Λ-vacuum initial data on compact Cauchy surfaces

CERS12 Talk

János Östör

Budapest University of Technology / Wigner RCP

Joint work with István Rácz

GR as a Cauchy Problem

need a predictive formulation

- $\mathcal{M} \cong_{top} \mathbb{R} \times \Sigma$ and apply a 3+1 decomposition
- Project $G_{ab} - \mathcal{G}_{ab} = 0$ onto Σ with n^a and $h^a{}_b$
- 4 constraint and 6 evolutionary equations

$$^{(3)}R + K^2 - K_{ab}K^{ab} - 2\mathfrak{e} = 0$$

$$D_b K^b{}_a - D_a K + \mathfrak{p}_a = 0$$

where $K_{ab} \sim \mathcal{L}_n(h_{ab})$ $\quad \mathfrak{e} = n^a n^b \mathcal{G}_{ab}$ $\quad \mathfrak{p}_a = -h^e{}_a n^f \mathcal{G}_{ab}$

- $\{\Sigma,\ h_{ab},\ K_{ab}\}$
- underdetermined system
- the constraints propagate

The conformal (elliptic) method

Lichnerowicz - York:

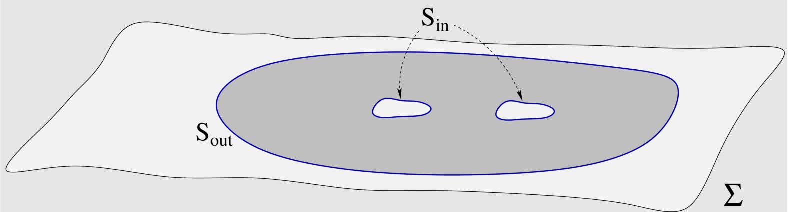
- $h_{ab} = \phi^4 \ \tilde{h}_{ab}$ $\qquad K_{ab} = \phi^{-2} \ \tilde{K}_{ab} + \frac{1}{3} \ h_{ab} \, K$
- $\tilde{K}_{ab} = \tilde{K}^{[L]}_{ab} + \tilde{K}^{[TT]}_{ab}$ $\qquad \tilde{K}^{[L]}_{ab} = 2 \, \tilde{D}_{(a} X_{b)} - \frac{2}{3} \tilde{h}_{ab} \tilde{D}^m X_m$

$$\tilde{\Delta}\phi - \frac{1}{8}\tilde{R}\phi + \frac{1}{8}\tilde{K}_{ab}\tilde{K}^{ab} \ \phi^{-7} - \left(\frac{1}{12}K^2 - \frac{1}{4}\mathfrak{e}\right)\phi^{-5} = 0$$

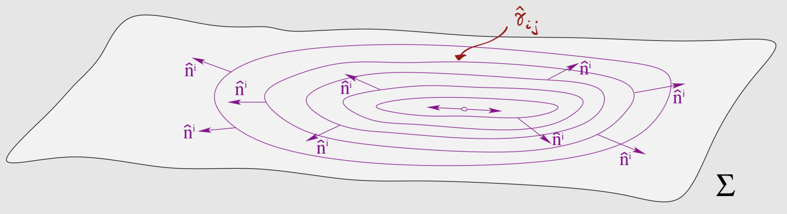
$$\tilde{\Delta} \, X_a + \frac{1}{3}\tilde{D}_a(\tilde{D}^b X_b) + \tilde{R}_a{}^b X_b - \frac{2}{3}\phi^6 \tilde{D}_a K - \phi^{10} \ \mathfrak{p}_a = 0$$

$$\Big(h_{ab}, K_{ab}\Big) \longleftrightarrow \Big(\phi, \tilde{h}_{ab}, K, X_a, \tilde{K}^{[TT]}_{ab}\Big)$$

- can produce every initial data set
- well established, long literature
- simple if $\widetilde{h}_{ab} = \delta_{ab}$ and $K = \text{const.}$
- implicit, no direct control over physical parameters
- well-posedness on a case by case basis, unphysical solutions



