Radial and non-radial oscillation modes of compact stars

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Related papers: 1908.02808 & 1904.00907

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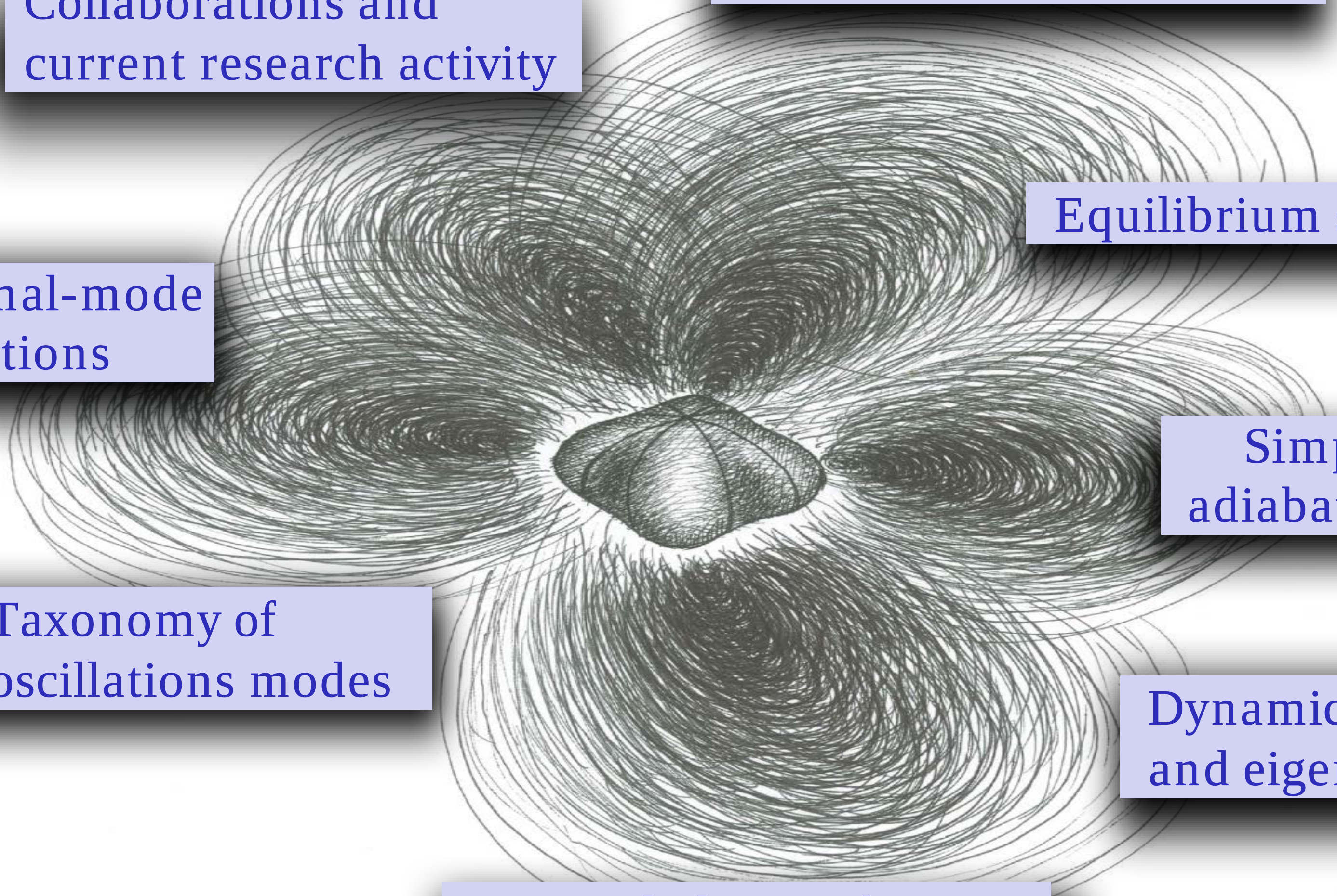


- Sources of NS oscillation
- Collaborations and current research activity
- Equilibrium stellar models
- Polar normal-mode oscillations
- Simplest case: Linear adiabatic radial oscillation
- Taxonomy of oscillations modes
- Dynamical stability and eigenfrequencies
- Non-radial perturbations & their parity

Sources of oscillations

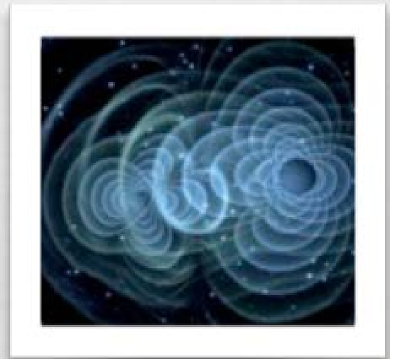
- **Supernova explosion:** triggers all kinds of oscillation modes

- **Starquakes** caused by craks in the crust or magnetic reconfiguration

- **Accretion** triggers oscillations

- **Tidal forces** in binary mergers

- **Oscillation modes** are *unstable* to gravitational wave emission → ***r*-mode** or ***f*-mode** oscillations



