Adiabatic equatorial inspirals of a spinning body into a Kerr black hole

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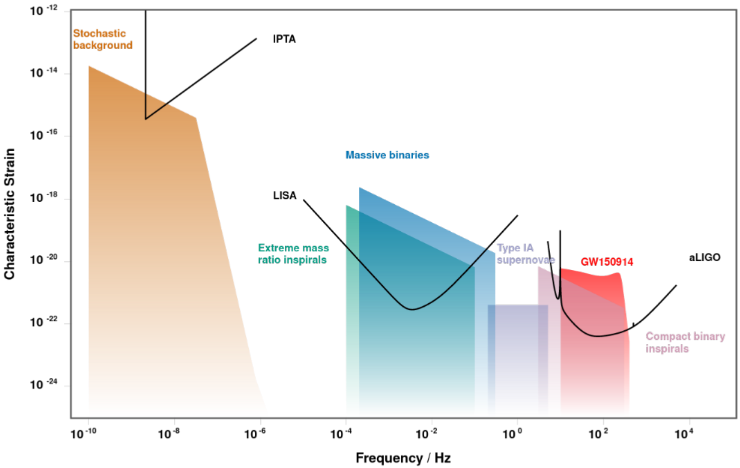
Introduction

- Motivation: calculation of gravitational-wave templates for the detection of GWs
- Extreme mass ratio inspirals with spinning secondary
- Calculation of phase-shifts between EMRI with spinning and non-spinning body

Gravitational waves

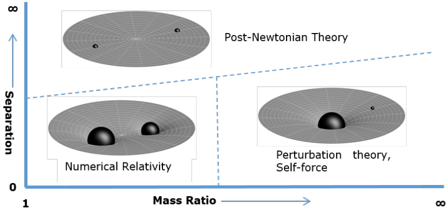
- Disturbances of the curvature of space-time
- Sources: black hole binaries, supernovae, pulsars, . . .
- LISA: future space-based GW detector
- Detecting GWs in mHz bandwidth
- Overlaping singnals: matched filtering
- Accurate templates must be generated



http://gwplotter.com/

Extreme mass ratio inspirals

- Extreme mass ratio inspiral: stellar mass BH/NS orbiting a supermassive black hole
- Mass ratio $q = \mu/M = 10^{-7}$–10^{-4}
- Energy and angular momentum loss due to gravitational radiation reaction
- Emitting GWs to infinity
- Possible to detect with LISA
- Opportunity to study strong gravitation around BH
- Phase of the GW: $\Phi(t) = \Phi_0(t)q^{-1} + \Phi_1(t) + \mathcal{O}(q)$
- Secondary spin contribution in Φ_1



https://en.wikipedia.org/wiki/Extreme_mass_ratio_inspiral

Spinning particle in the Kerr spacetime

- Pole-dipole stress-energy tensor

$$T^{\mu\nu} = \frac{1}{\sqrt{-g}} \left(\frac{P^{(\mu}v^{\nu)}}{v^t} \delta^3(x^i - x_p^i(t)) - \nabla_\alpha \left(\frac{S^{\alpha(\mu}v^{\nu)}}{v^t} \delta^3(x^i - x_p^i(t)) \right) \right)$$

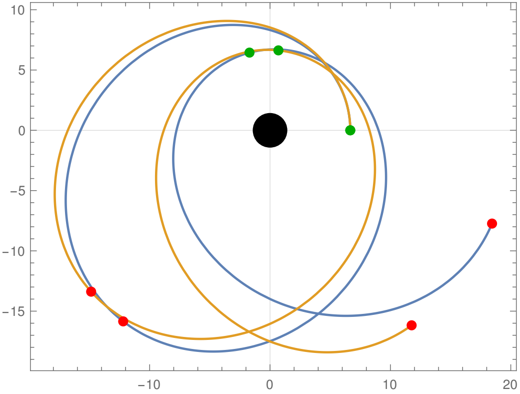
- Mathisson-Papapetrou-Dixon equations for the four-momentum P^μ and spin tensor $S^{\mu\nu}$
- Constants of motion:
  - $\mu = \sqrt{-P^\mu P_\mu}$
  - $\sigma = \sqrt{S^\mu S_\mu}/(\mu M) \leq q \ll 1$
  - $E = -\xi^\mu_{(t)} P_\mu + \xi^{(t)}_{\mu;\nu} S^{\mu\nu}/2$
  - $J_z = \xi^\mu_{(\phi)} P_\mu - \xi^{(\phi)}_{\mu;\nu} S^{\mu\nu}/2$

