The Schrödinger-Newton-Hooke equation — existence and stability of the bound states

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Plan

- Motivation
- Existence of the bound states
- Stability of the bound states

$$\begin{cases} i\partial_t \psi &= -\Delta\psi + |\mathbf{x}|^2\psi + v\psi \\ \Delta v &= |\psi|^2 \end{cases}$$

$$i\partial_t\psi = -\Delta\psi + |\mathbf{x}|^2\psi - \left(\int_{\mathbb{R}^d} \frac{|\psi(t,y)|^2}{|x-y|^{d-2}}dy\right)\psi$$

Schrödinger-Newton-Hooke (SNH) equation

- Atomic physics
- Plasma physics
- Solid state physics
- Quantum gravity

+

- An example of a spatially confined system
- Investigations of the AdS stability

(Energy) critical dimension is *d*=6.

Motivation

Dynamics of spatially confined systems

Soliton resolution conjecture

for generic initial data, after a sufficiently long time the solution separates into a collection of decoupled solitons and radiation escaping to infinity

Energy cascade

the energy initially focused in the lower modes is gradually moving to the higher ones

the transfer of energy to higher modes is usually accompanied by the concentration of mass in lower modes

SEPARATION IN MOMENTUM SPACE

Motivation

Stability of the anti-de Sitter spacetime

Equations

Einstein equation $$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Klein-Gordon equation $$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \frac{m^2c^2}{\hbar^2}\phi = 0$$

$$T_{\mu\nu} = \frac{\hbar^2}{2m}\left[\partial_\mu\phi\partial_\nu\bar{\phi} + \partial_\mu\bar{\phi}\partial_\nu\phi - g_{\mu\nu}\left(\partial_\lambda\phi\partial^\lambda\bar{\phi} + \frac{m^2c^2}{\hbar^2}|\phi|^2\right)\right]$$

Motivation

Stability of the anti-de Sitter spacetime

Ansatz

$$\phi(t,x) = e^{-\frac{imc^2}{\hbar}t}\, u(t,x)$$

$$ds^2 = -c^2\left(1 + \frac{2A(t,x)}{c^2}\right) dt^2 + \left(1 + \frac{2B(t,x)}{c^2}\right)\sum_{j=1}^{d}(dx^j)^2$$

Nonrelativistic Limit

$$\lim_{c\to\infty} \Lambda c^2 = -\frac{d(d-1)}{2}\,\omega^2$$

Result

$$\begin{cases} i\hbar\,\partial_t u & = -\frac{\hbar^2}{2m}\Delta u + \frac{1}{2}m\omega^2|x|^2 u + Vu \\ \Delta V & = \frac{8\pi G(d-2)}{d-1}|u|^2 \end{cases}$$

Existence

$$i\partial_t \psi = -\Delta\psi + |\mathbf{x}|^2\psi - \left(\int_{\mathbb{R}^d} \frac{|\psi(t,y)|^2}{|x-y|^{d-2}} dy\right)\psi$$

Stationary state
$\psi = e^{-i\omega t} u$

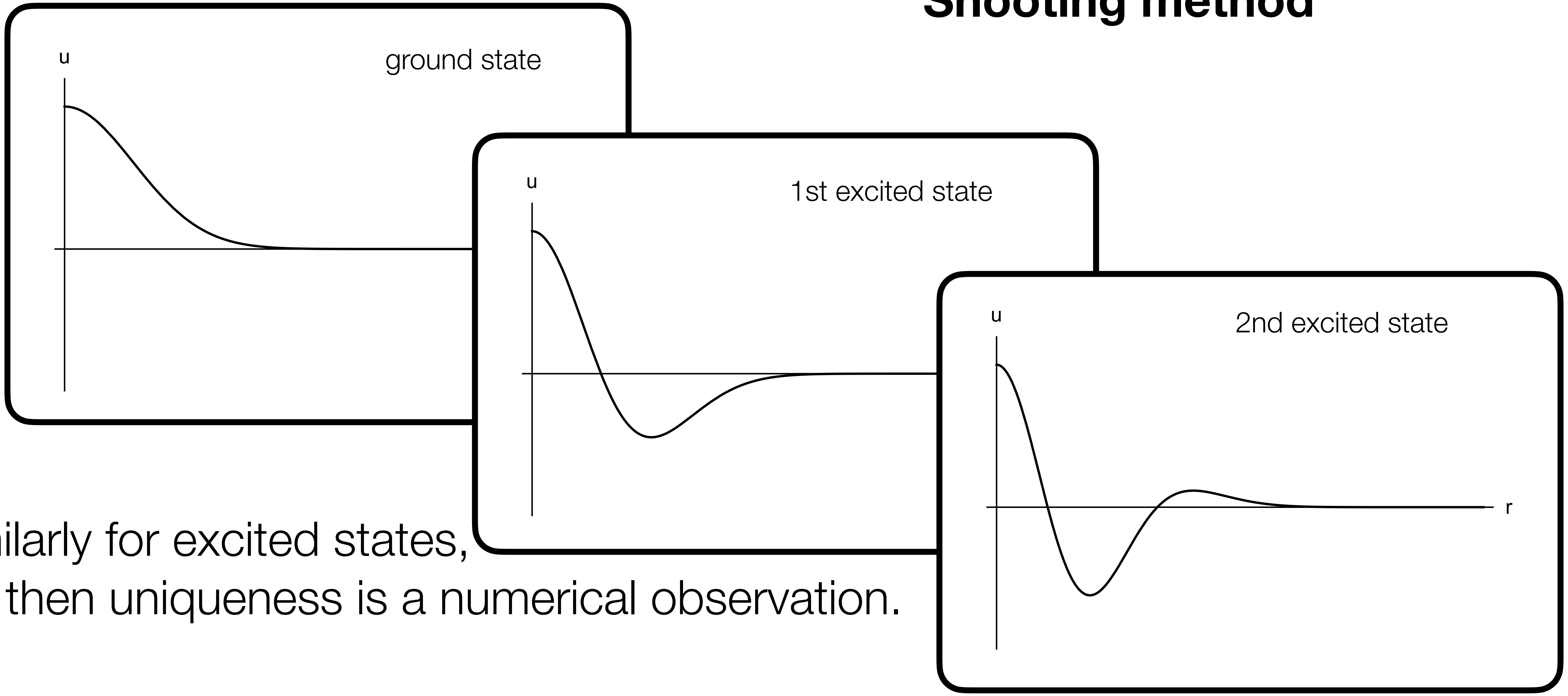
Spherical symmetry

$$\omega u = -u'' - \frac{d-1}{r}u' + r^2 u - \frac{1}{d-2}\left(\int_0^\infty \frac{u(s)^2\, s^{d-1}}{\max\{r^{d-2}, s^{d-2}\}} ds\right) u$$