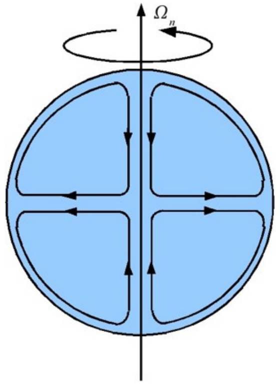
Neutron stars as GW sources

“Burst” emission

Binary neutron star mergers
(our safest bet for detection)

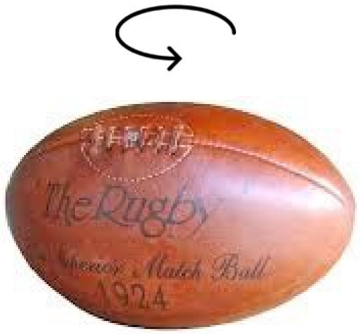
Magnetar flares
(likely too weak)

Pulsar glitches
(likely too weak)

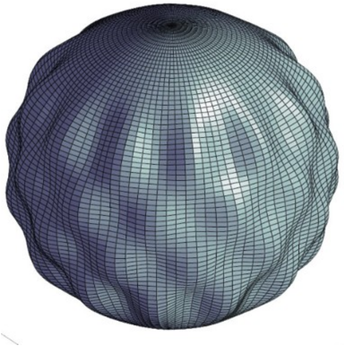


Continuous emission

Non-axisymmetric mass
quadrupole (“mountains”)



Fluid part (oscillations)



EQUILIBRIUM STELLAR MODEL

Einstein equations: $G_{\mu\nu} = 8\pi T_{\mu\nu}$
Energy–momentum conservation: $\nabla_\mu T^{\mu\nu} = 0$

Energy–momentum tensor (perfect fluid):

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}$$

The energy density and the pressure of the fluid are related by an ***equation of state***: $p = p(\rho)$ (zero temperature)

THERMODYNAMIC PROPERTIES

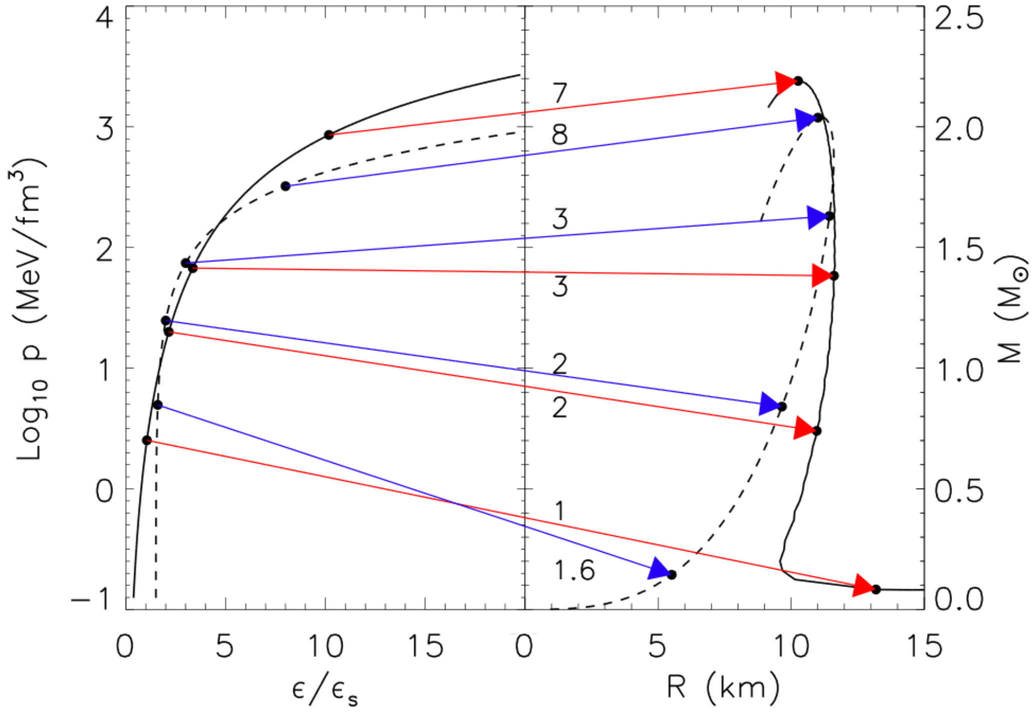
Gravitational mass: $\frac{dm}{dr} = 4\pi r^2 \rho$

Gravitational potential: $\frac{d\nu}{dr} = \frac{2m+8\pi r^3 p}{r(r-2m)}$

Hydrostatic equilibrium: $\frac{dp}{dr} = -\frac{(\rho+p)(m+4\pi r^3 p)}{r^2(1-2M/r)}$

(Tolman–Oppenheimer–Volkoff equation)

STRUCTURE



At the stellar center ($r = 0$):
- $M(0) = 0$: the mass function vanish
- $\rho_0 \equiv \rho(0)$: central density is freely specified

