & \text{Euclidean} \\ +1 & \text{positively curved} \end{cases}$$



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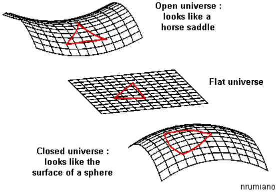

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Possible to allow locally homogeneous spaces

$$M = \{\mathbb{H}^3/\Gamma, \mathbb{T}^3, \mathbb{S}^3\} \tag{6}$$

FLRW models

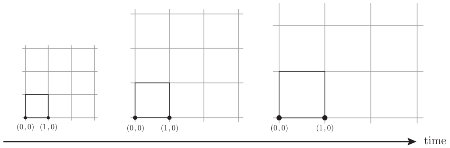
Summary:

$$I \times M, \quad g = -dt_c^2 + a(t_c)^2 h_{ab} dx^a dx^b \tag{7}$$

$a(t_c)$ is the **scale factor**

- it scales the spatial distance between fundamental observers

$$a(t_c) \times \mathrm{dist}_h((0,0),(1,0))$$



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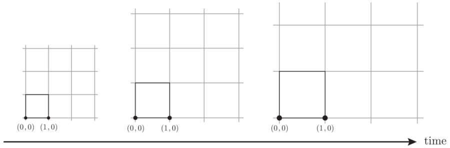

- **expanding** universe if $\dot{a} > 0$

Call $\{x^1, x^2, x^3\}$ co-moving coordinates.

Co-moving observer if $u^\mu = (1, 0, 0, 0)$ in these coordinates.

Matter model

Suppose an observer moving with timelike 4-velocity u^μ.

Look at stress-energy tensor $T_{\mu\nu}$ as measured by the observer

$$T_{\mu\nu} = \rho u_\mu u_\nu + q_\mu u_\nu + q_\nu u_\mu + P\Pi_{\mu\nu} + \pi_{\mu\nu} \tag{8}$$

$\Pi_{\mu\nu} = (g_{\mu\nu} + u_\mu u_\nu)$, so split into parts g-$||$ or g-$\perp$ to u^μ.

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Observer measures energy density ρ, pressure P, energy flux q^μ etc.

10 cpts of $T_{\mu\nu}$ $\longrightarrow$

$$\rho = T_{\mu\nu} u^\mu u^\nu$$
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etc.

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Symmetry of the FLRW model forces

$$T_{\mu\nu} = \rho u_\mu u_\nu + P(g_{\mu\nu} + u_\mu u_\nu), \quad u^\mu u_\mu = -1 \tag{9}$$

This is the algebraic form of a **perfect fluid**, with energy density $\rho(t_c)$, pressure $P(t_c)$, and 4–velocity $u^\mu = (1, 0, 0, 0)$.

Simplified Einstein-Euler equations

$$\begin{aligned} \mathrm{Ric}[g]_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} \mathsf{R}[g] + \Lambda g_{\mu\nu} &= 8\pi\big((\rho + P)u_\mu u_\nu + P g_{\mu\nu}\big) \\ \nabla_\mu\big((\rho + P)u^\mu u^\nu + P g^{\mu\nu}\big) &= 0 \end{aligned} \tag{10}$$

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the Raychaudhuri equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p) + \frac{\Lambda}{3} \tag{12}$$

and the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho + \frac{\Lambda}{3} - \frac{\kappa}{a^2} \tag{13}$$

$\dot{} = \frac{d}{dt_c}$

Equation of state

Close the system through an **equation of state** $P = f(\rho, n, s)$
- particle number density n, entropy per particle s...

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Often consider a **linear barotropic** equation of state

$$P = K\rho, \quad K \leq 1. \tag{14}$$

Important cases

- $K = 0$ **dust**, galactic epoch, neighbouring volume elements exert no action on each other
- $K = \frac{1}{3}$ **radiation** fluid, pre-decoupling epoch, energy density dominated by the kinetic energy, Stefan–Boltzmann law
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Frequenty see K written as speed of sound

$$K = c_s^2 = \left.\frac{dP}{d\rho}\right|_s \tag{15}$$

Also have polytropic equations of state (stars) with $P = C\rho^{\gamma}$.

