EigenForm

by Louis H. Kauffman

Abstract: This essay works with Heinz von Foerster's concept of eigenform, wherein an object is seen to be a token for those behaviours that lend the object its apparent stability in a changing world. In this essay we describe von Foerster's model for eigenforms and recursions and put this model in the context of mathematical recursions, fractals, set theory, logic, quantum mechanics, the lambda calculus of Church and Curry, and the categorical framework of fixed points of Lawvere. This concept of eigenform is at once mathematical, philosophical and phenomenological, suggesting a new framework for understanding the entangled relationship of an observer and the observer's world. From the point of view of the eigenform there is an interlock wherein the world and the observer cocreate each other, and are, in the form, identical.

Key Words: cybernetics, recursion, eigenform, object, eigenbehaviour, eigenvalue, set.

I. Introduction

This essay discusses the notion of eigenform as explicated by Heinz von Foerster in his paper (von Foerster, 1981). A strong source of Heinz's work on eigenforms and self-reference is his earlier consideration of second order cybernetics in the form of "cybernetics of cybernetics." We consider the possibility of a field of intellectual endeavor that can investigate itself, and the way in which such a turn can come about. Lets begin by listing some of these turns inward:

Cybernetics of Cybernetics

Mathematics of Mathematics

Computation of Computation

Linguistics of Linguistics

Magic of Magic

Logic of Logic

Geometry of Geometry

Pattern of Pattern

Will of Will

Teaching of Teaching

Learning of Learning

Meaning of Meaning

Heinz performs the magic trick of convincing us that the familiar objects of our existence can be seen to be nothing more than tokens for the behaviors of the organism that apparently create stable forms. These stabilities persist, for that organism, as an observing system. This is not to deny an underlying reality that is the source of objects, but rather to emphasize the role of process, and the role of the organism in the production of a living map, a map that is so sensitive that map and territory are conjoined. Von Foerster's book and papers (von Foerster, 1981) were instrumental in pioneering the field of second order cybernetics.

The notion of an eigenform is inextricably linked with second order cybernetics. One starts on the road to such a concept as soon as one begins to consider a pattern of patterns, the form of form or the cybernetics of cybernetics. Such concepts appear to close around upon themselves, and at the same time they lead outward. They suggest the possibility of transcending the boundaries of a system from a locus that might have been within the system until the circular concept is called into being. But then the boundaries have turned inside out, and the inside is the outside.

Forms are created from the concatenation of operations upon themselves and objects are not objects at all, but rather indications of processes. Upon encountering an object as such a form of creation, you are compelled to ask: How is that object created? How is it designed? What do I do to produce it? What is the network of productions? Where is the home of that object? In what context does it exist? How am I involved in its creation?

Taking Heinz's suggestion to heart, we find that an object is a symbolic entity, participating in a network of interactions, taking on its apparent solidity and stability from these interactions. We ourselves are such objects, we as human beings are "signs for ourselves", a concept originally due to the American philosopher C. S. Peirce (Kauffman, 2001). In many ways Heinz's eigenforms are mathematical companions to Peirce's work. We will not follow this comparison in the present essay, but the reader familiar with Peirce is encouraged to do so.

Here is description of the contents of this paper.

Sections 2 and 3 of this paper are a discussion of the nature of object as we tend to describe it in language, leading from the descriptions and assumptions that we use in the everyday world to the somewhat different concepts of object that occur in scientific work, particularly in the physics of the very small. We indicate how the concept of object arising as a token for the interaction and mutual production of the observer, and the observed intermediates among these points of view. Section 4 gives a short exposition of the mathematical form of Heinz's model. Once this mathematics of eigenforms is on the table, one can discuss how it is related to our use and concept of the notion "object."

Section 5 points out just how the von Foerster eigenform and the eigenvector of quantum physics are related. We give some hints about using eigenforms to understand quanta. The view of physics from the eigenforms is a reversal of epistemology, a complete turning of the world upside down. Heinz has tricked us into considering the world of our experience and finding that it is our world, generated by our actions. The world has become apparently objective through the self-generated stabilities of those very actions.

Section 6 is a recounting of a conversation of the author and Ranulph Glanville in which Ranulph asked "Does every recursion have a fixed point?" In the language of the present essay this is the question

"Does every process have an eigenform?". The obvious answer is no, but the answer that comes from Heinz's model is yes! There is always an ideal eigenform. The challenge is to integrate that form into the context of one's living.

