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A note on non time-symmetric initial data sets

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Constraint Equations

An initial data set is a triple (M, g, K), where (M, g) is a Riemannian manifold and K a symmetric 2-tensor that satisfy the *constraint equations*

$$\begin{aligned} R(g) + (\mathrm{tr}_g K)^2 - |K|_g^2 + 2\Lambda &= 16\pi\boldsymbol{\mu}, \\ \mathrm{div}_g(K - ((\mathrm{tr}_g K)g)) &= 8\pi\mathbf{J}, \end{aligned}$$

for a function $\boldsymbol{\mu}$ and one-form $\mathbf{J}$ on M, where $R(g)$ denotes the scalar curvature of (M, g) and Λ is called the cosmological constant.

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for a function $\boldsymbol{\mu}$ and one-form $\mathbf{J}$ on M, where $R(g)$ denotes the scalar curvature of (M, g) and Λ is called the cosmological constant.
Initial data sets arise naturally in the context of General Relativity as spacelike hypersurfaces (M, g) of a spacetime $(\overline{M}, \overline{g})$ with second fundamental form K and (future) timelike unit normal $\vec{n}$, and

$$\boldsymbol{\mu} = G(\vec{n}, \vec{n}) \qquad \mathbf{J} = G(\vec{n}, \cdot),$$

where G denotes the Einstein tensor of $(\overline{M}, \overline{g})$.

A family of initial data sets

In the following, we will always assume that (M, g, K) is of the form $M = [r_0, \infty) \times \mathbb{S}^{n-1}$ with

$$g = N(r,\cdot)^2 dr^2 + r^2\sigma(r),$$
$$K = k(r,\cdot)N(r,\cdot)^2 dr^2 + p(r,\cdot)r^2\sigma(r),$$

where N, k, p are differentiable functions on M, and $\{\sigma(r)\}_{r\in[r_0,\infty)}$ is a family of metrics on $\mathbb{S}^{n-1}$ that satisfy

- (exponentially fast) decay towards the round metric $\mathrm{d}\Omega^2$ as $r \to \infty$,
- $\mathrm{tr}_\sigma \sigma' = 0$. (cf. Mantoulidis–Schoen [4])

Rotationally symmetric case:

Assume additionally that N, k, p only depend on r and $\sigma(r) = \mathrm{d}\Omega^2$.

A family of initial data sets

In this case, the constraint equations can be written as (cf. Bartnik [1], Rácz [6]):

$$
\begin{aligned}
\frac{2(n-1)}{r}\partial_r N =& \frac{2N^2}{r^2}\Delta_{\sigma(r)}N - \frac{R(\sigma(r))}{r^2}N^3 + \frac{(n-1)(n-2)}{r^2}N + \frac{N}{4}|\sigma'|^2_{\sigma(r)} \\
&- (2(n-1)kp + (n-1)(n-2)p^2)N^3 + (16\pi\boldsymbol{\mu} - 2\Lambda)N^3, \\
(n-1)\partial_r p =& \frac{(n-1)}{r}(k-p) - 8\pi\mathbf{J}_0, \\
\frac{(k-p)}{N}\frac{\partial}{\partial x^I}N =& (n-2)\frac{\partial}{\partial x^I}p + \frac{\partial}{\partial x^I}k + 8\pi\mathbf{J}_I.
\end{aligned}
$$

Asymptotic flatness

An initial data set (M, g, K) within the above familiy is asymptotically flat, iff

$$\begin{aligned} N = 1 + O_2(r^{-a}),\ k = O_2(r^{-b}),\ p = O_2(r^{-b}), \\ \tau_{AB} := \sigma(r)_{AB} - \mathrm{d}\Omega^2_{AB} = O_2(r^{-a}), \end{aligned}$$

for $a > \frac{n-2}{2}$ and $b > \frac{n}{2}$, and furthermore $\boldsymbol{\mu}$, $\mathbf{J} \in \mathcal{L}^1(M)$.
For $n = 3$ we find

$$E_{ADM} = \frac{1}{16\pi} \lim_{r\to\infty} \int_{\mathbb{S}^2} \frac{2}{r}(N^2 - 1) r^2 \, \mathrm{d}V_{\mathbb{S}^2},$$

$$P_{ADM,i} = \frac{1}{4\pi} \lim_{r\to\infty} \int_{\mathbb{S}^2} p \frac{x^i}{|x|} r^2 \, \mathrm{d}V_{\mathbb{S}^2}$$