Immediate observations

- $R = 4\Lambda + (1 - 3K)\rho$
  If $\rho \to \infty$ then curvature blow-up.

- Integrating up the continuity equation gives

$$\frac{\dot{\rho}}{\rho} = -3(1 + K)\frac{\dot{a}}{a} \quad \Rightarrow \quad \rho(t_c)a(t_c)^{3(1+K)} = c_0 = \text{const.}$$

  Energy density dilutes (at a rate depending on the type of fluid) with increasing scale factor.

- Friedmann equation becomes

$$\dot{a}(t_c)^2 = \frac{8\pi}{3}\frac{c_0}{a(t_c)^{1+3K}} - \kappa + \frac{\Lambda}{3}a(t_c)^2 \tag{16}$$

  e.g. if $\kappa = -1$ then $a(t_c)$ grows at least linearly.

Special solutions

- Einstein static universe

$$
\begin{aligned}
&\kappa = +1, \quad \Lambda = 0 \\
&g = -dt_c^2 + a_0^2(dr^2 + (\sin r)^2\, d\Omega^2) \qquad (17) \\
&\rho = 3a_0^{-2}, \quad P = -a_0^{-2}, \quad K = -1/3
\end{aligned}
$$

Include $\Lambda > 0$ to remedy negative pressure.

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  Include $\Lambda > 0$ to remedy negative pressure.

- Flat FL matter-dominated universe:

$$\begin{aligned} &\kappa = 0, \quad \Lambda = 0 \\ &g = -dt_c^2 + t_c^{\frac{4}{3(1+K)}}(dr^2 + r^2\, d\Omega^2) \\ &\rho = \tfrac{4}{3(1+K)^2 t^2} > 0 \end{aligned} \tag{18}$$

  Example of **decelerating** expansion since $\dot{a} > 0, \ddot{a} < 0$.

  Dust case $K = 0$ called Einstein-de Sitter

Special solutions

- de Sitter universe:

$$\begin{aligned} &\kappa = 0,\ \Lambda > 0, \\ &g = -dt_c^2 + \exp(2\sqrt{\Lambda}t_c)(dr^2 + r^2\, d\Omega^2) \end{aligned} \tag{19}$$

Example of **accelerated** expansion since $\dot{a} > 0, \ddot{a} > 0$.

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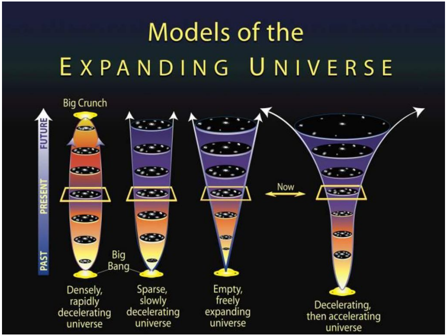
  Example of **accelerated** expansion since $\dot{a} > 0, \ddot{a} > 0$.

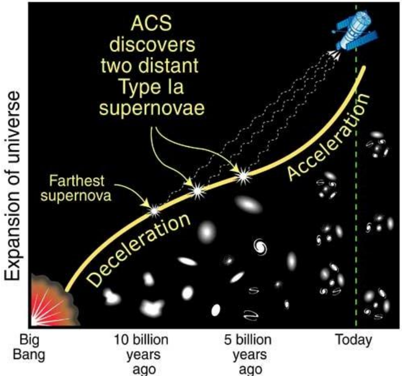
- Open Milne vacuum universe:

$$\begin{aligned} &\kappa = -1,\ \Lambda = 0, \\ &g = -dt^2 + t^2(dr^2 + (\sinh r)^2\, d\Omega^2) \\ &\rho = P = 0 \end{aligned} \tag{20}$$

  Example of **zero-accelerated** expansion since $\dot{a} > 0, \ddot{a} = 0$.

