# Relativistic Theory of Elastic Bodies in the Presence of Gravitational Waves

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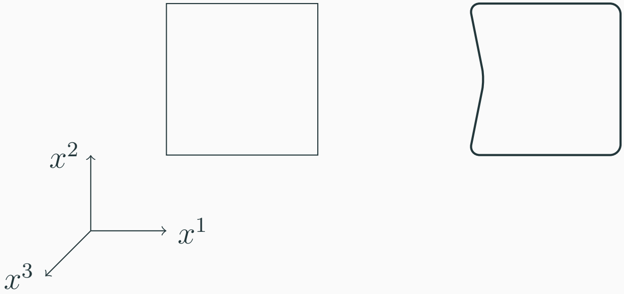
## Key points of interest

general relativistic behavior/description of an elastic material

response to the incidence of a gravitational wave

# Classical (linear) Elasticity



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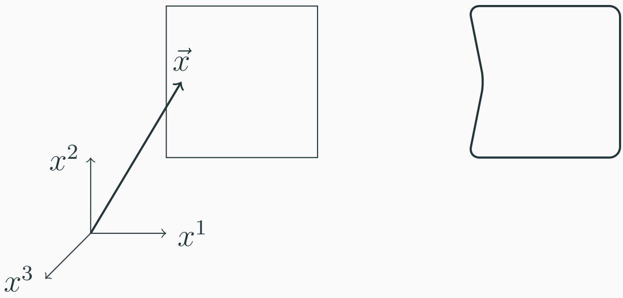

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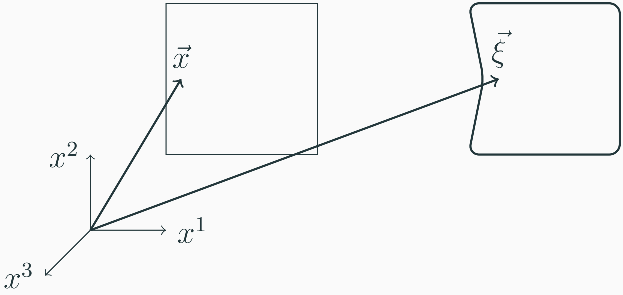

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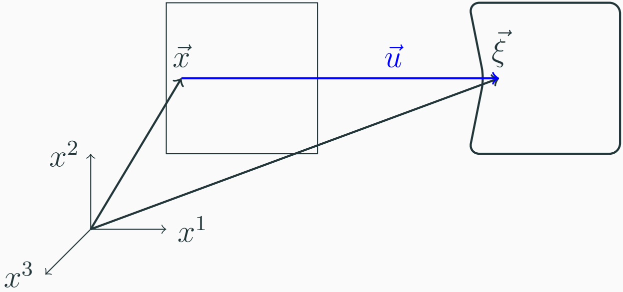


configuration $\vec{x}$ in terms of displacement: $\vec{x} = \vec{\xi} - \vec{u}$

# General Relativistic Elasticity

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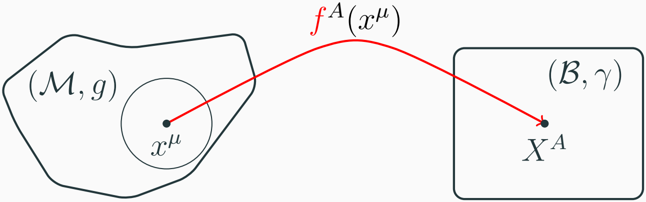


configuration map: $f : \mathcal{M} \to \mathcal{B} : x^\mu \mapsto f^A(x^\mu) := X^A$

deformation gradient: $\partial_\mu f^A(x^\nu) : T_x\mathcal{M} \to T_X\mathcal{B}$

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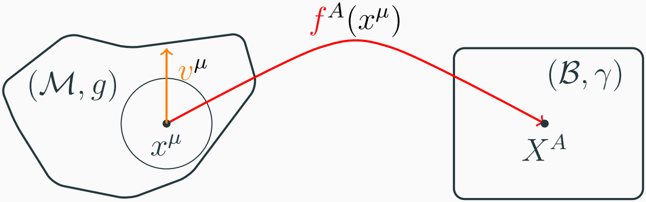


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deformation gradient: $\partial_\mu f^A(x^\nu) : T_x\mathcal{M} \to T_X\mathcal{B}$