In the next three sections we consider questions about eigenforms in relation to mathematical objects such as numbers (sections 7, 8 and 9) and linguistic objects such as names.

Section 10 describes how to construct eigenforms without an excursion to infinity, by the fixed point method of the logicians Church and Curry. We point out that the construction of eigenforms in the sense of Heinz's model can be done without the idealized excursion to infinity (used by Heinz) via a device invented by Church and Curry in the 1930's that is commonly called the "lambda calculus". This device involves first constructing an intermediate operator (that I like to call the "gremlin") who acts as a catalyst for a process. When gremlins meet they produce eigenforms!

Section 11 shows how differentiation in formal calculus is related to the inductive construction of number, and to eigenforms for the derivative. We discuss numbers as eigenforms, and here it will become immediately apparent that with number one has a perfect example of how an eigenform is both an object and an operator, how it is both specific and ideal. This harks back to our discussion of quantum mechanics and eigenforms.

In sections 11 and 12 we have created a model for a set theoretic world that contains a parable of the mental and the physical. The purely mental world is the class of sets generated from the empty set. The purely physical world is the class of sets generated from the special objects **a** that are eigenforms for the operation of framing, so that **a = {a}.** The interface of mental and physical occurs as these realms touch in the limit of nested parentheses

$$\mathbf{W} = \{\{\{\{\{\{\{...\}\}\}\}\}\}\}.$$

W is an amphibian living in both worlds. **W** is the eigenform that crosses the boundary from the mental to the physical.

In section 14 we consider naming and self-reference and return to Heinz's definition of "I". We end Section 14 with a discussion of how self-reference occurs in language, through an indicative shift that welds the name of a person to his/her physical presence and shifts the indication of that name to a metaname. More could be said at this point, as the indicative shift is a linguistic entry into the world of Godelian sentences and the incompleteness of formal systems. We emphasize the natural occurrence of eigenforms in the world of our linguistic experience and how this occurrence is intimately connected to our structure as self-observing systems.

In section 15 we show how eigenforms and imaginary Boolean values emanate naturally from the category theorist William Lawvere's amazing proof of Cantor's theorem on subsets of an infinite set. The paper ends with a short epilogue in Section 16.

II. Objects

What is an object? At first glance, the question seems perfectly obvious. An object is a thing, a something that you can pick up and move and manipulate in three dimensional space. An object is three dimensional, palpable, like an apple or a chair, or a pencil or a cup. An object is the simplest sort of entity that can be subjected to reference. All language courses first deal with simple objects like pens and tables. La plume est sur la table.

An object is separate from me. It is "out there". It is part of the reality separate from me. Objects are composed of objects, their parts. My car is made of parts. The chair is a buzzing whirl of molecules. Each molecule is a whirl of atoms. Each atom a little solar system of electrons, neutrons and protons. But wait! The nucleus of the atom is composed of strange objects called quarks. No one can see them. They do not exist as separate entities. The electrons in the atom are special objects that are not separate from each other and from everything else. And yet when you observe the electrons, they have definite locations.

The physicist's world divides into quantum objects that are subject to the constraints of the uncertainty principle, and classical objects that live in the dream of objective existence, carrying all their properties with them.

A classical object has a location at a given time. You can tell where it is. You can tell a story of where it has been. If the classical object breaks up into parts, you will be able to keep track of all the parts. Yet electrons and positrons can meet each other and disappear into pure energy! Should we allow objects to disappear? What sort of an object is the electromagnetic field of radio and television signals that floods this room?

Is my thought to be thought of as an object? Can I objectify my thought by writing it down on paper or in the computer? Am I myself an object? Is my body an object in the three dimensional space? Is the space itself an object? Objects have shape. What is the shape of space? What is the shape of the physical universe. What is the shape of the Platonic universe?

If a person (a thought, feeling and symbol object) were to read this section with the hope of finding a clear definition of object, he/she might be disappointed. Yet Heinz von Foerster has suggested the enticing notion that "objects are tokens for eigenbehaviours. There is a behaviour between the perceiver and the object perceived and a stability or repetition that "arises between them". It is this stability that constitutes the object (and the perceiver). In this view, one does not really have any separate objects, objects are always "objects perceived", and the perceiver and the perceived arise together in the condition of observation.