For an initial data set (M, g, K) within the above familiy, and for $\Sigma_r := \{r\} \times \mathbb{S}^{n-1}$, we have

$$\left|\vec{\mathcal{H}}\right|_{\overline{g}}^2 = \frac{(n-1)^2}{r^2}\left(\frac{1}{N^2} - r^2 p^2\right).$$

In particular, for $n = 3$, their Hawking energy is given as

$$m_H(\Sigma_r) = \frac{r}{2}\left(1 - \frac{1}{4\pi}\int_{\mathbb{S}^2}\left(\frac{1}{N^2} - r^2 p^2\right)\,\mathrm{d}V_{\sigma(r)}\right).$$

Penrose-type total energy bounds

We now assume that (M, g, K) is asymptotically flat with $n = 3$ and additionally require

- the (DEC), i.e. $\mu \geq |J|_g$,
- $r_0 N(r_0) |p(r_0)| = 1$ (the inner boundary is a (generalized) apparent horizon),
- $\{\Sigma_r\}_{r \in (r_0, \infty)}$ satisfies a strictly outer untrapped condition

$$rN |p| < 1$$

Recall that

$$m_H(\Sigma_{r_0}) = \sqrt{\frac{|\Sigma_{r_0}|}{16\pi}},$$

and $\lim_{r \to \infty} m_H(\Sigma_{r_0}) = E_{ADM}$.

Penrose-type total energy bounds

Proposition

Under the above assumptions, we have

$$\frac{\partial}{\partial r} m_H(\Sigma_r) \geq 0.$$

Proof:

Compute that

$$\frac{\partial}{\partial r} m_H(\Sigma_r) \geq \frac{1}{8\pi} \int \left(\frac{|\nabla^\sigma N|^2}{N^2} + \frac{s^2}{8N^2} |\sigma'|_\sigma^2 \right) + \int s^2 (\boldsymbol{\mu} - N^{-1} |\mathbf{J}_0|)$$

$$\geq \int s^2 (\boldsymbol{\mu} - |\mathbf{J}|_g)$$

Penrose-type total energy bounds

Corollary

Let (M, g, K) satisfy all of the above. Then

$$\sqrt{\frac{|\Sigma_{r_0}|}{16\pi}} \leq E_{ADM},$$

and equality holds, if and only if (M, g, K) embedds into the Schwarzschild spacetime as a rotationally symmetric slice.

- Since rigidity implies rotational symmetry, we have $E_{ADM} = m_{ADM}$ (i.e. $|P_{ADM}| = 0$),
- Proof of rigidity by reducing to rotational symmetry (cf. Mars [5]),
- the monotonity of m_H does not involve the full dominant energy scalar $\boldsymbol{\mu} - |\mathbf{J}|_g$

Penrose-type total energy bounds

Consider the quantity

$$\Phi(r) := \int_{r_0}^{r} \int_{\mathbb{S}^2} |\mathbf{J}|_g - N^{-1} |\mathbf{J}_0| \, \mathrm{d}V_{\sigma(r)},$$

which is non-negative and well-defined for $r \to \infty$ since $\mathbf{J}$ integrable. Notice thate $\Phi(r) \equiv 0$ in rotational symmetry and moreover

$$\frac{\partial}{\partial r}(m_H(\Sigma_r) - \Phi(r)) \geq \int s^2(\mu - |\mathbf{J}|_g) \geq 0$$

under the above assumptions.

Penrose-type total energy bounds

Corollary

Let (M, g, K) satisfy all of the above. Then

$$\sqrt{\frac{|\Sigma_{r_0}|}{16\pi}} + \int_{r_0}^{\infty} \int_{\mathbb{S}^2} |\mathbf{J}|_g - N^{-1} |\mathbf{J}_0| \, \mathrm{d}V_{\sigma(r)} \leq E_{ADM},$$

and equality holds, if and only if (M, g, K) has vanishing dominant energy scalar, i.e. $\mu = |\mathbf{J}|_g$, with $|\mathbf{J}_0| = 0$, and $N = N(r)$, $\sigma(r) = d\Omega^2$.