The evolutionary (parabolic-hyperbolic) method



- $\Sigma \cong \mathbb{R} \times \mathcal{S}$ and apply a 2+1 decomposition
- project the constraints onto $\mathcal{S}$ with $\widehat{n}^{\,i}$ and $\widehat{\gamma}^{\,i}{}_j$
- $K_{ij} = \boldsymbol{\kappa}\, \widehat{n}_i \widehat{n}_j + 2\, \widehat{n}_{(i}\, \boldsymbol{k}_{j)} + \boldsymbol{K}_{ij}$ $\qquad \boldsymbol{K}_{ij} = \mathring{\boldsymbol{K}}_{ij} + \frac{1}{2}\, \widehat{\gamma}_{ij}\, \boldsymbol{K}$
- $\widehat{K}_{ij} \sim \mathcal{L}_{\widehat{n}}(\widehat{\gamma}_{ij})$ and $\overset{\star}{K}_{ij} = \widehat{N}\, \widehat{K}_{ij}$

$$\left(h_{ab},\ K_{ab}\right) \longleftrightarrow \left(\widehat{N},\ \widehat{N}^i,\ \widehat{\gamma}_{ij},\ \boldsymbol{\kappa},\ \boldsymbol{k}_i, \boldsymbol{K},\ \mathring{\boldsymbol{K}}_{ij}\right)$$

$$\left(h_{ab},\ K_{ab}\right) \longleftrightarrow \left(\widehat{N},\ \widehat{N}^{i},\ \widehat{\gamma}_{ij},\ \boldsymbol{\kappa},\ \boldsymbol{k}_{i}, \boldsymbol{K},\ \mathring{\boldsymbol{K}}_{ij}\right)$$

$$2\,\overset{\star}{K}\left[\,\partial_{\rho}\widehat{N} - \widehat{N}^{\,i}\ \widehat{D}_{i}\widehat{N}\,\right] - \widehat{N}^{\,2}\ \widehat{D}^{b}\ \widehat{D}_{b}\widehat{N} - \mathcal{A}\widehat{N} - \mathcal{B}\widehat{N}^{3} = 0$$

$$\mathcal{L}_{\widehat{n}}\boldsymbol{K} - \widehat{D}^{\,j}\boldsymbol{k}_{j} - \widehat{N}^{-1}\overset{\star}{K}\Big[\boldsymbol{\kappa} - \frac{1}{2}\boldsymbol{K}\Big] + \\ + \widehat{N}^{-1}\mathring{\boldsymbol{K}}_{ij}\overset{\star}{K}^{\,ij} + 2\dot{\widehat{n}}^{\,j}\boldsymbol{k}_{j} - \mathfrak{p}_{j}\widehat{n}^{\,j} = 0$$

$$\mathcal{L}_{\widehat{n}}\boldsymbol{k}_{i} - \widehat{D}_{j}\left[\frac{1}{2}\boldsymbol{K} + \boldsymbol{\kappa} - \mathring{\boldsymbol{K}}^{j}{}_{i}\right] + \widehat{N}^{-1}\,\overset{\star}{K}\,\boldsymbol{k}_{i} + \\ + \left[\boldsymbol{\kappa} - \frac{1}{2}\boldsymbol{K}\right]\dot{\widehat{n}}_{i} - \dot{\widehat{n}}^{\,j}\,\mathring{\boldsymbol{K}}_{ji} + \mathfrak{p}_{j}\widehat{\gamma}^{\,j}{}_{i} = 0$$

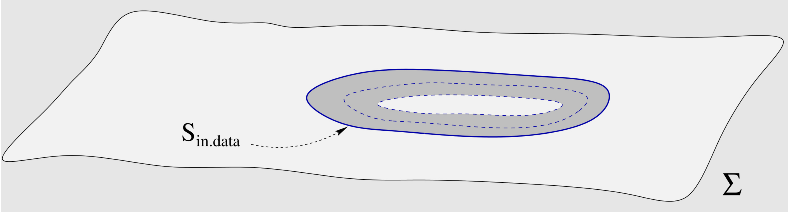
$$\left(h_{ab},\ K_{ab}\right) \longleftrightarrow \left(\widehat{N},\ \widehat{N}^i,\ \widehat{\gamma}_{ij},\ \boldsymbol{\kappa},\ \boldsymbol{k}_i, \boldsymbol{K},\ \mathring{\boldsymbol{K}}_{ij}\right)$$

for the variables $\left(\widehat{N},\ \boldsymbol{k}_i,\ \boldsymbol{K}\right)$: parabolic-hyperbolic