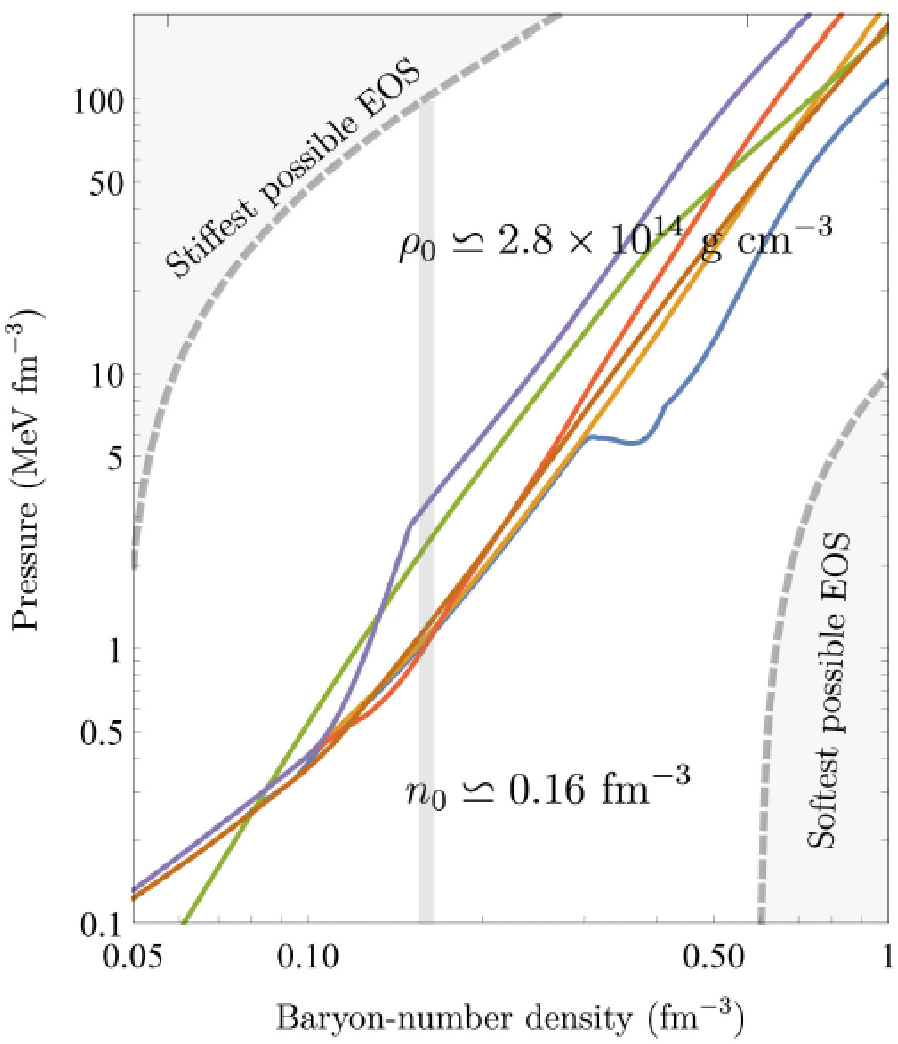
At the stellar surface ($r = R$):
- $M \equiv m(R)$: total mass of the star
- $p(R) = 0$: the isotropic pressure vanishes
- $e^{\nu(R)} = 1 - 2M/R$: normalizing the time coordinate at spatial infinity