Existence

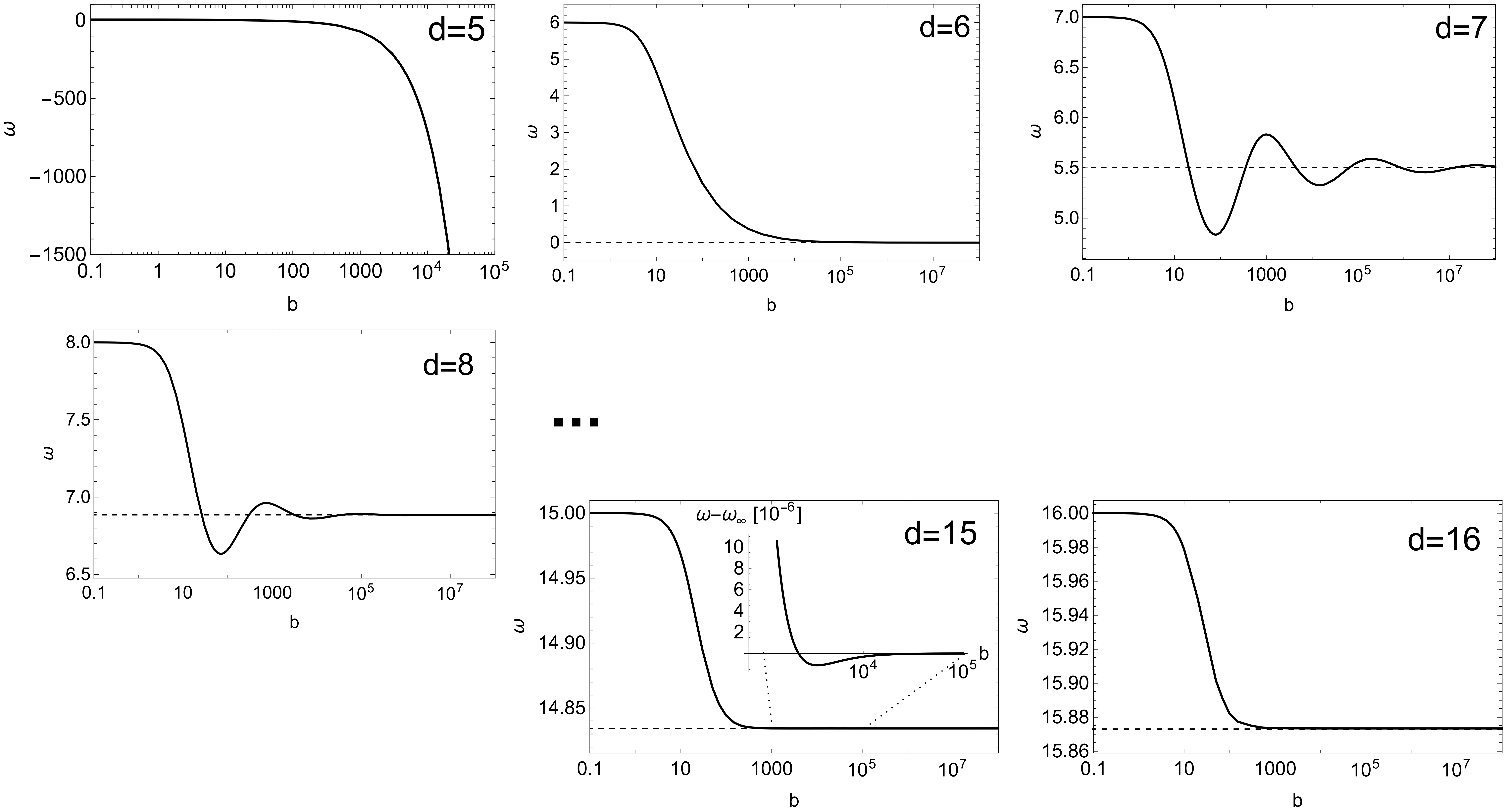
Let us fix $u(0)$. Then one can show that there exists a unique value of ω such that u is a ground solution.

Shooting method



Similarly for excited states,
but then uniqueness is a numerical observation.

