Local foliations of surfaces characterizing the center of mass

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Joint work with J. Metzger.

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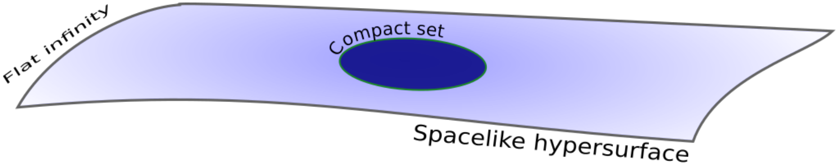
Setting and Center of mass

We consider a 3-dimensional initial data set (M, g, k), that is a 3-Riemannian manifold (M, g) and a symmetric 2-tensor k. Which is asymptotically flat.



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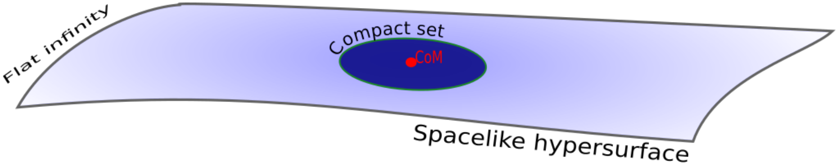

Time symmetric case ($k = 0$), Huisken-Yau

CMC center of mass

If $m > 0$, there exist a unique foliation $\{\Sigma^\sigma\}_{\sigma > \sigma_o}$ of constant mean curvature spheres.The center of each of the spheres is given by

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and the center of mass of the system is defined to be

$$\vec{C}_{CMC} := \lim_{\sigma \to \infty} \vec{c}(\Sigma^\sigma).$$

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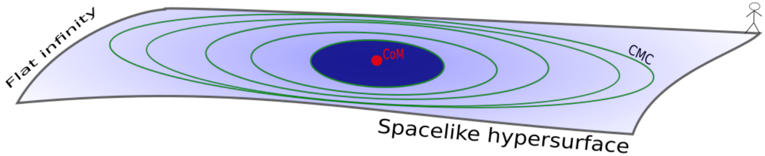

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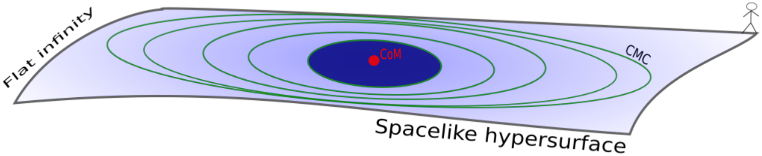

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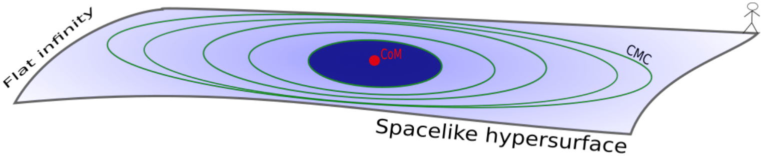

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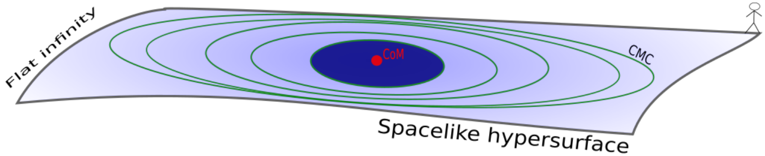

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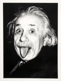
There are examples with $k \neq 0$ where CoM doesn't converge 

Constant expansion foliation

In this context Metzger found two unique foliations by 2-spheres $\{\Sigma_r^{\pm}\}_{r>r_0}$ of constant expansion, that is surfaces satisfying

$$\theta^{\pm}(\Sigma_r^{\pm}) = H(\Sigma_r^{\pm}) \pm P(\Sigma_r^{\pm}) = \frac{2}{r}$$

where H represents the mean curvature of the surface and P is the trace of the tensor k with respect to the induced metric on the surface g_{Σ_r}, $P = \mathrm{tr}_{g_{\Sigma_r}} k$.

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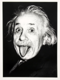
- But the center of mass does not converge 

STCMC center of mass

In 2018 Cederbaum and Sakovich propose a new definition of center of mass. They found an unique foliation of 2-spheres $\{\Sigma_r\}_{r>r_0}$ of spacetime constant mean curvature (STCMC), that is a foliation of spheres satisfying

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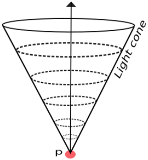
- This foliation gives a well defined center of mass. 🎉

Small sphere limit quasi-local energy

There are many different definitions of quasi-local mass, but any of these definitions should have the right asymptotics.