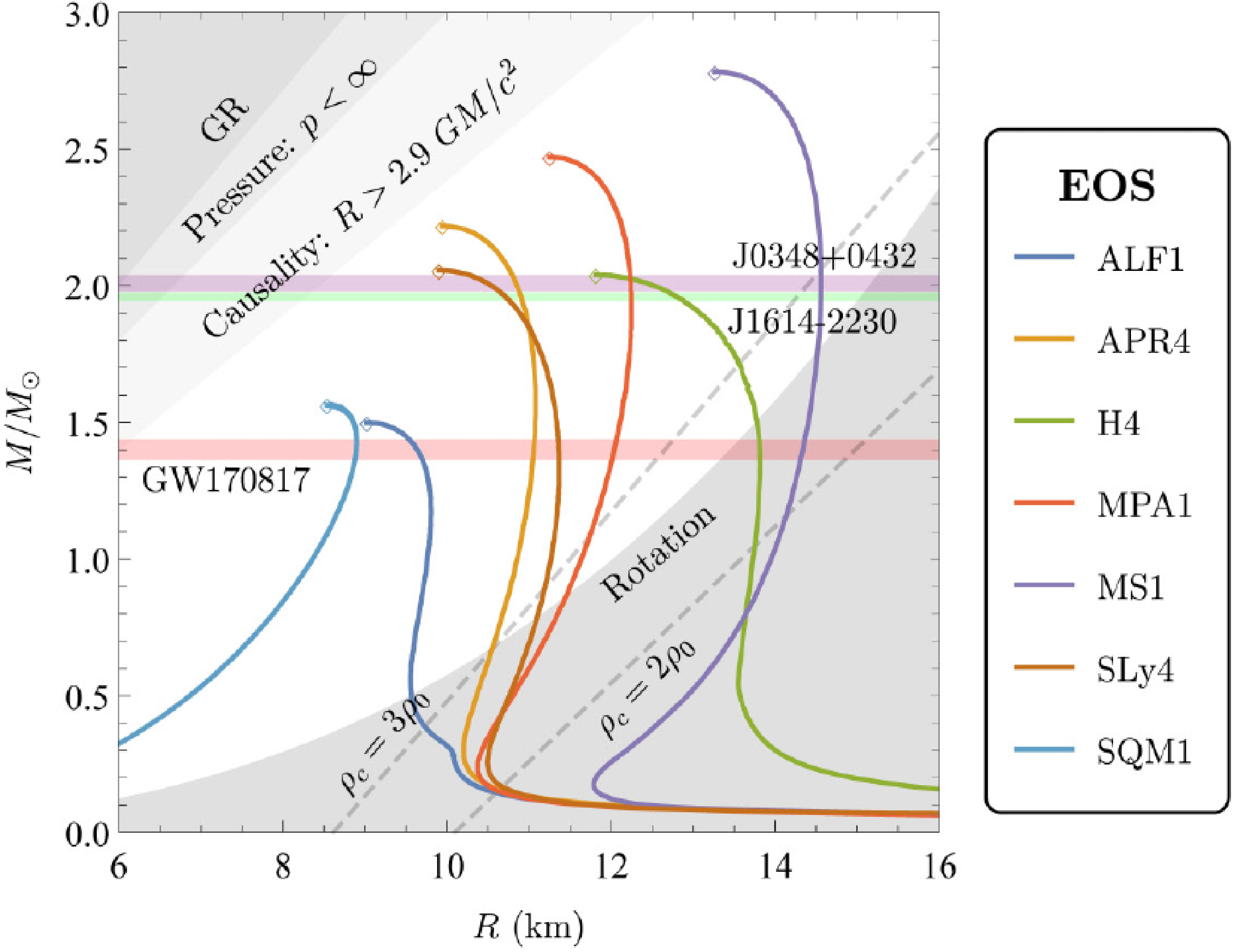
BOUNDARY CONDITIONS

Metric tensor: $(ds^2)_0 = -e^\nu dt^2 + e^\lambda dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$
where $m(r) \equiv r(1 - e^{-\lambda})/2$ is the „gravitational mass” inside radius r
Comoving coordinates: $u^\mu = e^{\nu/2}[-1,0,0,0]$
Normalization condition: $g_{\mu\nu} u^\mu u^\nu = 1$

FLUID 4-VELOCITY

REALISTIC TABULATED EOS MODELS AND ASSOCIATED NEUTRON STARS





Total gravitational mass–radius relations.

SIMPLEST CASE: LINEAR ADIABATIC RADIAL OSCILLATIONS

- **Perturbations in the fluid 4-velocity** can be expressed by

$$\delta u^{\mu}_{\text{radial}} = [e^{\nu_0/2}, -e^{\lambda_0-\nu_0/2}\delta u_r, 0,0] \quad \text{where} \quad \delta u_r = dr/dt$$

is associated with a displacement field in the Lagrangian representation: $\partial\xi/\partial t = \delta u_r$.

1. The perturbation equations are obtained from $\delta\left(\nabla_\mu T^{\mu\nu}\right) = 0,\ \delta(G_{\mu\nu} - 8\pi T_{\mu\nu}) = 0$. Then, it is straightforward to compute the linear perturbations of any equilibrium quantity ($\delta\rho, \delta p$, ...).

2. With the assumption of harmonic time dependence, Chandrasekhar (1964) showed that:

FUNDAMENTAL EQUATION FOR RADIAL PULSATION

$$\omega^2 e^{\lambda_0-\nu_0}(p_0+\varepsilon_0)\xi = \left[\frac{4}{r}\frac{dp_0}{dr} + 8\pi e^{\lambda_0}p_0(p_0+\varepsilon_0) - \frac{1}{p_0+\varepsilon_0}\left(\frac{dp_0}{dr}\right)^2\right]\xi - e^{-(\lambda_0+2\nu_0)/2}\frac{d}{dr}\left[e^{(\lambda_0+3\nu_0)/2}\frac{\Gamma p_0}{r^2}\frac{d}{dr}\underbrace{\left(r^2 e^{-\nu_0/2}\xi\right)}_{X}\right]$$

BOUNDARY CONDITIONS

a) The fluid at the center of the star is assumed to remain at rest: $X = 0$ at $r = 0$

b) The Lagrangian change in the pressure vanishes at the surface: $\delta p = e^{\nu_0/2}r^{-2}\Gamma p_0\frac{d}{dr}\left(r^2 e^{-\nu_0/2}X\right) \equiv 0$ at $r = R$

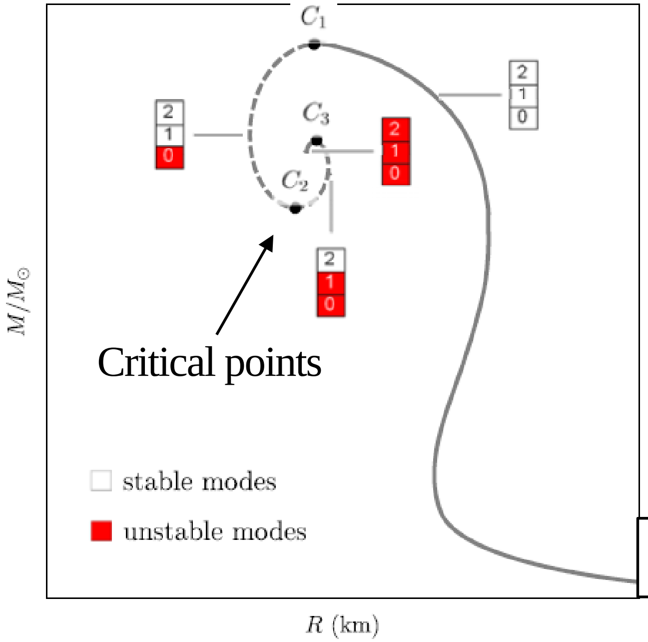
The fundamental equation together with its boundary conditions constitutes a Sturm–Liouville eigenvalue problem (SL-EVP) for a discrete set of scalar-valued eigenfunctions of radial displacement $\{X_0(r), X_1(r), \dots, X_j(r), \dots\}$ with their respective eigenvalues $\{\omega_0^2, \omega_1^2, \dots, \omega_j^2, \dots\}$.