Spinning particle in the equatorial plane

- Spin parallel to the z-axis
- Equations of motion in the equatorial plane: 3 ODE
- parametrization by eccentricity e, semi-latus rectum p

$$r_1 = \frac{Mp}{1+e} \qquad r_2 = \frac{Mp}{1-e}$$

$$E(p,e,\sigma) = E^{(\mathrm{g})}(p,e) + \sigma\,\delta E(p,e)$$
$$J_z(p,e,\sigma) = J_z^{(\mathrm{g})}(p,e) + \sigma\,\delta J_z(p,e)$$
$$\Omega_r(p,e,\sigma) = \Omega_r^{(\mathrm{g})}(p,e) + \sigma\,\delta\Omega_r(p,e)$$
$$\Omega_\phi(p,e,\sigma) = \Omega_\phi^{(\mathrm{g})}(p,e) + \sigma\,\delta\Omega_\phi(p,e)$$



Teukolsky equation

- GW from EMRI as perturbation of the background spacetime
- NP formalsm: perturbation of Weyl tensor projected on a tetrad $\Psi_4 = -C_{\alpha\beta\gamma\delta} n^\alpha \overline{m}^\beta n^\gamma \overline{m}^\delta$
- Teukolsky equation for the field variable $\psi = (r - ia\cos\theta)^4 \Psi_4$

$$\left(\frac{\left(r^2+a^2\right)^2}{\Delta} - a^2\sin^2\theta\right)\frac{\partial^2\psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\varphi} + \left(\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right)\frac{\partial^2\psi}{\partial\varphi^2}$$
$$- \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - 2s\left(\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right)\frac{\partial\psi}{\partial\varphi}$$
$$- 2s\left(\frac{M\left(r^2-a^2\right)}{\Delta} - r - ia\cos\theta\right)\frac{\partial\psi}{\partial t} + \left(s^2\cot^2\theta - s\right)\psi = 4\pi\Sigma T\,,$$

Solutions of TE

- Decomposition into Fourier modes

$$\psi = \sum_{l,m} \int \mathrm{d}\omega \psi_{lm\omega}(r) {}_sS^{a\omega}_{lm}(\theta) e^{-i\omega t + im\varphi}$$

- Radial equation solved using Green function formalism
  $\psi_{lm\omega}(r) = C^{+}_{lm\omega}(r) R^{+}_{lm\omega}(r) + C^{-}_{lm\omega}(r) R^{-}_{lm\omega}(r)$
- Periodicity of the radial motion: discrete frequencies $\omega_{mn} = m\Omega_\phi + n\Omega_r$

$$C^{\pm}_{lm\omega} = \sum_n C^{\pm}_{lmn} \delta(\omega - \omega_{mn})$$

- Linearization

$$C^{\pm}_{lmn} = C^{\pm(\mathrm{g})}_{lmn} + \sigma \, \delta C^{\pm}_{lmn}$$

Energy and angular momentum fluxes

- Strain at infinity $h_{\mu\nu} = h_+ e^+_{\mu\nu} + h_\times e^\times_{\mu\nu}$

$$h = h_+ - ih_\times = -\frac{2}{r}\sum_{l,m,n}\frac{C^+_{lmn}}{\omega^2_{mn}} S^{a\omega_{mn}}_{lm}(\theta) e^{-i\omega_{mn}(t-r^*)+im\phi}$$

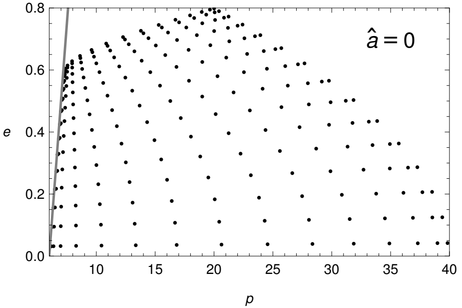
- Energy and angular momentum fluxes

$$\mathcal{F}^E = \left\langle \frac{\mathrm{d}E^\infty_{\mathrm{GW}}}{\mathrm{d}t} \right\rangle = \sum_{l=2}^{\infty}\sum_{m=-l}^{l}\sum_{n=-\infty}^{\infty}\frac{\left|C^+_{lmn}\right|^2}{4\pi\omega^2_{mn}}$$

$$\mathcal{F}^{J_z} = \left\langle \frac{\mathrm{d}J^\infty_{z\mathrm{GW}}}{\mathrm{d}t} \right\rangle = \sum_{l=2}^{\infty}\sum_{m=-l}^{l}\sum_{n=-\infty}^{\infty}\frac{m\left|C^+_{lmn}\right|^2}{4\pi\omega^3_{mn}}$$

Numerical calculation

- $C^{\pm(\mathrm{g})}_{lmn}$, $\delta C^{\pm}_{lmn}$ calculated using numerical integration
- Summed over l, m, n for given accuracy
- Repeated for grid-points in the $p - e$ plane
- $\mathcal{F}^E$, $\mathcal{F}^{J_z}$ interpolated using Chebyshev interpolation