• The small sphere limit: when evaluating the quasi-local mass on spheres approaching a point p in a spacetime along the null cone of p the leading term of the quasi-local mass should recover the energy density of the Einstein constrained equations.



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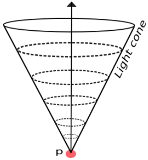

$$\lim_{r\to 0} \frac{M(S_r)}{r^3} \sim \mu \sim \mathrm{Sc}_p + (\mathrm{tr}\, k)^2 - |k|^2$$

Question

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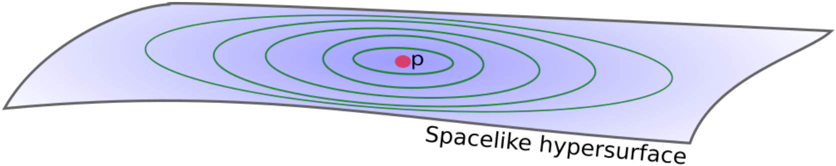
We will consider an $n+1$-dimensional initial data set (M, g, k).



Results: local constant expansion

For any tangent vector $V \in TM$ we define the *local constant expansion* 1*-form*

$$E(V) = \frac{n+2}{n+3} \nabla_V \operatorname{tr} k - 2 \operatorname{div} k(V)$$

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- If at p, $E = 0$, $k = 0$, ∇E is invertible and $C|(\nabla E)^{-1}|\left(|\nabla \mathrm{R}| + |\mathrm{Ric}||\nabla k|\right) < 1$ for some $C = C(n)$ then there exist an unique smooth foliation of constant expansion surfaces around p.

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If $k = 0$ on M then ∇E is not invertible.

Complicated constant expansion foliation

There is another way to obtain a local foliation but with more conditions on a 1-form $\hat{E}^{\pm}$ and a 2-tensor $\hat{C}$.

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- If at p, $E = \hat{E}^{\pm} = 0$, $k = \nabla E = 0$, $\operatorname{Hess} E = 0$ and $\nabla \hat{E}^{\pm} + \hat{C}$ is invertible and

$$C|(\nabla \hat{E}^{\pm} + \hat{C})^{-1}|\left(|\nabla k|\left(|\mathrm{Ric}| + |\nabla k| + |\nabla\nabla k|\right) + |\nabla\nabla\nabla k|\right) < 1$$

for some $C = C(n)$ then there exist an unique smooth foliation of constant expansion surfaces around p.

$$
\begin{aligned}
\hat{E}^{\pm}(V) = & -\frac{1}{2}\nabla_V \mathrm{R} \pm \frac{1}{3(n+3)(n+5)}\Big(-4\langle \mathrm{Ric}, \nabla k(V,\cdot)\rangle \\
& + \frac{2(n^2+6n+10)}{(n+3)}\langle \mathrm{Ric}(V,\cdot), \nabla \operatorname{tr} k\rangle - 2\langle \mathrm{Ric}, \nabla_V k\rangle \\
& - \frac{n^3+14n^2+52n+60}{n(n+3)}\mathrm{R}\nabla_V \operatorname{tr} k\Big)
\end{aligned}
$$

$$
\begin{aligned}
\hat{C}(V,W) := & \frac{4}{(n+3)(n+5)}(\langle \nabla_W k, 2\nabla k(V,\cdot) + \nabla_V k\rangle \\
& - \frac{2n+5}{(n+3)^2}(\nabla_V \operatorname{tr} k \nabla_W \operatorname{tr} k + 2\langle \nabla_W k(V,\cdot), \nabla \operatorname{tr} k\rangle))
\end{aligned}
$$

Results local STCMC

Definition

For any tangent vector $V \in TM$ we define the *local STCMC 1-form*

$$A(V) = \frac{n}{2}\nabla_V \mathrm{R} + \frac{1}{(n+5)}\Big[((n+1)(n+5)+1)\nabla_V\Big(\frac{(\operatorname{tr} k)^2}{2}\Big) + \nabla_V(|k|^2) - 2(n+4)\,\mathrm{div}\,(\operatorname{tr} k \cdot k(V,\cdot)) + 4\,\mathrm{div}\,(\langle k, k(V,\cdot)\rangle)\Big]$$

where $|k|^2 = k_{ij}k_{pq}g^{ip}g^{jq}$ and $\langle k, k(V,\cdot)\rangle = k_{ij}k_{pq}V^i g^{jp}$.

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- If at p, $A = 0$, ∇A is invertible and

$$C|(\nabla A)^{-1}|\,(|k|^2 + |\mathrm{Ric}|)|k||\nabla k| < 1$$

for some $C = C(n)$ then there exist an unique smooth foliation of STCMC surfaces around p.

Conclusions

The local STCMC 1-form A fully characterize these surfaces locally.

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However we haven't found any physical quantity related to the local STCMC 1-form.

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The End