3. To find the eigenfrequencies, we convert the boundary value problem to an initial value problem by "shooting" method!

DYNAMICAL STABILITY

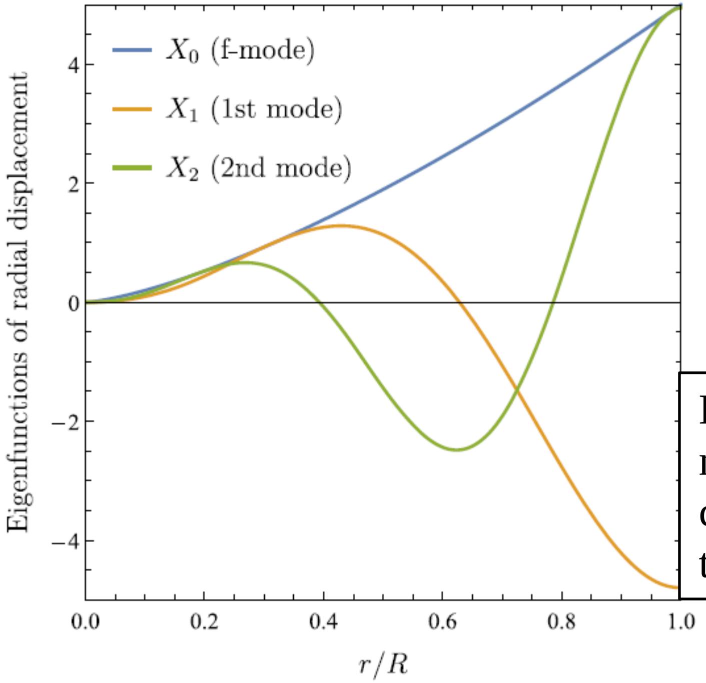
- ***If any of these*** ω_j^2 ***is negative*** for a particular star, the frequency is purely imaginary and therefore any perturbation of the star ($\sim e^{i\omega t}$) will grow exponentially in time. ⇨ ***dynamically unstabile***

- If $\omega_j^2 > 0$, ***the star is stable*** against adiabatic radial perturbations (up to the jth excited oscillation mode)