$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$.


Models of the
EXPANDING UNIVERSE
Big Crunch
FUTURE
PRESENT
PAST
Big Bang
Now
Densely, rapidly decelerating universe
Sparse, slowly decelerating universe
Empty, freely expanding universe
Decelerating, then accelerating universe


ACS discovers two distant Type Ia supernovae
Farthest supernova
Deceleration
Acceleration
Expansion of universe
Big Bang
10 billion years ago
5 billion years ago
Today

Outside of symmetry

Study Einstein-rel. Euler equations as an initial value problem.

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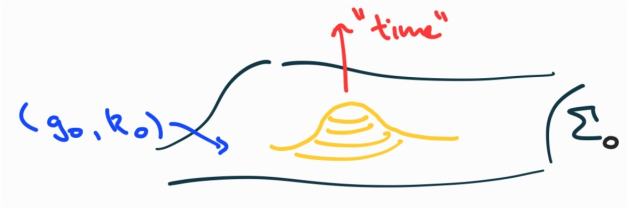
u^\alpha \nabla_\alpha \ln \rho + (1 + K) \nabla_\alpha u^\alpha &= 0 \\
u^\alpha \nabla_\alpha u^\mu + \tfrac{K}{1+K} \Pi^{\mu\alpha} \nabla_\alpha \ln \rho &= 0 \\
\Pi^{\mu\nu} &:= g^{\mu\nu} + u^\mu u^\nu
\end{aligned}
\tag{21}
$$



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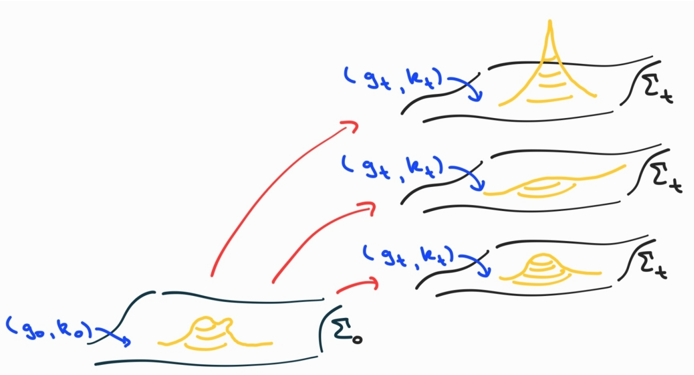

Initial value problem

Simpler problem: fix $g_{\mu\nu} = \mathring{g}_{\mu\nu}$ a Lorentzian metric and only consider the relativistic Euler equations.

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Simpler problem: restrict to **irrotational fluids**

$$u_\mu = -\zeta^{-1}\nabla_\mu\Phi, \qquad \zeta = \big(- \nabla_\mu\Phi\nabla^\mu\Phi\big)^{1/2} \tag{22}$$

which have vanishing vorticity and enthalpy ζ.

The rel.-Euler equations reduce to a quasilinear wave equation

$$a^{\mu\nu}\nabla_\mu\nabla_\nu\Phi = 0 \tag{23}$$

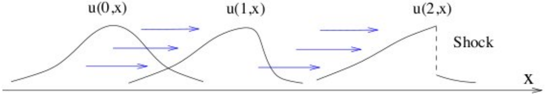
where a is called the **acoustic metric**

$$a^{\mu\nu} = g^{\mu\nu} + \tfrac{1-K}{K} u^\mu u^\nu \tag{24}$$

Literature

- In the $v \lll c$ limit of relativistic Euler equations we get the Euler equations for a perfect compressible fluid.

  Smooth solutions develop singularities (gradients $\to \infty$).