- local solution guaranteed
- direct control over physical parameters

$$h_{ab} = \begin{pmatrix} \widehat{N}^2 + \widehat{N}_k \widehat{N}^k & \widehat{N}_i \\ \widehat{N}_i & \widehat{\gamma}_{ij} \end{pmatrix}$$

- no gauge fixing (arbitrary foliation and topology of $\mathcal{S}$)



Simplifications

Assume spherical symmetry

- $\mathcal{S} =$ metric $\mathbb{S}^2$
- $\widehat{N}^{\,i} = \boldsymbol{k}_{\,i} = 0$
- $\widehat{D}_i = 0$

$$\overset{\star}{K}\,\partial_\rho \widehat{N} - \widehat{N}\left[\partial_\rho \overset{\star}{K} + \frac{1}{2}\left(\overset{\star}{K}^2 + \overset{\star}{K}_{ij}\,\overset{\star}{K}^{ij}\right)\right] + \\ + \widehat{N}^{\,3}\left[\frac{\widehat{R}}{2} + \boldsymbol{\kappa}\,\boldsymbol{K} + \frac{1}{4}\boldsymbol{K}^{\,2} - \mathfrak{e}\right] = 0$$

$$\partial_\rho \boldsymbol{K} - \left(\boldsymbol{\kappa} - \frac{1}{2}\boldsymbol{K}\right)\overset{\star}{K} = 0$$

- solve as ODEs for $\widehat{N}$, $\boldsymbol{K}$

Friedmann-Lemaitre-Robertson-Walker

Λ-vacuum and $\Sigma = \mathbb{S}^3$ e.g. Beig, Bizon, Walter : "Initial data from Killing Vectors"

$$ds^2 = -dt^2 + \underbrace{a(t)^2 \left[d\psi^2 + \sin^2\psi \, d\Omega^2 \right]}_{^{(3)}ds^2}$$

Strictly-Near case: $\{ \, \widehat{\gamma}_{ij} \, , \, \boldsymbol{\kappa} \}$ background form

$$\boldsymbol{K} = \boldsymbol{K}_{\,FLRW} + \frac{C_{\boldsymbol{K}}}{\sin\psi}$$

$$\widehat{N}(\psi \, ; C_{\boldsymbol{K}}, C_{\widehat{N}}) = ...$$

How to study solutions?

- $^{(3)}ds^2 = \widehat{N}^2 \, d\psi^2 + a(t)^2 \, \sin^2\psi \, d\Omega^2$ invariants
- conical singularity: $\lim_{\psi \to 0} \dfrac{\widehat{N}}{a(t)\cos\psi} \neq 1$

- $C^k(\Sigma)$ at the origin, even function in ψ
- asymptotic behaviour (e.g. Schwarzschild)

$\Longrightarrow$ only realistic **strictly-near** solution: background

$$\boldsymbol{K}(\psi) = \boldsymbol{K}_{\,FLRW} = 2\,\frac{\partial_t\, a(t)}{a(t)}$$

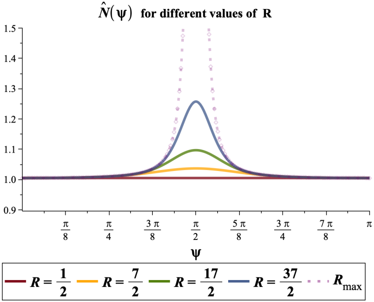
$$\widehat{N}(\psi) = \widehat{N}_{FLRW} = a(t)$$

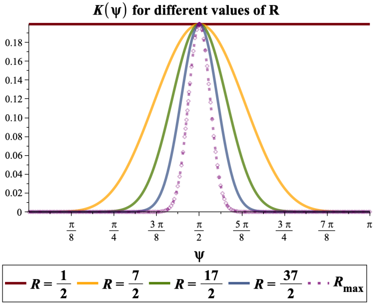
Modifying the freely specifiables: $\boldsymbol{\kappa} = \mathcal{R}\,\boldsymbol{K}$ $\quad \mathcal{R} \in \mathbb{R}$