- Can replace Φ by any $0 \leq f \leq \Phi$, and recover the full ridigity statement for a large class of examples,
- Can we relate the above integral to $|P_{ADM}|$? (For a choice of f?)

Construction of initial data sets

Recall that in this familiy of metrcis, the constraint equations can be written as

$$\begin{aligned}
\frac{2(n-1)}{r}\partial_r N =& \frac{2N^2}{r^2}\Delta_{\sigma(r)}N - \frac{R(\sigma(r))}{r^2}N^3 + \frac{(n-1)(n-2)}{r^2}N + \frac{N}{4}|\sigma'|^2_{\sigma(r)} \\
&- (2(n-1)kp + (n-1)(n-2)p^2)N^3 + (16\pi\boldsymbol{\mu} - 2\Lambda)N^3, \\
(n-1)\partial_r p =& \frac{(n-1)}{r}(k-p) - 8\pi\mathbf{J}_0, \\
\frac{(k-p)}{N}\frac{\partial}{\partial x^I}N =& (n-2)\frac{\partial}{\partial x^I}p + \frac{\partial}{\partial x^I}k + 8\pi\mathbf{J}_I.
\end{aligned}$$

The rotationally symmetric case

In rotational symmetry, the equations decouple and simplify to

$$
\begin{aligned}
\frac{2(n-1)}{r}\partial_r N =& \frac{(n-1)(n-2)}{r^2} N(1-N^2) + N^3(16\pi\boldsymbol{\mu} - 2\Lambda) \\
& - N^3(2(n-1)kp + (n-1)(n-2)p^2), \\
(n-1)\partial_r p =& \frac{(n-1)}{r}(k-p) - 8\pi \mathbf{J}_0, \\
0 =& \mathbf{J}_I.
\end{aligned}
$$

Also considered by Bartnik–Isenberg [2] in the context of dynamical horizons and by Csukás–Rácz [3] in a near Schwarzschild vacuum context.

The rotationally symmetric case

Setting $h(r) := 1 + \frac{2}{n(n-2)}\Lambda r^2 - \frac{1}{N^2}$, the constraint equations become

$$h'(r) = -\frac{(n-2)}{r}h + c_1(r),$$
$$p'(r) = -\frac{1}{r}p(r) + c_2(r).$$

with

$$c_1(r) := -r(2kp + (n-2)p^2) + \frac{r}{(n-1)}16\pi\boldsymbol{\mu},$$
$$c_2(r) = \frac{k}{r} - \frac{8\pi}{(n-1)}\mathbf{J}_0$$

with k, $\boldsymbol{\mu}$, $\mathbf{J}_0$ given.

The rotationally symmetric case

For k, $\boldsymbol{\mu}$, $\mathbf{J}_0$ are given (with appropiate decay), wa can first solve for p, then for h. This yields

$$\frac{1}{N^2} = 1 + \frac{2}{n(n-2)}\Lambda r^2 - \frac{1}{r^{n-2}}\left(C_0 - \int_r^\infty c_1(s)s^{n-2}\,\mathrm{d}s\right),$$

$$p = -\frac{1}{r}\int_r^\infty c_2(s)s\,\mathrm{d}s,$$

where the constant C_0 may be chosen freely, and N, p indeed satisfy the right decay, such that (M, g, K) is asymptotically flat.

The rotationally symmetric case

In the context of two-parameter foliations of spacetimes (cf. Rácz [7]), we also find the following result in rotational symmetry:
If we choose $\mathbf{J}_0 = 0$, so vanishing momentum density $\mathbf{J} \equiv 0$, then (M, g, K) embedds into $(\overline{M}, \overline{g})$ with $\overline{M} = \mathbb{R} \times I \times \mathbb{S}^{n-1}$ and

$$\overline{g} = -f \,\mathrm{d}t^2 + \frac{1}{f} \,\mathrm{d}r^2 + r^2 \,\mathrm{d}\Omega^2,$$

with $f(r) := \frac{1}{N^2} - r^2 p^2$.

Thank you!

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