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  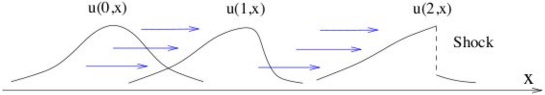

  Riemann 1880s, Friedrichs and Lax '60s, Sideris 1980s...

- Consider the relativistic Euler equations on **fixed** Minkowski

$$\mathring{g} = -dt^2 + dx^2 + dy^2 + dz^2$$

  Irrotational perturbations around the constant solution

$$\rho_B = \rho_0 > 0 \text{ on } \Sigma_0 \backslash B, \quad u_B^\mu = (1,0,0,0)$$

  lead to finite-time shock formation for many $P = f(\rho)$ [Christodoulou '07]

Literature

Situations of global existence on Minkowski do occur.

- if $H = \rho f''(\rho) + 2f'(\rho)$ vanishes on the constant state then no shocks form.

$$P = -A\rho^{-1}, A > 0, \text{ Chaplygin gas} \tag{25}$$

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Moving to cosmological models [Brauer–Rendall–Reula '94]

- Newton-Cartan-Ehlers cosmological model
- For $\Lambda > 0$ and (1) $P = C\rho^\gamma$ for $\gamma > 1$, or (2) $P = 0$, small perturbations of homogeneous solutions exist globally.
- For $\Lambda = 0$ and $a(t) = t^{2/3}$, trajectories of dust particles cross.

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- For $\Lambda = 0$ and $a(t) = t^{2/3}$, trajectories of dust particles cross.

Key idea: spacetime expansion induced by $\Lambda > 0$ can suppress shock formation in fluids.

Literature: $\Lambda > 0$

Think of Minkowski as

$$\mathring{g} = \eta = -dt^2 + 1^2(dx^2 + dy^2 + dz^2)$$

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Consider uniformly quiet fluid solutions to Einstein–Λ–rel. Euler

$$\begin{aligned} g_B &= -dt^2 + a_B(t)^2 h, \quad a_B(t) \sim \exp(\sqrt{\Lambda} t) \\ \rho_B &= \rho_0 a_B(t)^{-3(1+K)}, \quad u_B^\mu = (1, 0, 0, 0) \end{aligned} \tag{26}$$

"Theorem": such solutions are stable for

$$P = K\rho, \quad 0 \leq K \leq 1/3$$

Typically $M = g_{\mathbb{T}^3}$. Also $g_{\mathbb{S}^3}$.

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- [Rodnianski–Speck ‘09, Speck ‘11] $0 < K < 1/3$
- [Lübbe–Valiente Kroon ‘11] $K = 1/3$, [Hadžić–Speck ‘13] $K = 0$
- [Oliynyk ‘15, ‘20] $0 < K < 1/2$, [Friedrich ‘16] $K = 0$
- Also: [Ringström ‘08, ‘09]

Literature: power law inflation

Focus on fixed **power law** spacetimes

$$I \times \mathbb{T}^3, \quad g = -dt^2 + t^{2p}\delta_{ab}dx^a dx^b, \quad p > 0 \tag{27}$$

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[Speck '12] showed stability of homogeneous solutions to the relativistic Euler equations provided integrability conditions hold for the scale factor. See also [Wei JDE '18]

For example, if $K = 1/3$ and

$$\int_1^\infty \frac{1}{a(s)}\mathrm{d}s < \infty \quad \text{then perturbation exists globally}$$
$$\int_1^\infty \frac{1}{a(s)}\mathrm{d}s = \infty \quad \text{then shocks form in finite time}$$

Relies on conformal invariance of rel. Euler equations when $K = \frac{1}{3}$.