Existence



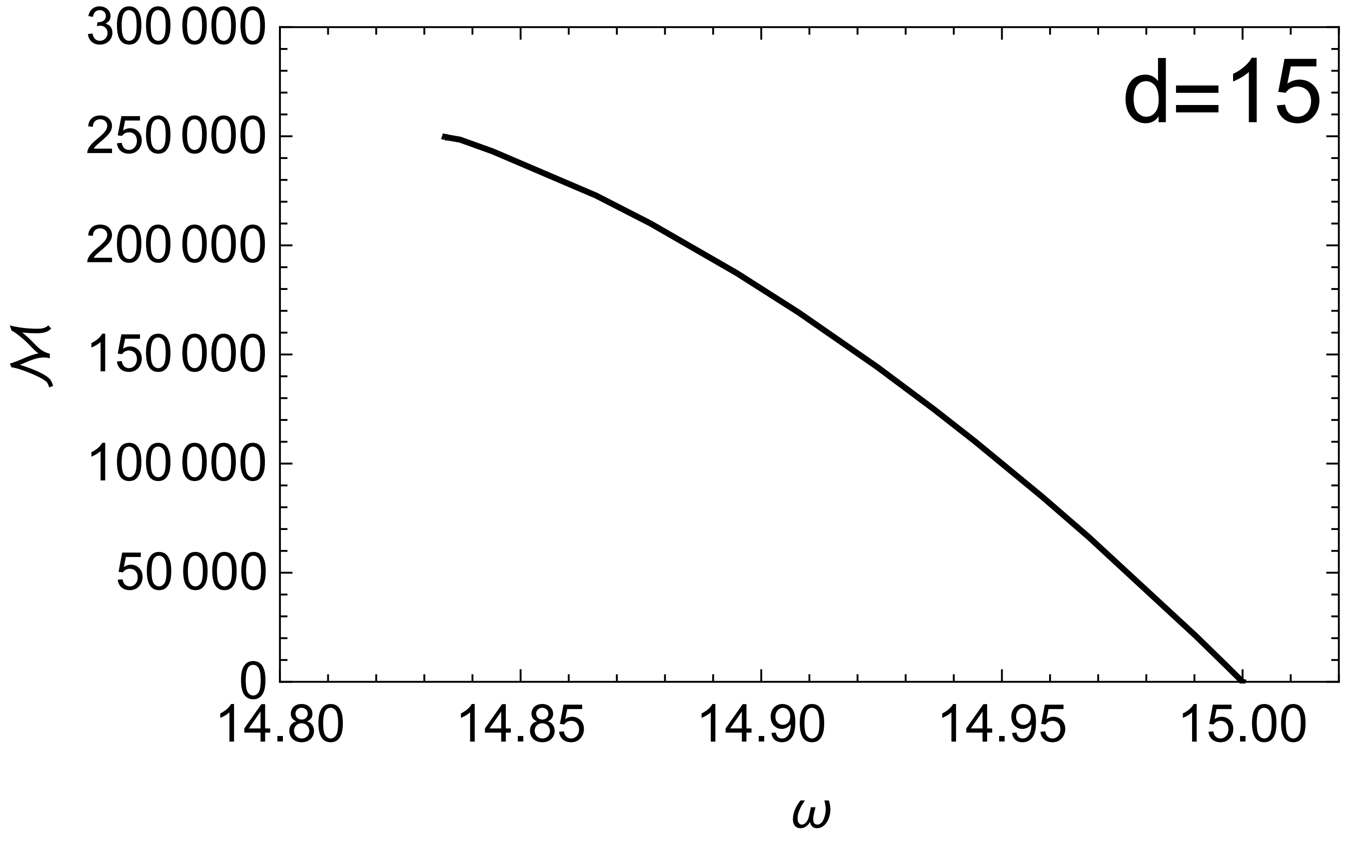
Stability

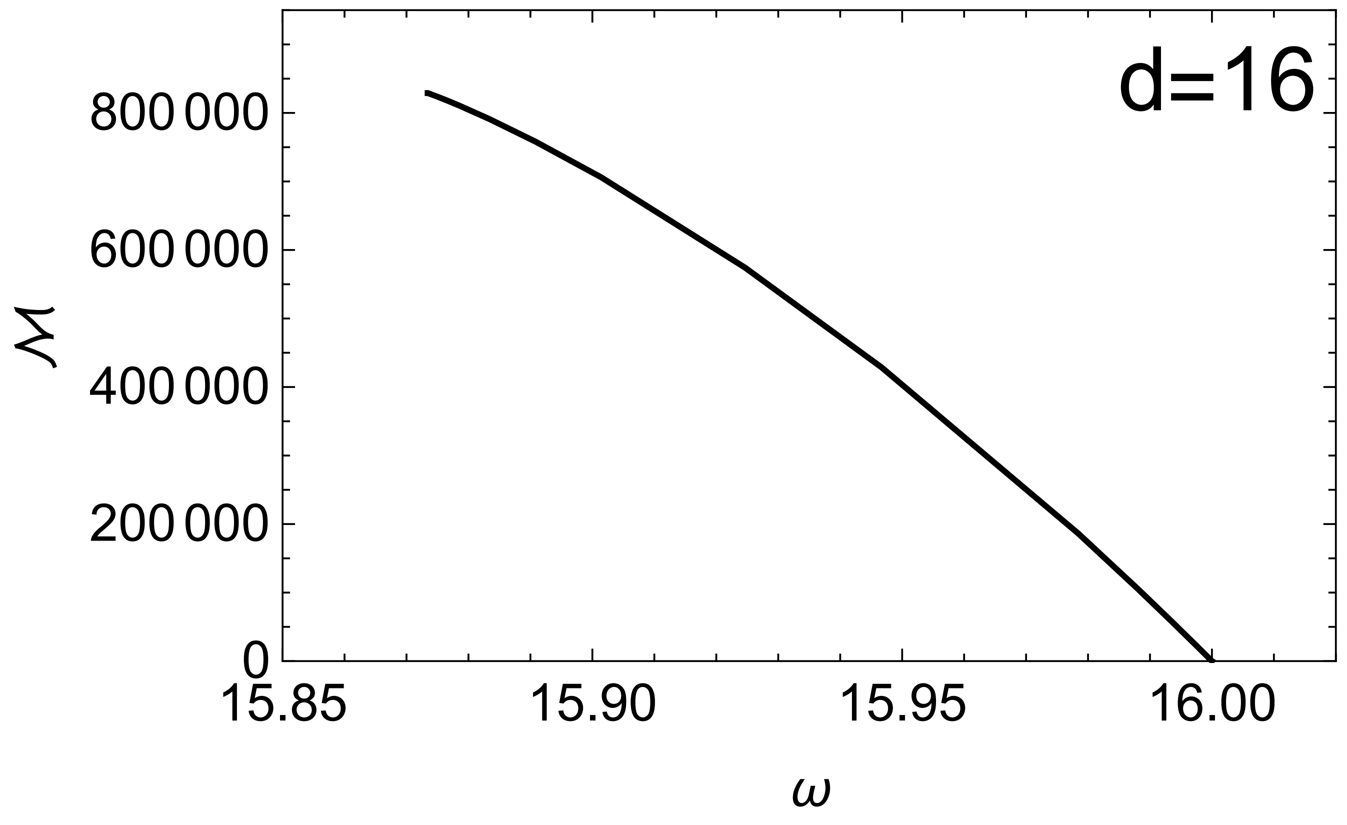
Vakhitov-Kolokolov criterion

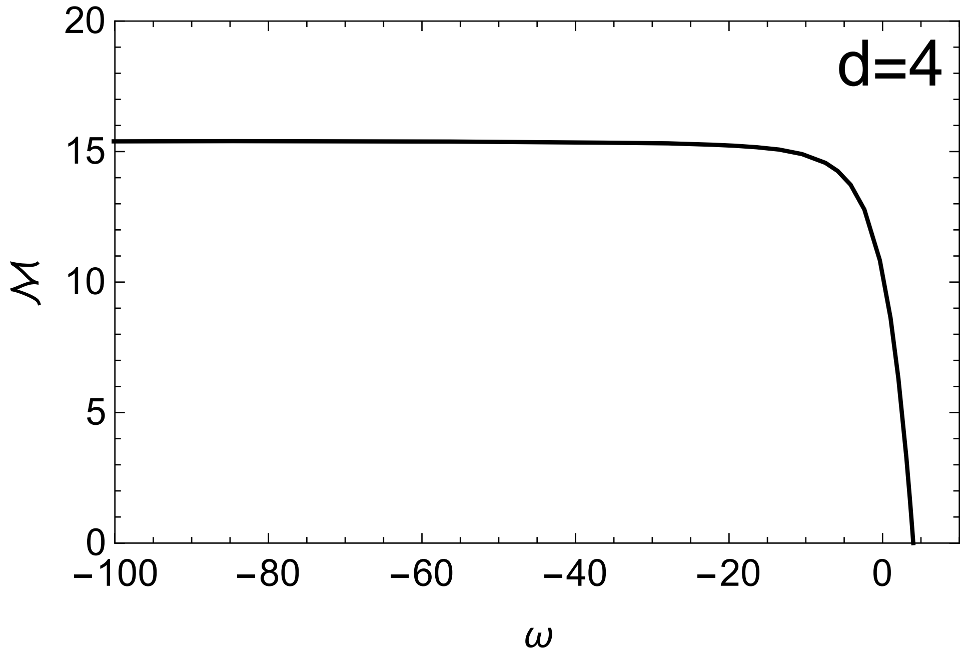