$\exists! v^\mu \in T_x\mathcal{M} : v^\mu \partial_\mu f^A(x^\nu) = 0$

# Spacetime Deformation Tensors

$(\mathcal{M}, g)$ $(\mathcal{B}, \gamma)$

$\gamma$

$$c_{\mu\nu} := \partial_\mu f^A \partial_\nu f^B \gamma_{AB}$$

$$(c_{\mu\nu})^{-1} = b^{\mu\nu}$$

Finger deformation tensor: $b^{\mu\nu}$

left Cauchy-Green deformation tensor: $c_{\mu\nu}$

# Material Deformation Tensors

$(\mathcal{M}, g)$ $\quad$ $(\mathcal{B}, \gamma)$

$g^{-1}$

$$B^{AB} = \partial_\mu f^A \partial_\nu f^B g^{\mu\nu}$$

$$C_{AB} = (B^{AB})^{-1}$$

right Cauchy-Green deformation tensor: $C_{AB}$

Piola deformation tensor: $B^{AB}$

# Strain tensor

$(\mathcal{M}, g)$ $(\mathcal{B}, \gamma)$

$g \longrightarrow E_{AB} := \frac{1}{2}\left[C_{AB} - \gamma_{AB}\right]$

$e_{\mu\nu} = E_{AB}\partial_\mu f^A \partial_\nu f^B \longleftarrow$

Almansi strain tensor: $e_{\mu\nu}$

Green-Lagrange strain tensor: $E_{AB}$

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wave equation (neglecting $\mathcal{O}(\epsilon^2)$)

$$\partial_\mu T^{\mu j} = \rho_0 \partial_{tt} u^j - \left[\mu \Delta u^j + (\mu + \lambda)\partial_{ik} u^k \delta^{ij}\right] = 0$$

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no gravitational wave contribution

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… No, because the boundary conditions are different from those of flat space.

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gravitational wave normal incidence: $h_{xx}|_{x=0} = \cos\omega t$

$$\sigma_{xx}\big|_{x=\pm\frac{L}{2}} = \partial_x \delta u\big|_{x=\pm\frac{L}{2}} + \beta(t)(\lambda + 2\mu) + \mu h_{xx}|_{x=0} \stackrel{!}{=} 0$$

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internal force as result of the gravitational wave

# Solution for the Displacement

$$u(t,x) = \left[-\frac{c_1}{\omega}\sec\left(\frac{\omega L}{2c_1}\right)\sin\left(\frac{x\omega}{c_1}\right)\right]\frac{c_2^2}{c_1^2}\cos(\omega t)\,.$$

$c_1 = \sqrt{\frac{\lambda+2\mu}{\rho_0}}$ and $c_2 = \sqrt{\frac{\mu}{\rho_0}}$

# Computations

Christoffel symbols (GW)

$$\Gamma^{\mu}_{\nu\rho} = \frac{\epsilon}{2}\left(h'^{\ \mu}_{\rho}\kappa_{\nu} + h'^{\ \mu}_{\nu}\kappa_{\rho} - h'_{\nu\rho}\kappa^{\mu}\right) + \mathcal{O}(\epsilon^2)$$

Cauchy stress (rod example)

$$\sigma^{ij}n_j = [\lambda(\partial_k u^k)\delta^{ij} + \mu(\partial^i u^j + \partial^i u^j)]n_j + \mu h^{ij}n_j = 0$$

# Weak field plane wave TT

Overview of the main assumptions considering gravitational waves:

1. Small perturbation close to the flat metric:
   $g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}$ with inverse $g^{\mu\nu} = \eta^{\mu\nu} - \epsilon h^{\mu\nu}$
2. plane wave: $h_{\mu\nu} = h_{\mu\nu}(\kappa_\alpha x^\alpha)$ with $\kappa_\mu = \omega_{GW}(-1, \vec{n})$
   where $|\vec{n}| = 1$ and $\eta^{\mu\nu}\kappa_\mu\kappa_\nu = 0$
3. Transverse and traceless (TT) gauge condition
   3.1 $h_{0\mu} = h_{\mu 0} = 0$ (synchronous gauge)
   3.2 $\eta^{\mu\nu} h_{\mu\nu} = 0$ (zero trace)
   3.3 $\partial^\mu h_{\mu\rho} = h^{'}_{\mu\rho}\kappa^\mu = 0$ (transversality condition)

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