Adiabatic inspirals

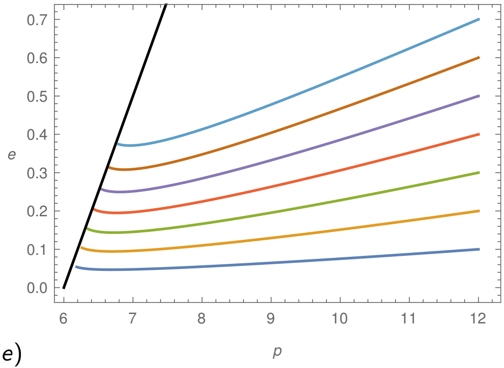
- Fluxes are very small: two-timescale approximation
- Flux-balance laws: $\mathcal{F}^E$ and $\mathcal{F}^{J_z}$ are equal to $-\dot{E}$, $-\dot{J}_z$
- Evolution of p, e

$$\begin{pmatrix} \frac{\mathrm{d}p}{\mathrm{d}t} \\ \frac{\mathrm{d}e}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} \frac{\partial E}{\partial p} & \frac{\partial E}{\partial e} \\ \frac{\partial J_z}{\partial p} & \frac{\partial J_z}{\partial e} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\mathrm{d}E}{\mathrm{d}t} \\ \frac{\mathrm{d}J_z}{\mathrm{d}t} \end{pmatrix}$$

- Linearization: $p(t) = p^{(\mathrm{g})}(t) + \sigma\, \delta p(t)$, $e(t) = e^{(\mathrm{g})}(t) + \sigma\, \delta e(t)$
- Evolution equations:

$$\frac{\mathrm{d}p^{(\mathrm{g})}}{\mathrm{d}t} = \dot{p}^{(\mathrm{g})}(p^{(\mathrm{g})}, e^{(\mathrm{g})}) \qquad \frac{\mathrm{d}\delta p}{\mathrm{d}t} = \delta\dot{p}(p^{(\mathrm{g})}, e^{(\mathrm{g})}, \delta p, \delta e)$$

$$\frac{\mathrm{d}e^{(\mathrm{g})}}{\mathrm{d}t} = \dot{e}^{(\mathrm{g})}(p^{(\mathrm{g})}, e^{(\mathrm{g})}) \qquad \frac{\mathrm{d}\delta e}{\mathrm{d}t} = \delta\dot{e}(p^{(\mathrm{g})}, e^{(\mathrm{g})}, \delta p, \delta e)$$



Waveform

- Waveform from inspiralling orbit

$$h(t) = -\frac{2}{r}\sum_{l,m,n}\frac{C^{+}_{lmn}(t)}{\omega^2_{mn}(t)} S^{a\omega_{mn}(t)}_{lm}(\theta) e^{-i\Phi_{mn}(t)+im\phi}$$

- GW phase $\Phi_{mn} = m\Phi_\phi + n\Phi_r$

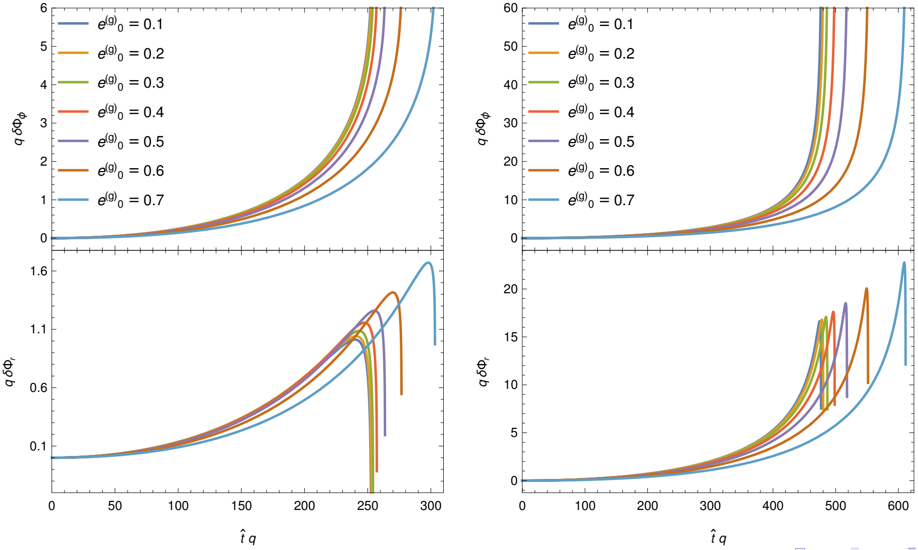
$$\Phi_{r,\phi}(t) = \int_0^t \Omega_{r,\phi}(p(t'), e(t'), \sigma)\mathrm{d}t'$$