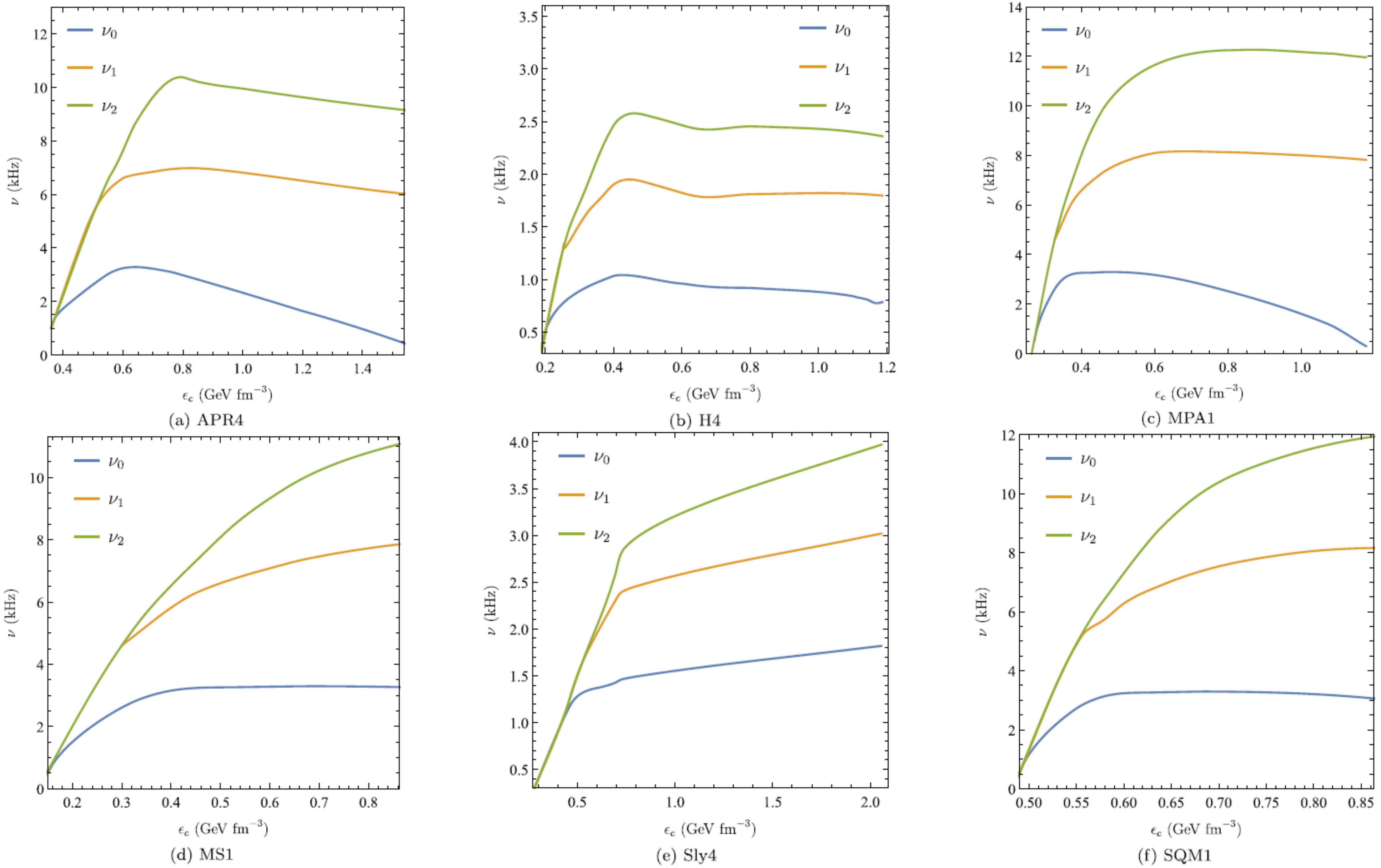


Schematic illustration of the unstable branch of the mass–radius relation. [Barta 2021, CQG **38**, 185002]

- The smallest eigenvalue ω_0^2 is associated with the *fundamental-mode frequency* of radial oscillations which *has no nodes* between the center and the stellar surface, whereas the first excited mode ($j = 1$) has a node, the second one ($j = 2$) has two, and so forth.



Eigenfunctions functions of radial displacement for the first three lowest-frequency oscillation modes $\{X_0(r), X_1(r), X_2(r)\}$ as a function of the fractional radius r/R obtained for SLy4 EoS at a central density $\rho_c = 0.547$ GeV fm^{-3}.The displacement amplitude has been renormalized such that $X_0 = 1$. [Barta 2021, CQG **38**, 185002]



The frequencies of the fundamental mode (ν_0) and the first two lowest-frequency excited modes (ν_1 and ν_2) of radial oscillation as functions of central density (ε_c) for each EoS of nucleonic state (APR4, MPA1, MS1, SLy4) and hybrid nucleon–hyperon–quark state (H4, SQM1).

INTERPRETATION OF RESULTS (RADIAL OSCILLATION)

1. ***The decay of the lowest-frequency eigenmodes is a general feature*** (irrespective of the particular EoS): The f-mode frequency drops toward zero as the particular stellar model approaches its dynamical stability limit which, indeed, is indicated by the presence of an eigenmode with zero-frequency.
   - The dynamical instability in stars with MPA1 and APR4 is exposed by the presence of a very low frequency of the *f*-mode, which has dropped to less than 5% of that of the first excited mode, at central energy densities associated with the maximal-mass stable configurations.

2. ***The oscillation frequency of higher modes is always larger than that of a lower stable mode*** and for all modes it appears to decrease as the central energy density approaches the smallest possible value $\varepsilon_{\mathrm{min}}$ of the particular stellar model
   - When the central energy density of NSs is approaching $\varepsilon_{\mathrm{min}}$, such compact objects *become approximately homogeneous* and due to their small mass.

3. ***Stellar models of softer EoSs have higher frequencies in the f-mode than the stiffer ones for the same central density***.
   - Stellar models of softer EoSs are generally associated with more centrally condensed stars with larger average densities.

NON-RADIAL FLUID DISPLACEMENT AND PERTURBATION

- The perturbation equations are obtained in the following way:

$$\delta\left(\nabla_\mu T^{\mu\nu}\right) = 0, \qquad \delta\left(G_{\mu\nu} - 8\pi T_{\mu\nu}\right) = 0$$

- **Perturbations** in the **4-velocity of a fluid element** δu^μ (associated with a mode) can be decomposed in radial and angular parts:

$$\delta \boldsymbol{u} = \sum_{l,m} \Big[\underbrace{W_l \hat{\mathbf{r}} + V_l \nabla Y_m^l}_{\textit{polar}\text{ part: parity }(-1)^l} + \underbrace{U_l(\hat{\mathbf{r}} \times \nabla Y_m^l)}_{\textit{axial}\text{ part: parity }(-1)^{l+1}}\Big] e^{i\omega t}$$

Parity is defined to be the ***change in sign*** under a combination of reflection in the equatorial plane and rotation by π.

where $W_l(r),\ V_l(r),\ U_l(r)$ are radial eigenfunctions.

➢ A linear perturbation of scalar quantities ($\delta\rho, \delta p$, etc.) can be written *as a sum of quasi-normal modes* that are characterized by the indices (l,m) of the spherical harmonic functions Y_m^l and harmonic time dependence $e^{i\omega t}$.