Compresence and Coalescence

The world appears to be the union of separate objects, each a union of ever-smaller particles until this reduction recedes beyond our abilitity to perceive differences. And yet, at the same time, the world appears as a unity within which all these apparently separated entities reside within the whole. For an observer these are two primary modes of perception -- compresence and coalesence. Compresence connotes the coexistence of separate entities together in one including space. Coalesence connotes the one space holding, in perception, the observer and the observed, inseparable in an unbroken wholeness. Coalesence is the constant condition of our awareness. Coalesence is the world taken in simplicity. Compresence is the world taken through the filter of language.

This distinction of compresence and coalesence drawn by Henri Bortoft (Bortoft, 1971) can act as a compass in traversing the domains of object and reference. Eigenform is a first step in towards a mathematical description of coalesence. In the world of eigenform, the observer and the observed are one in a process that recursively gives rise to each.

III. Shaping a World

We identify the world in terms of how we shape it. We shape the world in response to how it changes us. We change the world and the world changes us. Objects arise as tokens of behaviour that leads to seemingly unchanging forms. Forms are seen to be unchanging through their invariance under our attempts to change, to shape them.

Can you conceive of an object independent of your ability to perceive it? I did not say an object independent of your perception.
Lets assume that it is possible to talk of the tree in the forest where we are not. But how are we to speak of that tree? One can say, the tree is there. What does this mean? It means that there is a potentiality for that tree to appear in the event of the appearance of a person such as myself or yourself in the place called that forest.
What is the tree doing when I am not in the forest?
I will never know, but I do know that "it" obediently becomes treeish and located when "I" am "there". The quotation marks are indications of objects dissolving into relationships. Whenever "I" am present, the world (of everything that is the case) is seen through the act of framing. I imagine a pure world, unframed. But this is the world of all possibility. As soon as we enter the scene the world is filtered and conformed to become the form that frame and brain have consolidated to say is reality.

IV. Heinz's Eigenform Model

Heinz created a model for thinking about object as token for eigenbehaviour. This model examines the result of a simple recursive process carried to its limit.

For example, suppose that

$$F(X) = \boxed{X}$$

That is, each step in the process encloses the results of the previous step within a box.

Then the infinite concatenation of F upon itself is an infinite nest of boxes as shown below.

$$F(F(F(\ldots))) = \boxed{\boxed{\boxed{\boxed{\boxed{\ldots}}}}}$$

An infinite nest of boxes is invariant under the addition of one more surrounding box. Hence this infinite nest of boxes is a fixed point for the recursion. In other words, if X denotes the infinite nest of boxes, then

$$\mathbf{X = F(X).}$$

The infinite nest of boxes is one of the simplest eigenforms.
In the process of observation, we interact with ourselves and with the world to produce stabilities that become the objects of our perception. These objects, like the infinite nest of boxes, go beyond the specific properties of the world in which we operate. They attain their stability through this process of going beyond the immediate

world. Furthermore, we make an imaginative leap to complete such objects that are tokens for eigenbehaviours. It is impossible to make an infinite nest of boxes in the physical world. We do not make it. We imagine it. And in imagining that infinite next of boxes, we arrive at the eigenform that is the object for this process.

But does the infinite nest of boxes exist? Certainly it does not exist in this page or anywhere in the physical world with which the writer or presumably the reader is familiar. The infinite nest of boxes exists in the imagination!

Just so, an object in the world (cognitive, physical, ideal,...) provides a conceptual center for the investigation of a skein of relationships. An object can have varying degrees of existence just as does an eigenform. If we take Heinz's suggestion to heart, that objects are tokens for eigenbehaviors, then an object in itself is a symbolic entity, participating in a network of interactions, taking on its apparent solidity and stability from these interactions. In and of itself each object is as imaginary as a pure eigenform.

Heinz's model can be expressed (as indeed he did express it) in quite abstract and general terms. Suppose that we are given a recursion symbolically by the equation

$$\mathbf{X(t+1) = F(X(t)).}$$

Here X(t) denotes the condition of observation at time t. X(t) could be as simple as a set of nested boxes, or as complex as the entire configuration of your body in relation to the known universe at time t. Then F(X(t)) denotes the result of applying the operations symbolized by F to the condition at time t. You could, for simplicity, assume that F is independent of time. Time independence of the recursion F will give us simple answers and we can later discuss what will happen if the actions depend upon the time. In the time independent case we can write

$$\mathbf{J = F(F(F(...)))}$$

the infinite concatenation of F upon itself. We then see that

$$\mathbf{F(J) = J}$$

since adding one more F to the concatenation changes nothing. Thus **J,** the infinite concatenation of the operation upon itself, leads to a fixed point for **F**. **J** is said to be the eigenform for the recursion **F.** It is just like the nested boxes, and we see that every recursion has an eigenform. Every recursion has a fixed point.