Power law inflation

Exp. rate	Range of K	Behaviour	Ref.
$p > 1$	$0 < K < 1/3$	Stable	[Speck '12]
$p > \frac{1}{2}$	Dust $K = 0$	Stable	[Speck '12]
$p = 1$	Radiation $K = 1/3$	Shocks	[Speck '12]
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$p = 1$	$0 < K < 1/3$	Stable (irrot.)	$\swarrow$

Theorem (Fajman–Oliynyk–ZW '20)

The canonical homogeneous solutions to the rel.-Euler equations on **Milne-like** FLRW spacetimes $\mathbb{R} \times \mathbb{T}^3$ with metric

$$\mathring{g} = -dt^2 + t^2 \delta_{ab} dx^a dx^b$$

and $0 < K < 1/3$ are nonlinearly stable to sufficiently small *irrotational* perturbations.

- first non-dust fluid stabilization below accelerated expansion
- shows that Speck's shock formation is sharp

Einstein-Euler: zero-acceleration

Move to fluid–gravity system, although note that rel. Euler is the weakest link.

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Consider the generalised Milne model:

$$g = -dt_c^2 + \frac{t_c^2}{9}\gamma, \quad \mathrm{Ric}[\gamma] = -\tfrac{2}{9}\gamma \tag{28}$$

with M a compact, connected orientable n-manifold admitting a negative Einstein metric γ.

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- Example of **zero-accelerated** expansion ($\Lambda = 0$)
- Milne vacuum stability for $n \geq 3$: [Andersson–Moncrief '11]
- $n = 3$ then γ has constant negative sectional curvature hence hyperbolic, hence we have Mostow rigidity

Einstein–Dust: zero-acceleration

Start with ‘most likely to be stable’ case of dust.

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First example of fluid-gravity stabilization by **non-accelerated** spacetime expansion.

- Milne matter stability: [Andersson–Fajman '17], [Branding–Fajman–Kroncke '18], [Wang '18], [Fajman–ZW '19], [Barzegar–Fajman '20].
- most relevant dust work: $\Lambda > 0$ [Hadžić–Speck '13]

Vacuum stability: set up 1/2

- ADM variables:

$$\bar{g} = -N^2 dt^2 + g_{ab}(dx^a + X^a dt)(dx^b + X^b dt) \tag{29}$$

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$$\bar{g} = -N^2 dt^2 + g_{ab}(dx^a + X^a dt)(dx^b + X^b dt) \qquad (29)$$

- CMCSH gauge:

  mean curvature $\tau := g^{ab} k_{ab} = -\frac{3}{t_c}$ on background

  Choose a foliation whose slices have constant mean curvature

$$\begin{aligned} t &= \tau \\ g^{ij}(\Gamma^k_{ij}[g] - \Gamma^k_{ij}[\gamma]) &= 0 \end{aligned} \qquad (30)$$

  [Andersson–Moncrief '03]

Vacuum stability: set up 2/2

- Rescale variables using τ to account for spatial expansion. E.g.

$$g_{ij} := \tau^2 \tilde{g}_{ij} \tag{31}$$

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- Unknowns: lapse N, shift X, Riemannian metric g, and

$$\Sigma_{ab} = (k_{ab})^{TF} := k_{ab} - \tfrac{\tau}{3} g_{ab} \tag{32}$$

Background solution in this gauge is

$$(g_{ab}, \Sigma_{ab}, N, X^a)|_B = (\gamma_{ab}, 0, 3, 0)\,. \tag{33}$$

Vacuum stability: PDEs

The gauge gives elliptic equations for the lapse and shift variables

$$\begin{aligned} (\Delta - \tfrac{1}{3})N &= -1 + \dots \\ \Delta X^i + \mathrm{Ric}[g]^i{}_m X^m &= 0 + \dots \end{aligned} \tag{34}$$

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$$\begin{aligned}\partial_T(g-\gamma) &= 2N\Sigma - \mathscr{L}_X g + \dots \\ \partial_T \Sigma_{ab} &= -2\Sigma - 3N\mathcal{L}_{g,\gamma}(g-\gamma) + \nabla_a \nabla_b N + \dots\end{aligned} \tag{35}$$

where

$$\mathcal{L}_{g,\gamma} = -g^{ab}\nabla[\gamma]_a \nabla[\gamma]_b - 2\,\mathrm{Riem}[\gamma] \sim \nabla^2 \tag{36}$$

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Aim to prove solutions to the PDEs exist for $T \to \infty$.