- Linearization:

$$\Phi_{r,\phi} = \Phi^{(\mathrm{g})}_{r,\phi} + \sigma\,\delta\Phi_{r,\phi}$$

- Phase shift $\sigma\,\delta\Phi_{r,\phi}$ of the order of radians

Phase shifts



Summary

- For the detection of EMRI, waveform templatest must be generated with high accuracy
- The spin of the smaller body must be included
- We have calculated orbital quantities of spinning body linearized in the spin
- Using Teukolsky equation we calculated the GW fluxes to infinity and to the horizon
- We have calculated adiabatic inspirals and the phase shifts due to the secondary spin

Thank you

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Equations of motion

$$\Sigma_\sigma(\hat{r})\Lambda_\sigma(\hat{r})\frac{\mu}{\mathsf{m}}\frac{\mathrm{d}\hat{t}}{\mathrm{d}\hat{\tau}} = \frac{\mathrm{d}\hat{t}}{\mathrm{d}\lambda} = V^t(\hat{r})$$

$$\Sigma_\sigma(\hat{r})\Lambda_\sigma(\hat{r})\frac{\mu}{\mathsf{m}}\frac{\mathrm{d}\hat{r}}{\mathrm{d}\hat{\tau}} = \frac{\mathrm{d}\hat{t}}{\mathrm{d}\lambda} = V^r(\hat{r}) = \pm\frac{\mathsf{m}}{\mu}\sqrt{R_\sigma(\hat{r})}$$

$$\Sigma_\sigma(\hat{r})\Lambda_\sigma(\hat{r})\frac{\mu}{\mathsf{m}}\frac{\mathrm{d}\phi}{\mathrm{d}\hat{\tau}} = \frac{\mathrm{d}\hat{t}}{\mathrm{d}\lambda} = V^\phi(\hat{r})$$

- Bound equatorial orbits:

$$R_\sigma(\hat{r}_1) = 0, \quad R_\sigma(\hat{r}_2) = 0$$

$$R_\sigma = \left(\Sigma_\sigma(\hat{r})\hat{E} - \left(\hat{a} + \frac{\sigma}{\hat{r}}\right)(\hat{J}_z - (\hat{a}+\sigma)\hat{E})\right)^2 - \hat{\Delta}\left(\frac{\Sigma_\sigma^2(\hat{r})}{\hat{r}^2} + (\hat{J}_z - (\hat{a}+\sigma)\hat{E})^2\right)$$

Frequencies

- Reparametrization $r = \frac{pM}{1+e\cos\chi}$
- Radial period

$$T_r = 2\int_{r_1}^{r_2} \frac{V^t(r)}{\sqrt{R_\sigma(r)}}\mathrm{d}r = \frac{2\sqrt{1-e^2}}{p}\int_0^\pi \frac{V^t(r(\chi))}{\sqrt{J(\chi)}}\mathrm{d}\chi$$

- Azimuthal phase $\Delta\phi$ accumulated over one radial period
- BL frequencies $\Omega_r = \frac{2\pi}{T_r}$, $\Omega_\phi = \frac{\Delta\phi}{T_r}$

Inspirals

$$\frac{\mathrm{d}\delta p}{\mathrm{d}\hat{t}} = \left.\frac{\mathrm{d}\dot{p}}{\mathrm{d}\sigma}\right|_{\sigma=0} \equiv \delta\dot{p}(p^{(\mathrm{g})}(\hat{t}), e^{(\mathrm{g})}(\hat{t}), \delta p(\hat{t}), \delta e(\hat{t})) \,,$$

$$\frac{\mathrm{d}\delta e}{\mathrm{d}\hat{t}} = \left.\frac{\mathrm{d}\dot{e}}{\mathrm{d}\sigma}\right|_{\sigma=0} \equiv \delta\dot{e}(p^{(\mathrm{g})}(\hat{t}), e^{(\mathrm{g})}(\hat{t}), \delta p(\hat{t}), \delta e(\hat{t})) \,,$$

$$\left.\frac{\mathrm{d}f}{\mathrm{d}\sigma}\right|_{\sigma=0} = \left.\frac{\partial f}{\partial \sigma}\right|_{\sigma=0} + \frac{\partial f^{(\mathrm{g})}}{\partial p}\delta p + \frac{\partial f^{(\mathrm{g})}}{\partial e}\delta e$$

$$\delta\Phi_i = \int_0^t \left.\left(\frac{\partial \Omega_i}{\partial \sigma} + \frac{\partial \Omega_i^{(\mathrm{g})}}{\partial p}\delta p(t') + \frac{\partial \Omega_i^{(\mathrm{g})}}{\partial e}\delta e(t')\right)\right|_{\sigma=0, p=p^{(\mathrm{g})}(t'), e=e^{(\mathrm{g})}(t')} \mathrm{d}t'$$