Must the fixed points of our recursions always live in a new domain? Certainly not. For example, the number two is a fixed point for **f(x) = x*x -2**, and the number **1 + Sqrt(2)** is a fixed point for **F(x) = 2 + 1/x.** These numbers are part of the domain of real numbers usually assumed in working with numerical recursions.

This last example is worth comparing with the infinite nest of boxes. If we ask for a fixed point for $\mathbf{F(x) = 2 + 1/x}$ we are asking for an $\mathbf{x}$ such that $\mathbf{x = 2 + 1/x}$. Hence we ask for x such that $\mathbf{x*x = 2x + 1}$, a solution to a quadratic equation. And one verifies that $\mathbf{(1 + Sqrt(2))*(1+ Sqrt(2)) = 2(1 + Sqrt(2)) + 1}$. Hence $\mathbf{x = 1 + Sqrt(2)}$ is an example of a fixed point for $\mathbf{F(x)}$.
On the other hand, following the proof of the Theorem, we find

$$\mathbf{J= F(F(F(...))) = 2 + 1/(2 +1/(2 + 1/(2 + ...)))},$$

an infinite continued fraction that formally satisfies the equation $\mathbf{J = F(J)}$. In this case we can make numerical sense of the infinite construction. In general we are challenged to find a context in which the infinite concatenation of the operator makes sense.

The place where this sort of construction reaches a conceptual boundary is met in dealing with all solutions to a quadratic equation. There we can begin with the equation $\mathbf{x*x = ax + b}$ with roots $\mathbf{x = (a + Sqrt(a*a + 4b))/2}$ and $\mathbf{x = (a - Sqrt(a*a + 4b))/2}$.
If (a*a +4b) <0 then the roots are imaginary. On the other hand, we can rewrite the quadratic (dividing by $\mathbf{x}$ for x not zero) as
$\mathbf{x = a + b/x = f(x)}$.

Associating to this form of the quadratic the eigenform

$$\mathbf{E = f(f(f(f(...))))},$$

we have

$$E = a + 1/(b + 1/(a + 1/(b + ...))) \text{ with } f(E) = E.$$

Thus E is a formal solution to the quadratic equation, and the consecutive terms

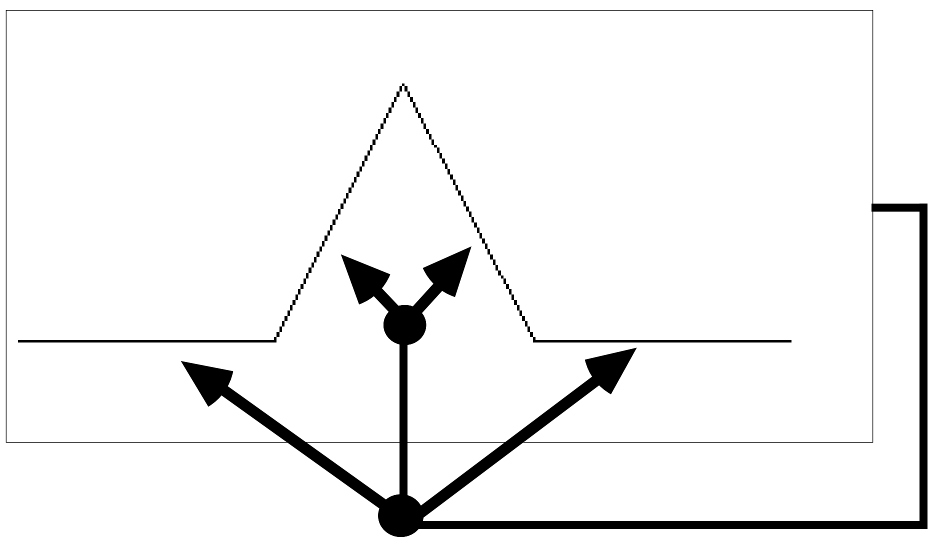
$$E1 = a, \; E2 = a + 1/b, \; E3 = a + 1/(b + 1/a), \; ...$$

will converge to one of the roots when the roots are real, but will oscillate with no convergence when the roots are imaginary. Nevertheless, this series and its associated eigenform are very closely related to the complex solutions, and the eigenform provides a conceptual center for the investigation of these relationships (Kauffman 1987, 1994).

