On the topology of gravitational instantons with symmetry

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Gravitational instantons

- Ricci-flat 4-manifolds (M, g) with **Riemannian** signature.
- Non-compact and complete, curvature decays "sufficiently fast".
- Different possibilities depending on volume growth rate: ALE ("asymptotically locally Euclidean"), AF ("asymptotically flat"), ALF ("asymptotically locally flat"), ALG, ALH.
- Some known examples are Taub–NUT (ALF), Taub-bolt (ALF), Eguchi–Hanson (ALE), Riemannian Kerr (AF) and Chen–Teo (AF).

Gravitational instantons

- It has long been conjectured that the only AF gravitational instantons are flat space $\mathbb{R}^3 \times S^1$ along with the Riemannian Kerr family (black hole uniqueness conjecture in Riemannian signature).[1]
- Proven wrong by the Chen–Teo instanton in 2011.[2]
- However, assuming the same topology as Riemannian Kerr, along with an S^1-symmetry with only isolated fixed points, we have uniqueness of Riemannian Kerr.[3]
- Recently, the Chen–Teo instanton was shown to be Hermitian,[4] and Hermitian AF/ALF instantons with toric symmetry were classified.[5]

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[1] Gibbons and Hawking 1979; Lapedes 1980.

[2] Chen and Teo 2011.

[3] Simon 1995.

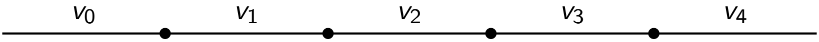
[4] Aksteiner and Andersson 2022.

[5] Biquard and Gauduchon 2021.

Symmetries

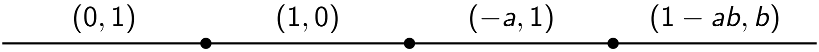
- All known AF, ALF and ALE examples have symmetries.
- S^1-symmetry: a Killing field generating a periodic flow (often assumed to have bounded norm).
- Toric ($S^1 \times S^1$) symmetry: two commuting Killing fields generating periodic flows.
- Restrict attention to AF/ALF from now on.

Toric symmetry



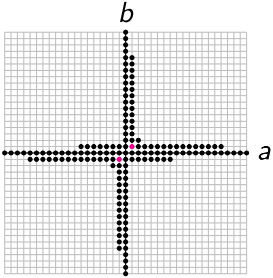
- We have two commuting Killing fields ξ_1 and ξ_2.
- We have an invariant of toric gravitational instantons, known as the **rod structure**.
- A sequence of vectors $v_i = (v_i^1, v_i^2) \in \mathbb{Z}^2$.
- Means that a linear combination $v_i^1\xi_1 + v_i^2\xi_2$ vanishes along a 2-surface in the manifold.
- The rod structure entirely determines the topology, given some assumptions (M is simply connected, etc.).

Obstructions to rod structures, an example



- A general rod structure with three turning points can be written in the way above, with $a, b \in \mathbb{Z}$.
- Many of the values of (a, b) can be ruled out using index theorems.

Obstructions to rod structures, an example



- From the assumption of Ricci-flatness, the Hitchin–Thorpe inequality rules out most values of (a, b).
- Pink corresponds to Chen–Teo.

S^1-symmetry

- We have a single Killing field ξ, whose vanishing locus is the fixed point set $\mathcal{Z}$.
- The connected components of $\mathcal{Z}$ are either 0-dimensional ("nuts") or 2-dimensional ("bolts").
- Hypersurfaces near infinity are circle fibrations over either S^2 or $\mathbb{R}P^2$; restrict attention to the case S^2.
- The circle fibration near infinity then has Euler number e.

The G-signature theorem

$$\operatorname{sign}[M] = \sum_{i=1}^{n_{\text{nuts}}} \epsilon(P_i) \prod_{j=\pm} \frac{1+g^{w_i^j}}{1-g^{w_i^j}} + \frac{4g}{(1-g)^2}\left(e - \sum_{i=1}^{n_{\text{bolts}}} B_i \cdot B_i\right) + \operatorname{sgn}(e),$$

where g is an indeterminate (!).

- Here, $\epsilon(P_i)$ and $w_i^{\pm}$ are invariants of the S^1-action, and $B_i \cdot B_i$ is the self-intersection number of the bolt B_i.
- Gives obstructions on the fixed point set $\mathcal{Z}$, along with the topology of M.

The G-signature theorem, an example

- Assume that M has exactly one nut, and any number of bolts.
- In the AF case, $e = 0$, and the G-signature theorem implies that
$$\sum_{i=1}^{n_{\mathrm{bolts}}} B_i \cdot B_i = \mathrm{sign}[M] = \pm 1.$$
- In particular, the number of bolts is nonzero.

The G-signature theorem, an example

- Assume that M has exactly one nut, and any number of bolts.
- In the ALF case, the fibration has Euler number $e \neq 0$.
- In this case, the G-signature theorem implies that

$$\sum_{i=1}^{n_{\text{bolts}}} B_i \cdot B_i = e + \epsilon(P_1) = e \pm 1$$

- In particular, the number of bolts must be nonzero, unless $e = \pm 1$.
- (The Taub–NUT instanton is ALF with $e = -1$, and has one nut and no bolts.)

Concluding remarks

- The methods are topological in nature.
- Give only topological results.
- Can, however, be an important step in uniqueness results involving the metric.

Thanks!

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