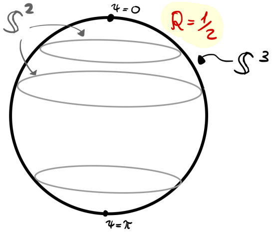
$$\boldsymbol{K}(\psi) = C_{\boldsymbol{K}}\,\left(\sin\psi\right)^{2\mathcal{R}-1}$$

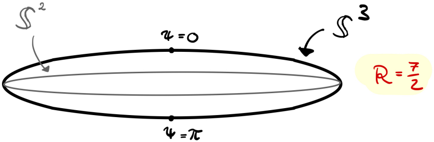
$$\widehat{N}\left(\psi\,;C_{\boldsymbol{K}},C_{\widehat{N}}\right) = \ldots$$

If $\quad C_{\boldsymbol{K}} = \frac{\partial_t\, a(t)}{a(t)} \qquad C_{\widehat{N}} = 0 \qquad \mathcal{R} \in \left\{\frac{1}{2},\ \frac{3}{2},\ \frac{5}{2}\ldots\ \mathcal{R}_{max}\right\}$


$\hat{N}(\psi)$ for different values of R
1.5
1.4
1.3
1.2
1.1
1.0
0.9
$\frac{\pi}{8}$ $\frac{\pi}{4}$ $\frac{3\pi}{8}$ $\frac{\pi}{2}$ $\frac{5\pi}{8}$ $\frac{3\pi}{4}$ $\frac{7\pi}{8}$ π
ψ
$R = \frac{1}{2}$ $R = \frac{7}{2}$ $R = \frac{17}{2}$ $R = \frac{37}{2}$ R_{max}


$K(\psi)$ for different values of R
0.18
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0
$\frac{\pi}{8}$ $\frac{\pi}{4}$ $\frac{3\pi}{8}$ $\frac{\pi}{2}$ $\frac{5\pi}{8}$ $\frac{3\pi}{4}$ $\frac{7\pi}{8}$ π
ψ
$R = \frac{1}{2}$ $R = \frac{7}{2}$ $R = \frac{17}{2}$ $R = \frac{37}{2}$ R_{max}


$\mathbb{S}^2$
$\psi = 0$
$R = 1/2$
$\mathbb{S}^3$
$\psi = \pi$


$\mathbb{S}^2$
$\psi = 0$
$\mathbb{S}^3$
$R = \frac{7}{2}$
$\psi = \pi$

Conformal FLRW

A slightly more general model:

$$ds^2 = -dt^2 + \underbrace{F(t,\psi)^2 \left[d\psi^2 + \sin^2\psi \, d\Omega^2\right]}_{^{(3)}ds^2}$$

EFE $\Longrightarrow$ $F(t,\psi) = a(t)\, B(\psi)$

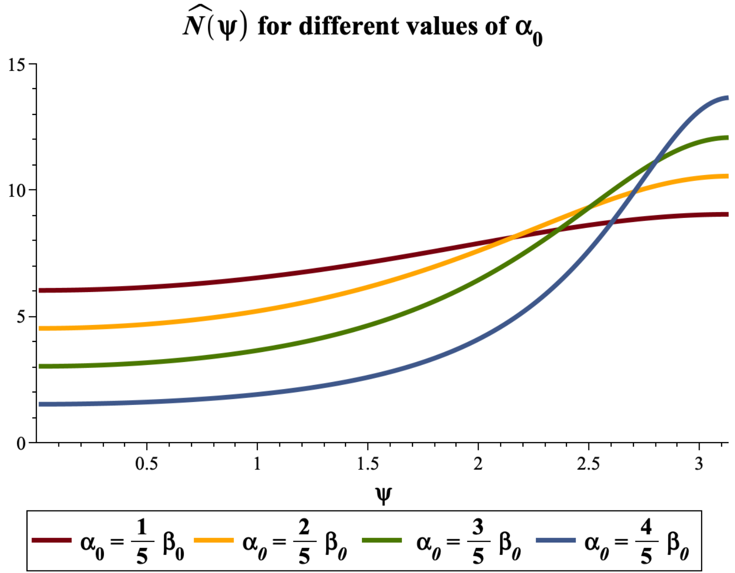
$$a(t) \sim \exp\left(\sqrt{\frac{\Lambda}{3}}t\right)\left(\beta_0^2 - \alpha_0^2\right) \qquad B(\psi) = \frac{1}{\alpha_0 \cos(\psi) + \beta_0}$$