We end this section with one more example. This is the eigenform of the Koch fractal (Kauffman, 1987). In this case one can write symbolically the eigenform equation

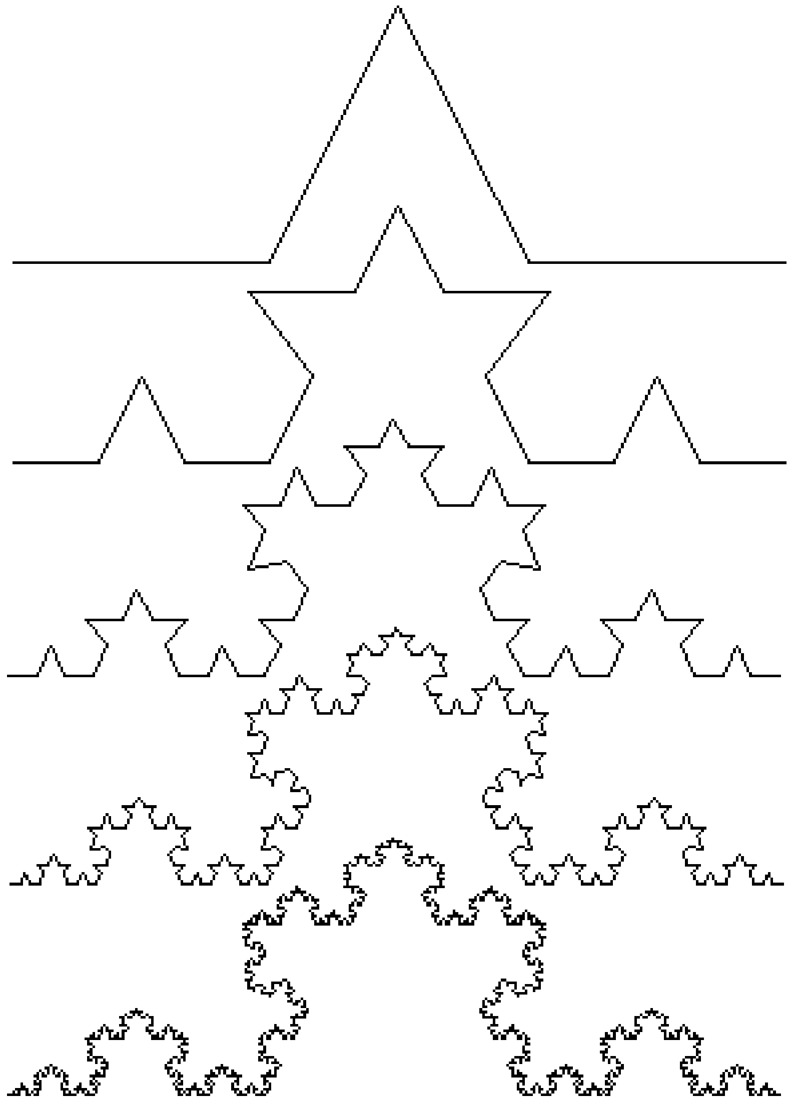
$$K = K \{ K \;\; K \} K$$

to indicate that the Koch Fractal reenters its own indicational space four times (that is, it is made up of four copies of itself, each one-third the size of the original. The curly brackets in the center of this equation refer to the fact that the two middle copies within the fractal are inclined with respect to one another and with respect to the two outer copies. In the figure below we show the geometric configuration of the reentry.



$$K = K \{ K K \} K$$

In the geometric recursion, each line segment at a given stage is replaced by four line segments of one third its length, arranged according to the pattern of reentry as shown in the figure above. The recursion corresponding to the Koch eigenform is illustrated in the next figure. Here we see the sequence of approximations leading to the infinite self-reflecting eigenform that is known as the Koch snowflake fractal.