If under some technical conditions, it holds: if

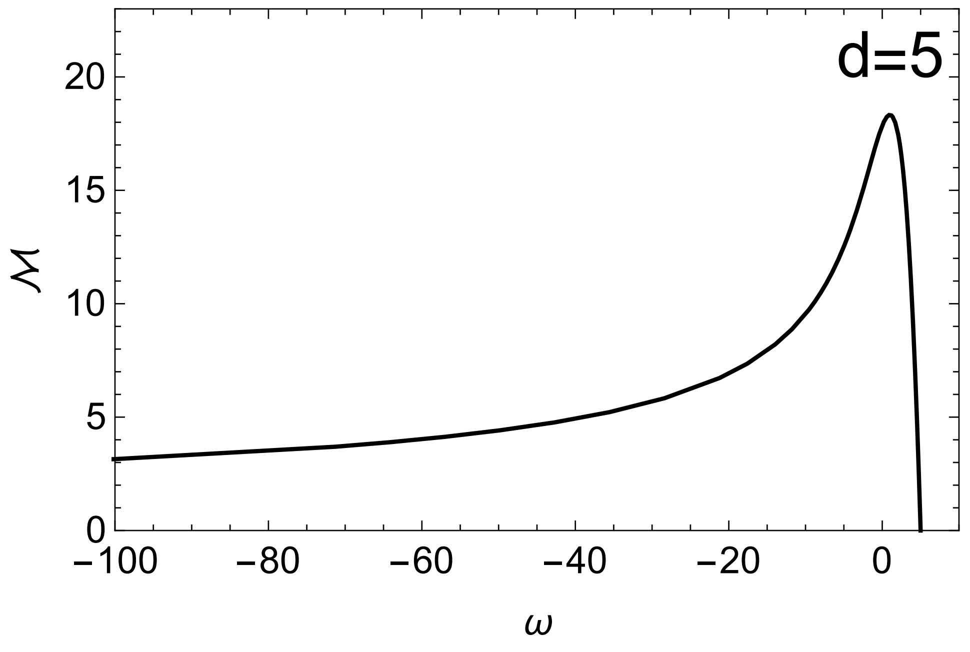
$$\mathcal{M}'(\omega) < 0$$

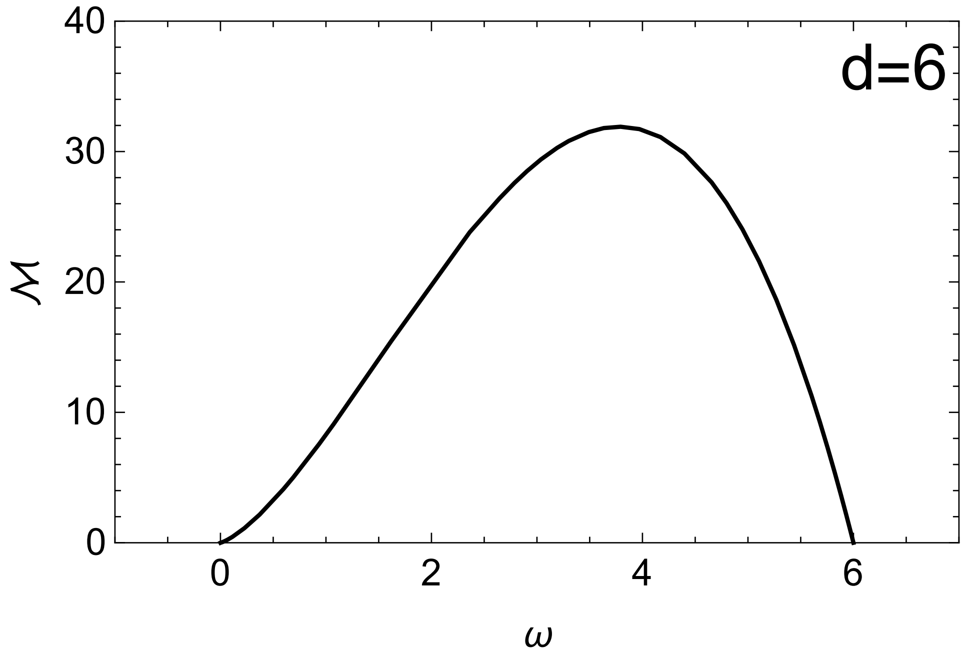
then this ground state is spectrally stable.