Vacuum stability: strategy

Aim to show

$$\|g-\gamma\|_s + \|\Sigma\|_{s-1} + \|N-3\|_{s+1} + \|X\|_{s+1} \to 0, \quad T \to \infty$$

with decay rates that imply future completeness. $\|\cdot\|_{H^s(M)} = \|\cdot\|_s$.

Use a coercive energy with correction terms. For example

$$\|g-\gamma\|_1^2 + \|\Sigma\|_0^2 \leq CE_1 \tag{37}$$

where E_1 roughly looks like

$$\begin{aligned} E_1 &\sim \int_M \Sigma^2 + \int (g-\gamma)\mathcal{L}_{g,\gamma}(g-\gamma) \\ &\sim \int_M (\partial_T g)^2 + \int (\nabla g)^2 \end{aligned} \tag{38}$$

and obeys a good estimate like

$$\partial_T E_1 \leq -2E_2 + \text{error} \tag{39}$$

Vacuum stability: corrected energies

For $c_E > 0$ depending on the smallest positive e'value of $\mathcal{L}_{\gamma,\gamma}$

$$E_2 := \frac{1}{2}\int_M \langle \Sigma, \mathcal{L}_{g,\gamma}\Sigma\rangle + \frac{9}{2}\int_M \langle \mathcal{L}_{g,\gamma}(g-\gamma), \mathcal{L}_{g,\gamma}(g-\gamma)\rangle + c_E \int_M \langle \Sigma, \mathcal{L}_{g,\gamma}(g-\gamma)\rangle \tag{40}$$

Add in correction term in order to get the full energy back

$$\begin{aligned} \partial_T E_2 &= \frac{1}{2}\int_M \langle \partial_T \Sigma, \mathcal{L}_{g,\gamma}\Sigma\rangle + \cdots \\ &= \frac{1}{2}\int_M \langle -2\Sigma + \cdots, \mathcal{L}_{g,\gamma}\Sigma\rangle + \cdots \\ &\leq -E_2 + C(E_2)^{3/2} \end{aligned} \tag{41}$$

Dust matter

Now include the matter. For example

$$(\Delta - \tfrac{1}{3})N = N\big(|\Sigma|_g^2 - \tau\eta\big) \qquad (42)$$

where $\eta = T^{\mu\nu}n_\mu n_\nu + g^{ab}T_{ab}$ is a matter contribution.

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Equations for the fluid variables

$$\partial_T \Sigma_{ab} = \text{geometry} + \tau\rho g_{ab} + \tau^3 \rho u_a u_b X + \ldots$$

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$$u^0 \partial_T u^b - \tau u^a \nabla_a u^b = u^a \Sigma_a^b + \ldots$$
$$u^0 \partial_T \rho - \tau u^a \nabla_a \rho = \tau\rho \nabla_a u^a + \Sigma_a^a + \ldots$$

suggest we instead have u, Σ at same order, but also ρ, Σ at same order and u one higher.

Follow idea of [Hadžić–Speck '13] and use a fluid/material derivative

$$\partial_{\mathbf{u}} = u^0 \partial_T - \tau u^a \nabla[\gamma]_a \simeq u^\alpha \nabla_\alpha \tag{43}$$

as differential operator in energy functionals.

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Matter PDEs look like

$$\begin{aligned} \partial_{\mathbf{u}} u^b &= u^a \Sigma_a^b + \ldots \\ \partial_{\mathbf{u}} \rho &= \tau \rho \nabla_a u^a + \Sigma_a^a + \ldots \end{aligned} \tag{44}$$

Commute Einstein equations with $\nabla^{N-1}\partial_{\mathbf{u}}$ instead of ∇^N.