Five stages of recursion are shown. To the eye, the last stage vividly illustrates how the ideal fractal form contains four copies of itself, each one-third the size of the whole. The abstract schema

$$\mathbf{K = K \{ K \ K \} \ K}$$

for this fractal can itself be iterated to produce a "skeleton" of the geometric recursion:

$$\mathbf{K} = \mathbf{K}\ \{\ \mathbf{K}\ \mathbf{K}\ \}\ \mathbf{K}$$
$$= \mathbf{K}\ \{\ \mathbf{K}\ \mathbf{K}\ \}\ \mathbf{K}\ \ \{\ \ \mathbf{K}\ \{\ \mathbf{K}\ \mathbf{K}\ \}\ \mathbf{K}\ \ \mathbf{K}\ \{\ \mathbf{K}\ \mathbf{K}\ \}\ \mathbf{K}\ \ \}\ \mathbf{K}\ \{\ \mathbf{K}\ \mathbf{K}\ \}\ \mathbf{K}$$
$$= \ldots$$

We have only performed one line of this skeletal recursion. There are sixteen **K**'s in this second expression just as there are sixteen line segments in the second stage of the geometric recursion. Comparison with this abstract symbolic recursion shows how geometry aids the intuition. The interaction of eigenforms with the geometry of physical, mental, symbolic and spiritual landscapes is an entire subject that is in need of deep exploration. Compare with (Kauffman, 1987).

It is usually thought that the miracle of recognition of an object arises in some simple way from the assumed existence of the object and the action of our perceiving systems. What is to be appreciated is that this is a fine tuning to the point where the action of the perceiver, and the perception of the object are indistinguishable. Such tuning requires an intermixing of the perceiver and the perceived that goes beyond description. Yet in the mathematical levels, such as number or fractal pattern, part of the process is slowed down to the point where we can begin to apprehend it. There is a stability in the comparison, in the one-to-one correspondence that is a process happening at once in the present time. The closed loop of perception occurs in the eternity of present individual time. Each such process depends upon linked and ongoing eigenbehaviors and yet is seen as simple by the perceiving mind.

V. Eigenform and Eigenvector -- Quantum Mechanics

In quantum mechanics, observation is modeled not by the eigenform but by its mathematical relative the eigenvector. An eigenvector **v** is a non-zero vector such that **Hv = kv** for some operator observable **H** and constant **k.** The constant **k** is what is regarded as the quantity that is observed (for example the energy of an electron). The operator **H** is taken to be a linear operator on a vector space. The vector space is an infinite dimensional Hilbert space. These are particular properties of the mathematical context of quantum mechanics. The **k** can be eliminated by replacing **H** by **G = H/k** (when **k** is non zero) so that **Gv = (H/k)v = (Hv)k = kv/k = v.** *In quantum mechanics observation is founded on the production of eigenvectors* ***v*** *with* ***Gv=v*** *where* ***v*** *is a vector in a Hilbert space and* ***G*** *is a Hermitian linear operator on that space.*

Heinz was certainly aware, as a practicing physicist, of this model of observation in quantum theory. His theory of eigenforms is a sweeping generalization of quantum mechanics that creates a context for understanding the remarkable effectiveness of that theory. If indeed the world of objects that we take to be an objective (sic) reality is, in fact, a world of tokens for eigenbehaviors, and if physics demands forms of observations that give numerical results, then the simplest example of such observation is the Hermitian observable in the quantum mechanical model.

This is a reversal of epistemology, a complete turning of the world upside down. Heinz has tricked us into considering the world of our experience and finding that it is our world, generated by our actions and that it has become objective through the self-generated stabilities of those actions. He has convinced us to come along with him, and see that all of cybernetics confirms this point of view. He has left the corollaries to us. He has not confronted the physicists and the philosophers head on. He has brought us into his world and let us participate in the making of it. And he has pointed to the genesis and tautological nature of quantum theory to those of us who might ask the question.

But he has also left the consequence of the question to us. For if the world is a world of eigenforms and most of them are in time oscillatory, and unstable, must we insist on stability at the level of our present perception of that world? In principle, there is an eigenform, but that form leads always outward into larger worlds and new understanding. In the case of quantum mechanics, the whole theory has the appearance of an elementary exercise, confirming the view point of objects as tokens for eigenbehaviors in a special case. Heinz leaves us with the conundrum of finding the more general physical theory that confirms that special case.

This dilemma is itself a special case of the dilemma that Heinz has given us. He said it himself many times. If you give a person an undecideable problem, the action of that person in attempting to solve the problem shows who is that person and what is the nature of his/her creativity.