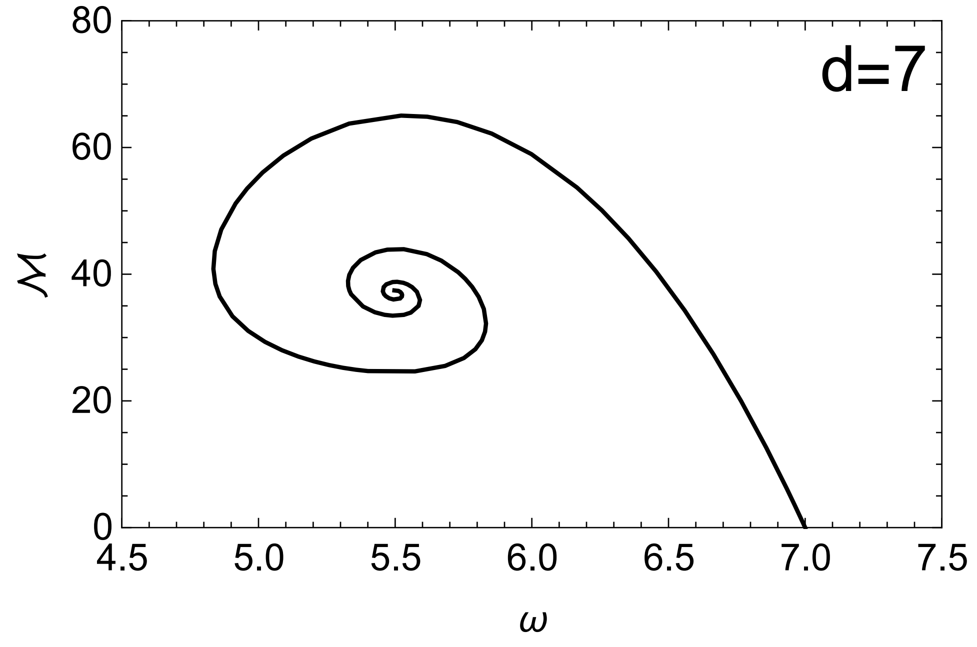












Stability

$$i\partial_t \psi = -\Delta\psi + |\mathbf{x}|^2\psi - \left(\int_{\mathbb{R}^d} \frac{|\psi(t,y)|^2}{|x-y|^{d-2}} dy\right)\psi$$

$$\psi(t,\mathbf{x}) = e^{-i\omega t}\left[u(\mathbf{x}) + f(t,\mathbf{x}) + ig(t,\mathbf{x})\right]$$

$$\partial_t f = -\Delta g + |\mathbf{x}|^2 g - \omega g + \left(\int_{\mathbb{R}^d} \frac{u(\mathbf{y})^2}{|\mathbf{x}-\mathbf{y}|^{d-2}} dy\right) g$$

$$\partial_t g = \Delta f - |\mathbf{x}|^2 f + \omega f - \left(\int_{\mathbb{R}^d} \frac{u(\mathbf{y})^2}{|\mathbf{x}-\mathbf{y}|^{d-2}} dy\right) f - 2\left(\int_{\mathbb{R}^d} \frac{u(\mathbf{y})f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d-2}} dy\right) u$$

Stability

$$\partial_t f = -\Delta g + |\mathbf{x}|^2 g - \omega g + \left(\int_{\mathbb{R}^d} \frac{u(\mathbf{y})^2}{|\mathbf{x}-\mathbf{y}|^{d-2}} dy \right) g$$

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$$- 2 \left(\int_{\mathbb{R}^d} \frac{u(\mathbf{y}) f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d-2}} dy \right) u$$

