The Hawking–Penrose Singularity Theorem for C^1-Lorentzian Metrics

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Singularity Theorems: General Remarks

Yield existence of incomplete causal geodesics under reasonable assumpt.

Theorem (Pattern singularity theorem, [Senovilla, 98])

A spacetime has incomplete causal geodesics if it satisfies

(E) an energy (i.e. curvature) condition,

(C) a causality condition,

(I) an initial/boundary condition.

"Proof":
(I) $\rightarrow$ initially, geodesics start focusing.
(E) $\rightarrow$ they focus even more $\rightarrow$ focal points.
(C) $\rightarrow$ no focal points.
Conclusion: Not all causal geodesics can exist for all times.

Singularity Theorems: Matters of Regularity

- Classical singularity theorems valid for C^2-spacetimes.
- Weak point: Extensions $\leftarrow$ regularity?
- Goal: Obstruct complete low regularity extensions
  $\rightarrow$ study singularity theorems for low reg. metrics.
- Ex.: Schwarzschild is C^0-inextendible to $\{r = 0\}$ [Sbierski, 18].
- In many examples (e.g. matched spacetimes): metric is below C^2.
- $g \in C^{1,1}$: Finite jumps in matter variables, bounded curvature, unique geodesics $\rightarrow$ exp-map, convex/normal neighborhoods; $\nexists$ notion of conjugate/focal points.
  (Ref.: Hawking theorem [KSSV15], Penrose theorem [KSV15], Hawking–Penrose theorem [GGKS18]).
- Recent new step: $g \in C^1$:
  -Hawking and Penrose singularity theorems [Graf, 20],
  -Gannon-Lee theorems [Schinnerl, Steinbauer, 21].

The Hawking–Penrose theorem for smooth spacetimes

Theorem ([Hawking, Penrose, 70])

Let (M, g) be a smooth spacetime satisfying the following:

(i) *(M, g) is chronological.*

(ii) $\mathrm{Ric}(X, X) \geq 0$ *for causal $X \in TM$ (SEC).*

(iii) *Genericity holds along each inextendible causal geodesic, i.e. $\forall\gamma\ \exists t$:* $R : [\gamma'(t)] \to [\gamma'(t)]$, $v \mapsto [R(v, \gamma'(t))\gamma'(t)]$ *is not identically zero.*

(iv) *There exist one of the following:*
   - (a) *a compact, achronal, edgeless set;*
   - (b) *a trapped point;*
   - (c) *a trapped surface;*

Then (M, g) contains incomplete causal geodesics.

(iv) with trapped submfds of arb. codimension [Galloway, Senovilla, 10].

From $C^{1,1}$ to C^1

- Geodesic eq. solvable, but not uniquely $\to$ geodesic branching.
- $\nexists$ exponential map, convex/normal neighborhoods.
- Geodesics are not locally maximizing.
- Levi-Civita connection $\Gamma \sim \partial g \in C^0$
  $\to$ curvature $\sim \partial\Gamma \in \mathcal{D}'^{(1)}$ (first order tensor distribution)
  $\to$ can insert C^1-vector fields into curvature tensors
  $\to$ needed for distributional genericity condition
  (loc. extending vector fields via parallel transport gives C^1-fields):
  Classically: $R(., \gamma'(t))\gamma'(t)$ not identically zero.
  Distributionally: $g(R(V,X)X,V) > c\ \forall$ local C^1-fields X, V close to γ', $\perp \gamma'$.

Hawking–Penrose in C^1: methods of proof

- "Analytical regularization": $g \star_M \rho_\varepsilon \xrightarrow{C^1} g$ ($\sim$ convolution).
- "Causal regularization": $\hat{g}_\varepsilon$, $\check{g}_\varepsilon$ with

$$\check{g}_\varepsilon \prec g \prec \hat{g}_\varepsilon \qquad (\prec \ldots \text{narrower lightcones})$$

  and $\check{g}_\varepsilon$, $\hat{g}_\varepsilon \xrightarrow{C^1} g$ [Chrusciel, Grant, 12].
- $\hat{g}_\varepsilon, \check{g}_\varepsilon$ and $g \star_M \rho_\varepsilon$ are compatible:
  Difference $\to 0$ in C^∞_{loc}, difference of curvatures $\to 0$ in C^0_{loc}, not just in $\mathcal{D}'^{(1)}$ [Graf, 20].
- Why both? E.g. energy conditions:
  - $\mathrm{Ric}[g] \geq 0$ in $\mathcal{D}'^{(1)} \to \mathrm{Ric}[g] \star_M \rho_\varepsilon \geq 0$ in C^∞
  - $\mathrm{Ric}[g] \star_M \rho_\varepsilon - \mathrm{Ric}[g \star_M \rho_\varepsilon] \to 0$ in C^0
  - $\mathrm{Ric}[g \star_M \rho_\varepsilon] - \mathrm{Ric}[\check{g}_\varepsilon] \to 0$ in C^0_{loc}

  $\Rightarrow$ almost energy condition for $\check{g}_\varepsilon$.

The C^1 Hawking–Penrose singularity theorem

Theorem ([Kunzinger, O., Schinnerl, Steinbauer, 21])

Let (M, g) be a C^1-spacetime satisfying the following:

(i) (M, g) is causal.

(ii) The distributional timelike and null energy conditions hold.

(iii) Distributional genericity holds along each inextendible causal geodesic.

(iv) (M, g) is maximally causally nonbranching.

(v) There exist one of the following:

- *(a) a compact, achronal, edgeless set;*
- *(b) a trapped point in the support sense;*
- *(c) a trapped C^0-surface in the support sense;*
- *(d) a trapped C^0-submanifold of codimension $2 < m < \dim M$ whose support submanifolds satisfy the distributional Galloway-Senovilla curvature condition.*

Then (M, g) contains incomplete causal geodesics.

Outlook: How low can we go?

- *Causally*: Very low (cone structures, [Minguzzi, 19]).
- *Analytically*: Ultimate goal = Geroch-Traschen metrics. Probably worth it: H^s-regularity (IVP).
- *Synthetically*: Lorentzian length spaces [KS, 17] (singularity theorems for LLS in [GKS18], [CM20]).
- *Compatibility*:
  Causal $\leftrightarrow$ analytic: OK.
  Causal $\leftrightarrow$ synthetic: OK.
  Analytic $\leftrightarrow$ synthetic: Unknown!

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