For the **strictly-near** case:

$$\boldsymbol{K}(\psi) = \boldsymbol{K}(\psi)_{FLRW} = 2\frac{\partial_t F}{F}$$

$$\widehat{N} = \widehat{N}_{FLRW} = F(t,\psi)$$

- 2 free parameters (+open conditions)


$\widehat{N}(\psi)$ for different values of α_0
15
10
5
0
0.5
1
1.5
2
2.5
3
ψ
$\alpha_0 = \frac{1}{5}\beta_0$
$\alpha_0 = \frac{2}{5}\beta_0$
$\alpha_0 = \frac{3}{5}\beta_0$
$\alpha_0 = \frac{4}{5}\beta_0$

Lemaitre-Tolman-Bondi

- Even more general, anisotropic
- alternative cosmological model, non-simultaneous BB
- usually formulated for dust and in "r" coord.:

$$ds^2 = -dt^2 + \frac{\left(R(t,r)_{,r}\right)^2}{1 + 2\,E(r)}\,dr^2 + R(t,r)^2 d\Omega^2$$

Instead:

- assume spherical ansatz, Λ-vacuum:

$$ds^2 = -dt^2 + f(t,\psi)^2 dr^2 + g(t, psi)^2 d\Omega^2$$

EFE $\Longrightarrow$

$$f(t,\psi) = \frac{\alpha_{,\psi}\ \exp\left(\sqrt{\Lambda}\, t\right) + \beta_{,\psi}\ \exp\left(-\sqrt{\Lambda}\, t\right)}{\sqrt{1 - 4\,\Lambda\,\alpha\,\beta}}$$

$$g(t,\psi) = \alpha\ \exp\left(\sqrt{\Lambda}\, t\right) + \beta\ \exp\left(-\sqrt{\Lambda}\, t\right)$$

- $\alpha(\psi),\ \beta(\psi)$ free functions ! (+ open conditions)
- can correspond to metric-FLRW and conformally-deformed-FLRW :

$$\alpha\ ,\ \beta \sim \sin(\psi)$$

and

$$\alpha\ ,\ \beta \sim \frac{1}{\alpha_0\ \cos\psi + \beta_0}\ \sin\psi$$

Strictly-near constraints for arbitrary $\alpha,\ \beta$:

- Explicit solutions exist !!

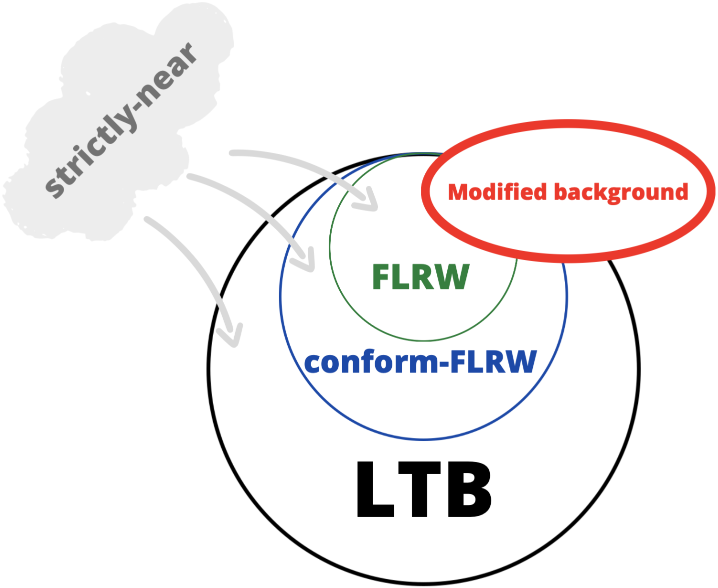
$$\boldsymbol{K}(\psi) \sim \frac{\alpha\ \exp\left(2\sqrt{\Lambda}\,t\right)\ -\beta}{\alpha\ \exp\left(2\sqrt{\Lambda}\,t\right)\ +\beta} + \frac{C_{\boldsymbol{K}}}{\alpha\ \exp\left(2\sqrt{\Lambda}\,t\right)\ +\beta}$$