$$\partial_t \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

Stability

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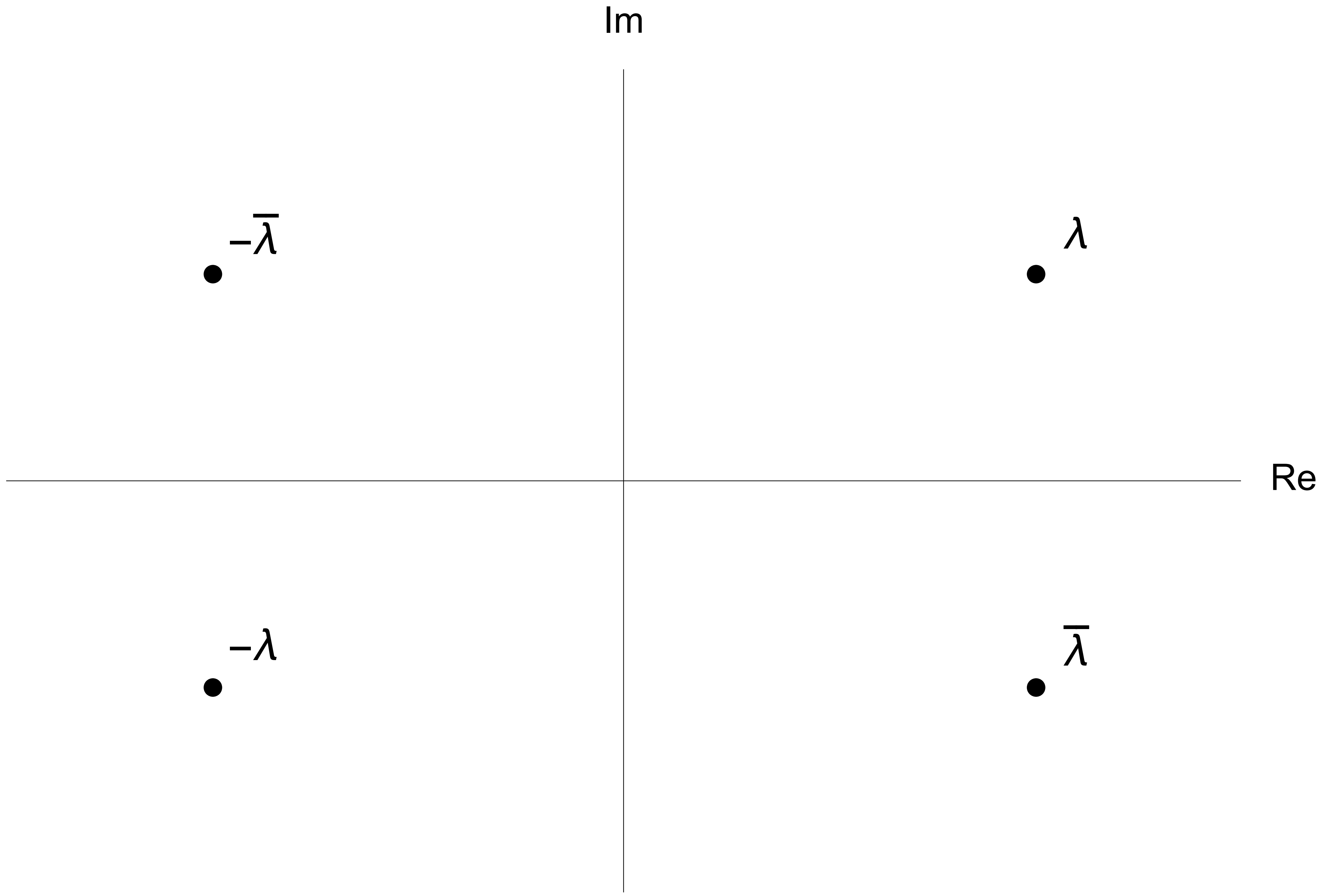
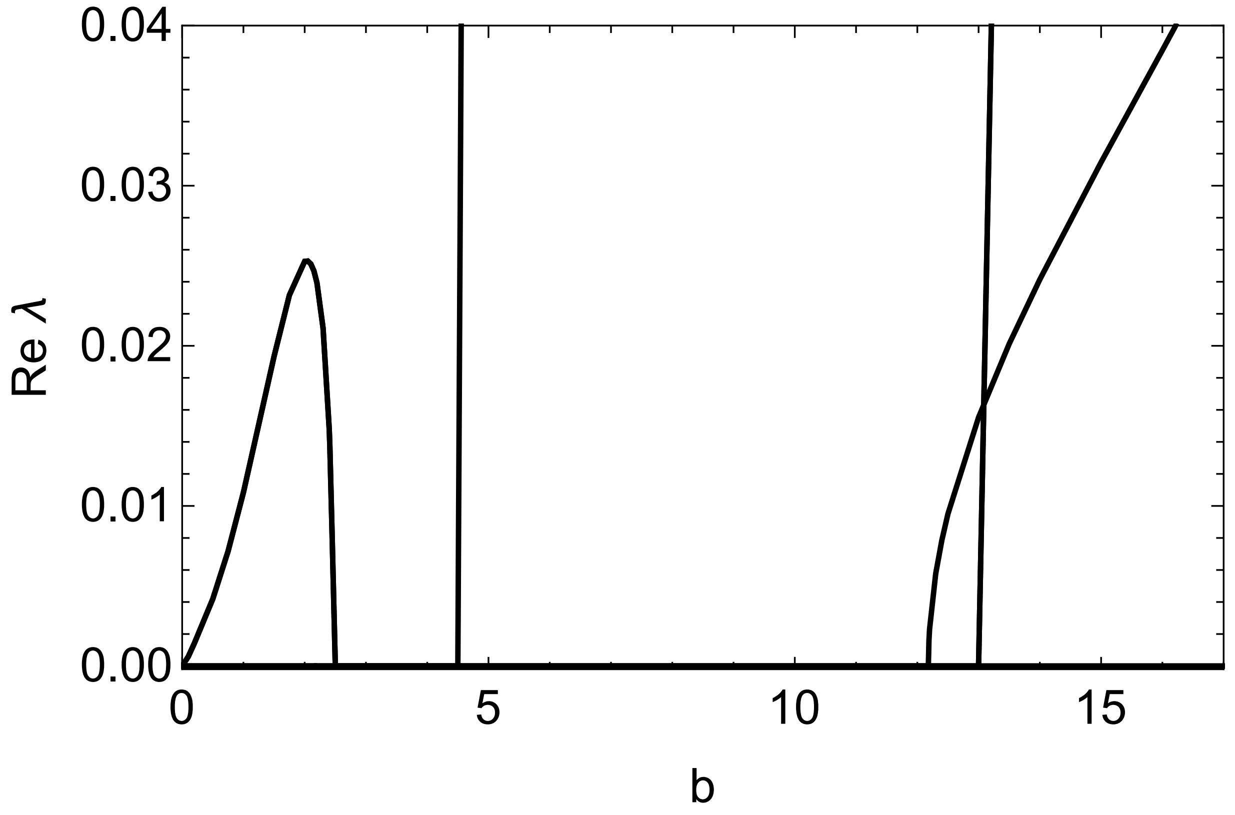