Adapt our bootstrap conditions

$$\begin{aligned} \|g-\gamma\|_s + \|\Sigma\|_{s-1} + \|N-3\|_s + \|X\|_s + \|\rho\|_{s-2} &\to 0 \\ \|\partial_T N\|_{s-1} + \|\partial_T X\|_{s-1} &\to 0 \\ \|u^a\|_{s-1} &\sim e^{\mu T} \end{aligned}$$

Adapt our geometric energy functionals. For example

$$E_2 := \frac{1}{2}\int_M \langle \Sigma, \mathcal{L}_{g,\gamma}\Sigma\rangle + \frac{9}{2}\int_M \langle \mathcal{L}_{g,\gamma}(g-\gamma), \mathcal{L}_{g,\gamma}(g-\gamma)\rangle$$
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$$\tilde{E}_2 = \frac{1}{2}\int_M \langle \partial_{\mathbf{u}}(\Sigma), \partial_{\mathbf{u}}\mathcal{L}_{g,\gamma}\Sigma\rangle + \frac{9}{2}\int_M \langle \partial_{\mathbf{u}}\mathcal{L}_{g,\gamma}(g-\gamma), \partial_{\mathbf{u}}\mathcal{L}_{g,\gamma}(g-\gamma)\rangle$$
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Recall

$$\begin{aligned}\partial_T \Sigma_{ab} &= \text{geometry} + \tau\rho g_{ab} + \tau^3 \rho u_a u_b X + ... \\ \partial_{\mathbf{u}}\rho &= \tau\rho\nabla_a u^a + \Sigma_a^a + ...\end{aligned}$$

When we take a time derivative, we end up replacing

$$\partial_{\mathbf{u}}\mathcal{L}_{g,\gamma}(\partial_T\Sigma) \simeq \partial_{\mathbf{u}}\mathcal{L}_{g,\gamma}(\tau\rho) \simeq \tau\mathcal{L}_{g,\gamma}(\partial_{\mathbf{u}}\rho)$$

Key: don't have to estimate $\nabla\rho$ in this equation since already contained in $\partial_{\mathbf{u}}\rho$

Need extra ideas for some top-order lapse and shift estimates.

$$\partial_{\mathbf{u}} \mathcal{L}^k_{g,\gamma}(\mathscr{L}_X g) \sim \partial_{\mathbf{u}} \nabla^{2k+1}(X)$$

gives problems, so we instead write:

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\end{aligned}$$

This now has too many derivatives on the metric perturbation but the structure in the energy is symmetric so we can conclude by IBP

$$\begin{aligned}
&\int_M X^k\nabla_k\left(\partial_{\mathbf{u}}\mathcal{L}^k_{g,\gamma}(g-\gamma)\right)\cdot\partial_{\mathbf{u}}\mathcal{L}^k_{g,\gamma}(g-\gamma) \\
&\sim \int \nabla_k X^k\cdot\left(\partial_{\mathbf{u}}\mathcal{L}^k_{g,\gamma}(g-\gamma)\right)^2
\end{aligned}$$

Outlook

Summary:

- The homogeneous solutions to the relativistic Euler equations with $0 < K < 1/3$ on Milne-like FLRW spacetimes

$$\mathbb{R} \times \mathbb{T}^3, \quad g = -dt^2 + t^2 \delta_{ab} dx^a dx^b$$

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Future work:

- Case $\frac{1}{3} < K < 1$ very much open. [Oliynyk '20], [Fournodavlos '21]
- Does Speck's blow-up result still hold if $(g_M, M) = (\delta, \mathbb{T}^3)$ replaced with negatively curved manifold?
- Can we study deccelerating cases $p < 1$, perhaps with $p = p(K)$?
- Milne-fluid stability when $0 < K < 1/3$.