The **frequency** ω is a *complex number:*

1. real part corresponding to the frequency of oscillations:

   $\mathrm{Re}(\omega) = \omega_n\sqrt{1-\zeta_d^2}$, where ω_n is the natural frequency

2. imaginary part to the relaxation time: $1/\tau = \mathrm{Im}(\omega) = -\omega_n\zeta_d$

PARITY OF PERTURBATIONS

- A general non-stationary asymmetric spacetime:

$$ds^2 = -e^{\nu}dt^2 + e^{\mu_2}dr^2 + e^{\mu_3}d\theta^2 + e^{\psi}(d\varphi - \omega dt - q_2 dr - q_3 d\theta)^2$$

- Two different types (or parity) of perturbations of the spherically symmetric metric:
  1. **Polar (or "magnetic-type") perturbation** has "even parity" $\pi = (-1)^l$. It gives small increments to the already nonzero metric coefficients $(e^{\nu}, e^{\mu_2}, e^{\mu_3}, e^{\psi})$.
  2. **Axial (or "electric-type") perturbation** has "odd parity" $\pi = (-1)^{l+1}$. This perturbation induces ***frame dragging*** and ***imparts a rotation*** to the compact star. It gives small values to the metric coefficients (ω, q_2, q_3) that were zero in $(ds^2)_0$.

In non-rotating stars (i.e. up to $O(\Omega)$) the **polar** and **axial perturbations** remain completely ***decoupled***.

- ➢ Further more, for small-amplitude motions ***there is no coupling*** between the various spherical harmonics

- The geometry of spacetime inside and around the equilibrium configuration fluctuates in a manner described by 10 independent components ($h_{\mu\nu} = h_{\nu\mu}$).
  $\Rightarrow ds^2 = (ds^2)_0 + h_{\mu\nu} dx^\mu dx^\nu$
- The small-amplitude motion of the perturbed configuration is described by the Lagrangian displacements ξ^i.

The entire theory of non-radial pulsations consists of the study of the "equations of motion" which governs the 13 functions $\begin{cases} \xi^i(t,r,\theta,\varphi) - 3 \text{ Lagrangian displacement vector field} \\ \mathrm{h}_{\mu\nu}(t,r,\theta,\varphi) - 10 \text{ metric perturbation} \end{cases}$

- Using an appropriate gauge (Regge–Wheeler):

„Odd-parity" (or axial) mode: $\pi = (-1)^l\|_{l=2n+1}$	„Even-parity" (or polar) mode: $\pi = (-1)^l\|_{l=2n}$
$\xi_r = \xi_\theta = 0,\ \ \xi_\phi = U(r,t) \sin\theta\, \partial_\theta P_l\,(\cos\theta)$	$\xi^r = r^{-2} e^{-\lambda/2} W P_l,\ \xi^\theta = -r^{-2} V \partial_\theta P_l,\ \ \xi^\theta = 0$
$h_{\nu\mu}^{\mathrm{axial}} = \begin{pmatrix} 0 & h_1 & 0 & h_0 \\ h_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ h_0 & 0 & 0 & 0 \end{pmatrix} \sin\theta\, \partial_\theta P_l\,(\cos\theta)$	$h_{\nu\mu}^{\mathrm{polar}} = \begin{pmatrix} e^\nu H_0 & H_1 & 0 & 0 \\ H_1 & e^\lambda H_0 & 0 & 0 \\ 0 & 0 & r^2 K & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta\, K \end{pmatrix} Y_m^l$
Equation of motion: *fluid displacement* *metric perturbations* • set of coupled equations for $U(r,t)$ and $h_0(r,t), h_1(r,t)$ • characterized by a ***stacionary***, ***differential rotation***	**Equation of motion**: • set of eqs. for $V(r,t), W(r,t)$ and $H_0(r,t), H_1(r,t), K(r,t)$ • characterized by ***gravitational radiation***

perturbation of odd parity cannot change p or ρ $\Rightarrow$ ***cannot cause stellar pulsation!*** ***reduced to only 5 functions!***
(p and ρ are scalar fields; and all scalar spherical harmonics are of even parity)

With the perturbed (polar mode) metric tensor:

$$
\begin{aligned}
ds^2 = &-e^{\nu}(1 + r^{\ell} H_0 Y^{\ell}_m e^{i\omega t})dt^2 - 2i\omega r^{\ell+1} H_1 Y^{\ell}_m e^{i\omega t} dt dr + \\
&+ e^{\lambda}(1 - r^{\ell} H_0 Y^{\ell}_m e^{i\omega t})dr^2 + r^2(1 - r^{\ell} K Y^{\ell}_m e^{i\omega t})(d\theta^2 + \sin^2\theta d\phi^2)
\end{aligned}
$$

In an appropriate gauge $\xi^t = 0$, and the other components of the displacement 3-vector are given by

$$
\xi^r = r^{l-1} e^{-\lambda/2} W Y^l_m e^{i\omega t}, \qquad \xi^\theta = -r^{l-2} V \partial_\theta Y^l_m e^{i\omega t}, \qquad \xi^\varphi = -r^l \, (r \sin\theta)^{-2} V \partial_\theta Y^l_m e^{i\omega t}
$$

lead to a ODE system of a set of 4 equations:

$$
\begin{aligned}
H_1' &= -r^{-1}[l + 1 + 2Me^{\lambda}r^{-1} + 4\pi r^2 e^{\lambda}(p - \rho)]H_1 + r^{-1}e^{\lambda}[H_0 + K - 16\pi(\rho + p)V] , \\
K' &= r^{-1}H_0 + \tfrac{1}{2}l(l + 1)r^{-1}H_1 - [(l + 1)r^{-1} - \tfrac{1}{2}\nu']K - 8\pi(\rho + p)e^{\lambda/2}r^{-1}W , \\
W' &= -(l + 1)r^{-1}W + re^{\lambda/2}[\gamma^{-1}p^{-1}e^{-\nu/2}X - l(l + 1)r^{-2}V + \tfrac{1}{2}H_0 + K] , \\
X' &= -lr^{-1}X + (\rho + p)e^{\nu/2}\{\tfrac{1}{2}(r^{-1} - \tfrac{1}{2}\nu')H_0 + \tfrac{1}{2}[r\omega^2 e^{-\nu} + \tfrac{1}{2}l(l + 1)r^{-1}]H_1 \\
&\quad + \tfrac{1}{2}(\tfrac{3}{2}\nu' - r^{-1})K - \tfrac{1}{2}l(l + 1)\nu' r^{-2}V - r^{-1}[4\pi(\rho + p)e^{\lambda/2} + \omega^2 e^{\lambda/2 - \nu} - \tfrac{1}{2}r^2(r^{-2}e^{-\lambda/2}\nu')']W\}
\end{aligned}
$$

The five perturbation function $H_0, H_1, K, V,$ ***and*** W ***are not all independent!***

As a consequence of Einstein's equation, these functions must satisfy the following relationship

$$
\begin{aligned}
[3M + \tfrac{1}{2}(l + 2)(l - 1)r + 4\pi r^3 p]H_0 = \; & 8\pi r^3 e^{-\nu/2}X - [\tfrac{1}{2}l(l + 1)(M + 4\pi r^3 p) - \omega^2 r^3 e^{-(\lambda+\nu)}]H_1 \\
& + [\tfrac{1}{2}(l + 2)(l - 1)r - \omega^2 r^3 e^{-\nu} - r^{-1}e^{\lambda}(M + 4\pi r^3 p)(3M - r + 4\pi r^3 p)]K
\end{aligned}
$$

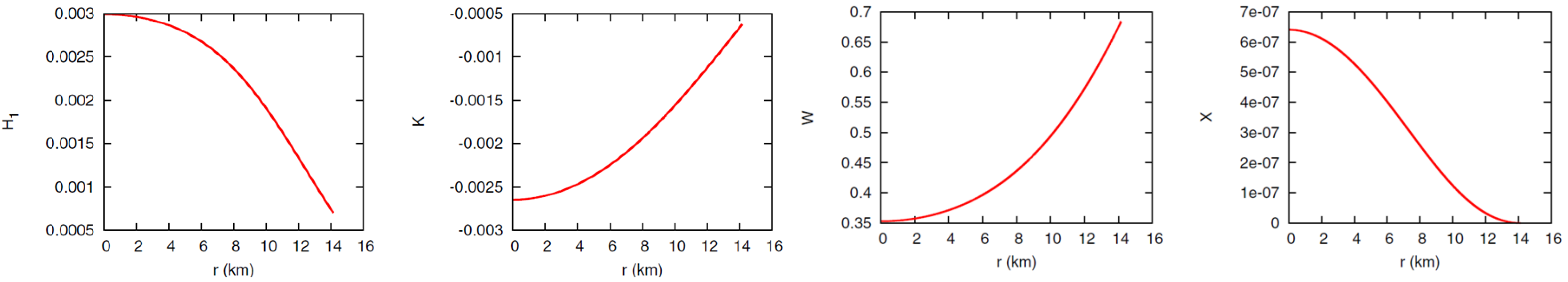
algebraic relation

In this equation the perturbation function X is defined by

$$X = \omega^2(\epsilon + p)e^{-\nu/2}V - \frac{p'}{r}e^{(\nu-\lambda)/2}W + \frac{1}{2}(\epsilon + p)e^{\nu/2}H_0$$

and V is to be thought of as the linear combination of H_1, K, W, and X obtained by eliminating H_0:

$$V = \omega^{-2}(\rho + p)^{-1}e^{\nu}\left[e^{-\nu/2}X + r^{-1}p'e^{-\lambda/2}W - \tfrac{1}{2}(\rho + p)H_0\right]$$



Behavior of the perturbation functions of the ODE system inside the star.

PAST AND PRESENT COLLABORATIONS

Period	Partner institution	Collaborators	Research topic and scientific activity
April 2018	EBERHARD KARLS UNIVERSITÄT TÜBINGEN	Kostas Kokkotas	**Linear adiabatic radial oscillations of neutron stars:** • Study of "shooting" method for finding quasi-normal modes. • Comparison of preliminary numerical results.
Oct. 2021 – present	l'Observatoire de Paris – LUTH Laboratoire de l'Univers et de ses Théories	Philippe Grandclément Jérôme Novak Éric Gourgoulhon	**For NS models (including fast-rotating or magnetized) and import EoS tables directly from CompOSE:** • **LORENE** (set of C++ classes) to solve partial differential equations by means of multi-domain spectral methods. • **KADATH** library (a more generic spectral solver), designed to describe functions as a finite sum of orthogonal functions known as the basis functions.
Sept. 2021 – present	WIGNER	György Wolf Mátyás Vasúth Balázs Kacskovics Gyula Fodor	Research project "*Nuclear matter properties from heavy-ion collisions to compact stars*", supported by OTKA grant agreement No. K138277

Thank you very much for your kind attention!

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