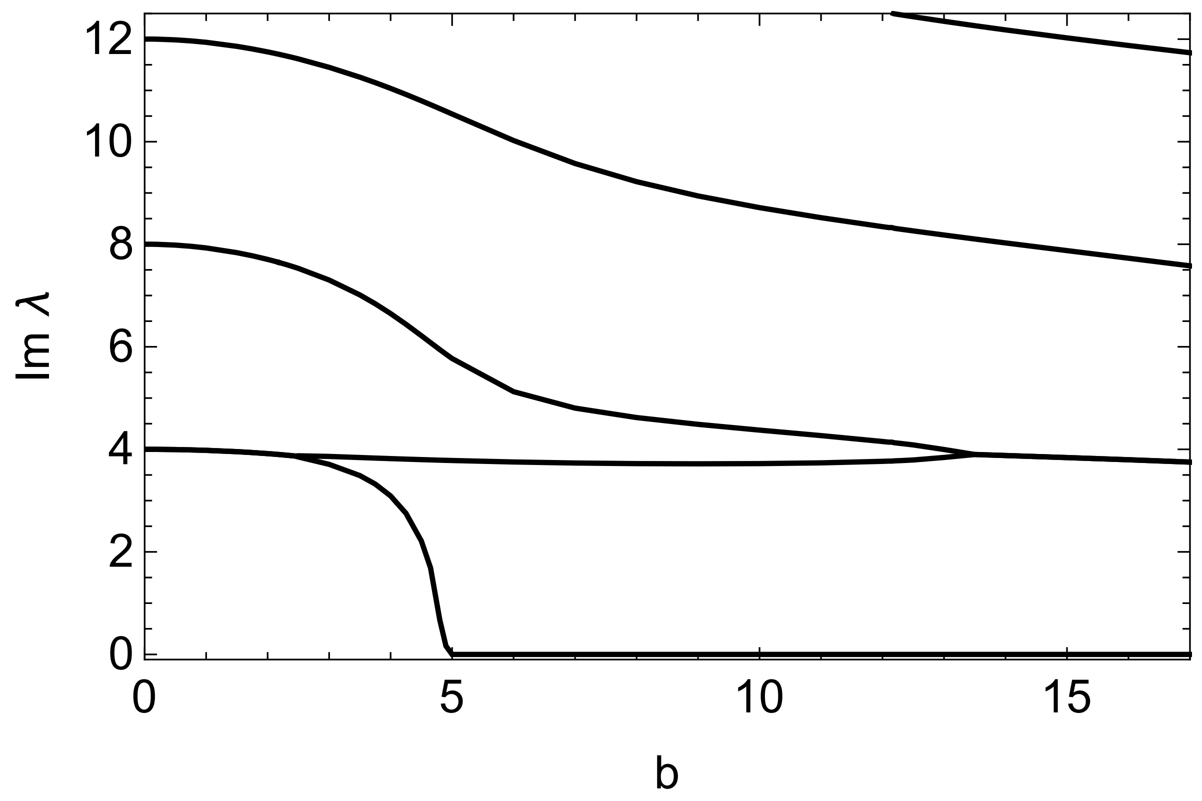
Stability

Gross-Pitaevskii equation $$i\partial_t \psi = -\Delta\psi + |x|^2\psi - |\psi|^2\psi$$