$$\widehat{N}^2 = \frac{\left(\alpha\,\beta_{,\psi}^2 + 2\alpha_{,\psi}\,\beta\,\beta_{,\psi}\right)e^{2\mu\,t} + \left(2\,\alpha\,\beta_{,\psi}\,\alpha_{,\psi} + \beta\,\alpha_{,\psi}^{\,2}\right)\ e^{4\mu\,t}}{\mathcal{A}\,e^{2\mu\,t} + \alpha\,\mathcal{B}\,e^{4\mu\,t} + \left(C_{\widehat{N}} - C_{\boldsymbol{K}}\,\mu\,\beta^2 + \frac{C_{\boldsymbol{K}}}{4}\,\beta\right)} +$$

$$+\frac{\alpha_{,\psi}^{\,2}\,\alpha\,e^{2\mu\,t} + \beta_{,\psi}^{\,2}\,\beta}{\mathcal{A}\,e^{2\mu\,t} + \alpha\,\mathcal{B}\,e^{4\mu\,t} + \left(C_{\widehat{N}} - C_{\boldsymbol{K}}\,\mu\,\beta^2 + \frac{C_{\boldsymbol{K}}^2}{4}\,\beta\right)}$$

$$\mathcal{A} = \beta+\alpha\left(\frac{C_{\boldsymbol{K}}}{4}-4\mu^2\beta^2\right) \qquad \mathcal{B} = \alpha\,C_{\boldsymbol{K}}\,\mu-16\,\alpha\,\beta\,\mu^2+4 \qquad \mu \equiv \sqrt{\Lambda/3}$$

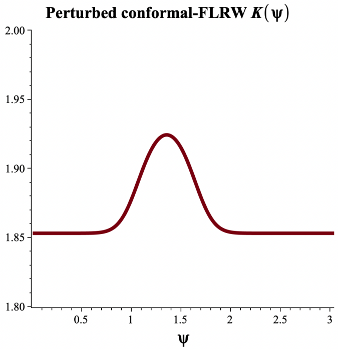
- 2 free functions and 2 constants
- can analytically construct lot of initial data sets

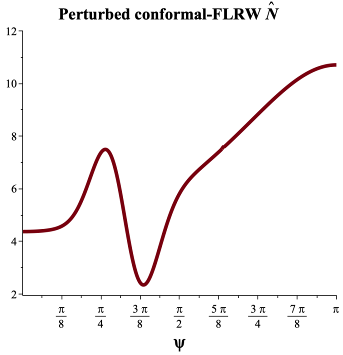


Can choose to have small deviations from known solutions:

- "perturbed" conformally-deformed FLRW:

$$\alpha = \alpha_{CD-FLRW} + \{\text{compact gaussian}\} \qquad \beta = \beta_{CD-FLRW}$$





- gaussian, compact bump function, extra periodic term ...

Summary & Outlook

- novel method of solving the constraints
- applied to compact initial data (sparse literature)
- produces interesting solutions, a proof of concept so far

Violating the spherical symmetry:

- fills Σ with gravitational waves
- time evolution ?

- Rácz: Constraints as evolutionary system (C.Q.G **33** 015014)
- Beig et al.: Vacuum initial data on $\mathbb{S}^3$ from Killing vectors (C.Q.G **36** 215017)
- Beyer, Frauendiener, Ritchie: Asymptotically flat vacuum initial data sets from a modified parabolic-hyperbolic formulation of the Einstein vacuum constraint equations (Phys.Rev.D **101**, 084013)