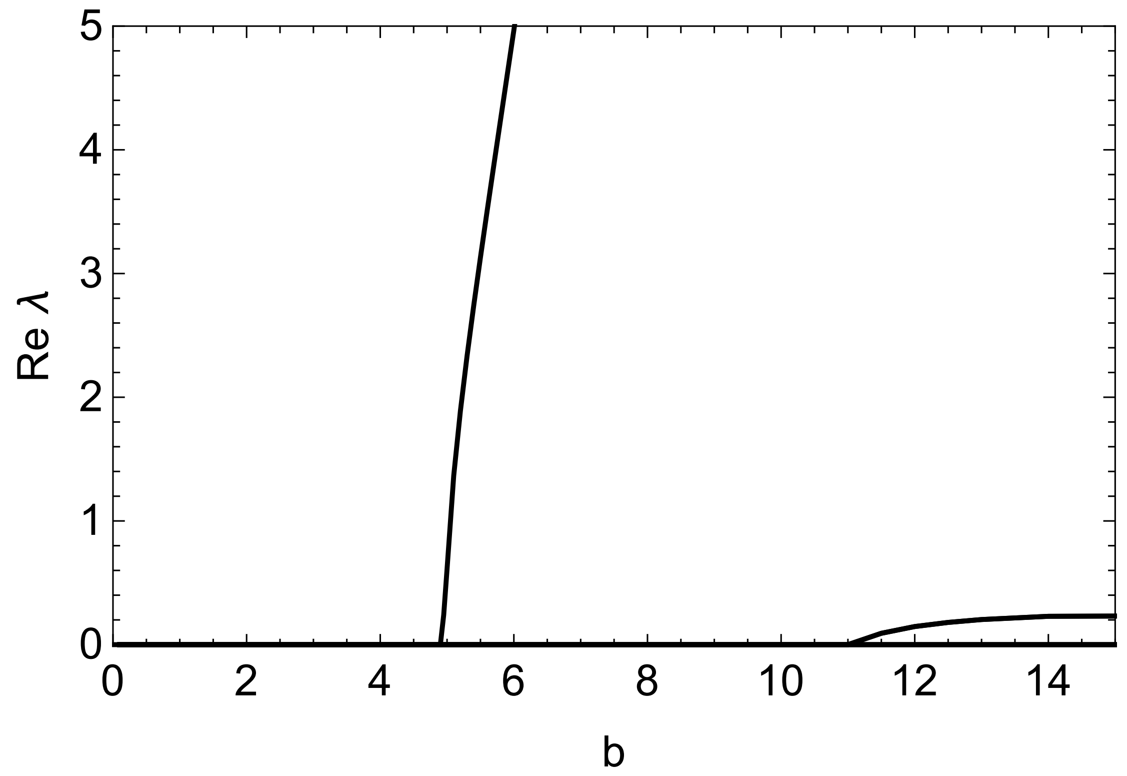
Critical dimension: d=4

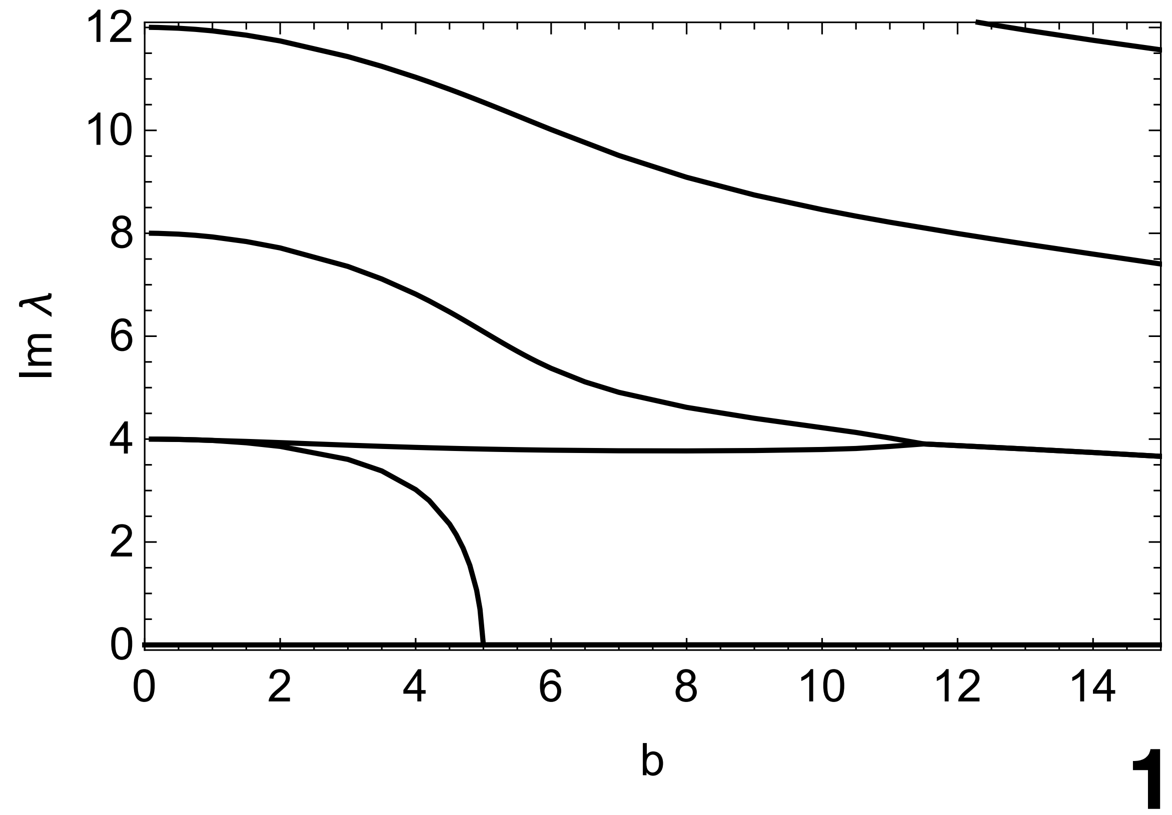
d=5
1st excited state





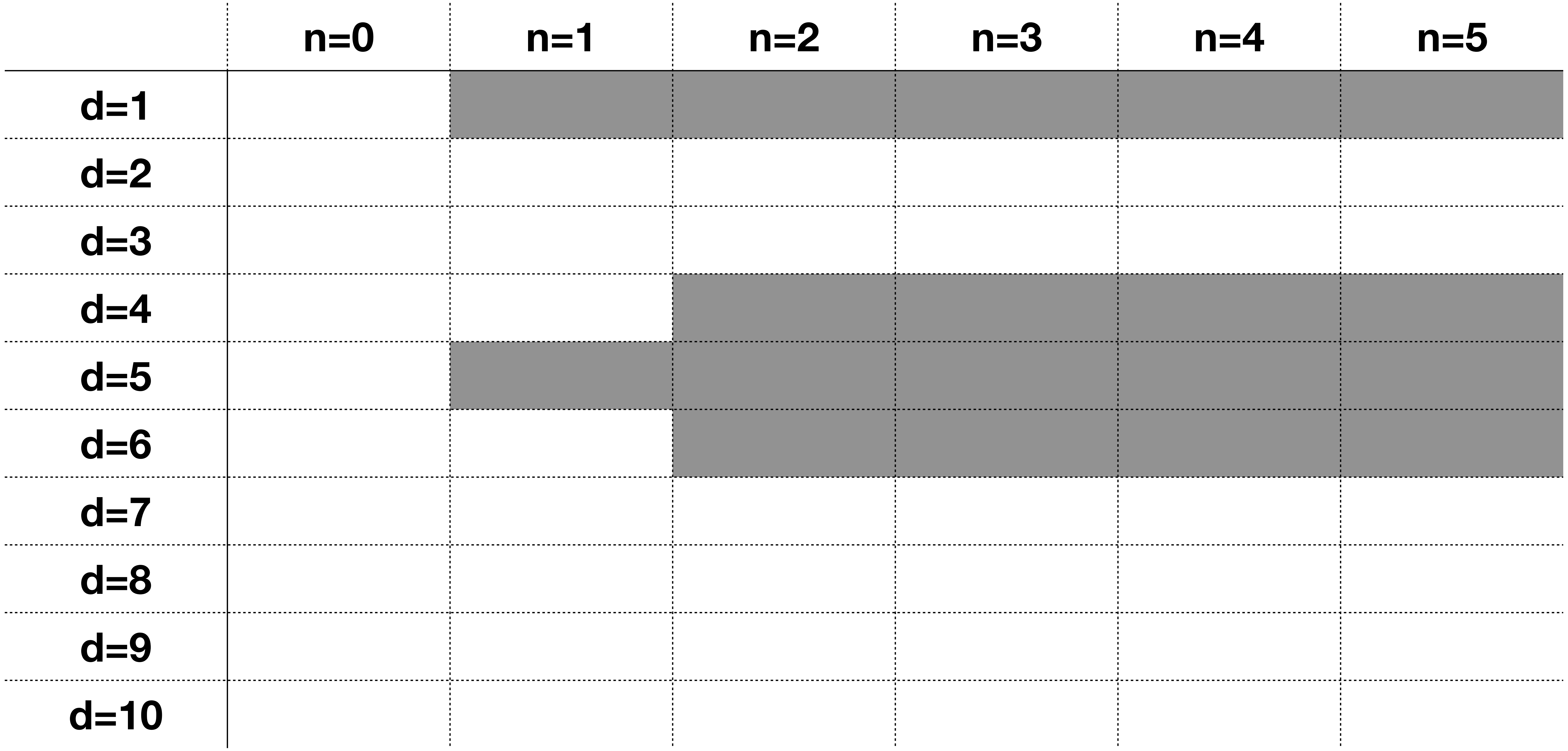
d=6
1st excited state





Stability

Instability for n-th excited state with small b



Summary

- SNH system can be obtained as a nonrelativistic limit of AdS perturbation.
- This system (similarly to other nonlinear Schrödinger equations) was not really studied in supercritical dimensions.
- One can show that for every $u(0)$ there exists a whole ladder of excited states with a unique ground state.
- There are interesting changes in behaviour of the system in dimensions 7 and 16.
- Stationary solutions of this system seem to have interesting stability properties, both for small and large initial data.
- **What is left:**
  Investigation of the full dynamical evolution.

Thank you for your attention!

The End