VI. A Conversation with Ranulph Glanville

This essay has its beginnings in a conversation with Ranulph Glanville. Ranulph asked "Does every recursion have a fixed point?", hoping for a mathematician's answer. And I said first, "Well no, clearly not, after all it is common for processes to go into oscillation and so never come to rest." And then I said, "On the other hand, here is the

Theorem: Every recursion has a fixed point.

Proof. Let the recursion be given by an equation of the form

$$\mathbf{X' = F(X)}$$

where X' denotes the next value of X and F encapsulates the function or rule that brings the recursion to its next step. Here F and X can be any descriptors of actor and actant that are relevant to the recursion being studied. Now form

$$\mathbf{J = F(F(F(F(...)))),}$$

the infinite concatenation of F upon itself.
Then we see that

$$F(J) = F(F(F(F(F(\ldots))))) = J.$$

Hence J is a fixed point for the recursion and we have proved that every recursion has a fixed point. QED"

Ranulph said "Oh yes I remember that! Can I quote your proof?", and I said "Certainly, but you will have to make your attribution to Heinz and his paper "Objects: Tokens for (Eigen-)Behaviors" (von Foerster, 1981), for that is where I came to appreciate this result, although I first understood it via the book "Laws of Form" (Spencer-Brown, 1969) by G. Spencer-Brown."

And I went on to say that this theorem was in my view a startling magician's trick on Heinz's part, throwing us into the certainty of an eigenform (fixed point) corresponding to any process and at the same time challenging us to understand the nature of that fixed point in some context that is actually relevant to the original ground of conversation. Ranulph agreed, and our emails settled back into the usual background hum.

Audio-Activity and the Social Context

I kept thinking about that question, and wondering about finding a good mathematical example. Then I remembered learning about the "audio-active sequence" of numbers from John H. Conway (Conway, 1985). This is a number sequence that begins

1, 11, 21, 1211, 111221, 312211, 13112221, 1113213211, ...

Can you find the next number in the sequence?
If you read them out loud, the generating idea becomes apparent.

one, one one, two ones, one two, one one, ...

Each term in the sequence is a description of the digits in the previous member of the sequence. The recursion goes back and forth between number and description of number. What happens as this recursion goes on and on?
Here is a bit more of it:

1

1 1

2 1

1 2 1 1

1 1 1 2 2 1

3 1 2 2 1 1

1 3 1 1 2 2 2 1

1 1 1 3 2 1 3 2 1 1

3 1 1 3 1 2 1 1 1 3 1 2 2 1

1 3 2 1 1 3 1 1 1 2 3 1 1 3 1 1 2 2 1 1

1 1 1 3 1 2 2 1 1 3 3 1 1 2 1 3 2 1 1 3 2 1 2 2 2 1

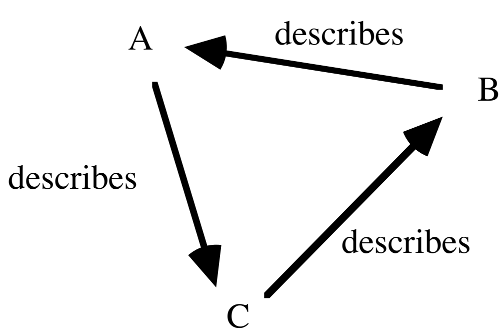
3 1 1 3 2 2 2 1 2 3 2 1 1 2 1 1 1 3 1 2 2 1 1 3 1 2 1 1 3 2 1 1

1 3 2 1 1 3 3 2 1 1 1 2 1 3 1 2 2 1 1 2 3 1 1 3 1 1 2 2 2 1 1 3 1 1 1 2 2 1 1 3 1 2 2 1

Now you can begin to see that there is a approach to a triple of infinite sequences, each describing the next, with the first describing the last. This triple is the limiting condition of the audio-active sequence. In one sense the audio-active sequence oscillates among these three sequences (in the limit), and yet in another sense this triplet of infinite sequences is the eigenform in back of the audio-activity!

A = 11131221131211132221...
B = 3113112221131112311332...
C = 132113213221133112132123...

The triple of infinite sequences are built by continually cycling the self-description through the three sequences. This leads to a definite and highly unpredictable buildup of the three infinite sequences, A,B, C such that B describes A, C describes B and A describes C!



This triplication is the eigenform for the recursion of the audio-active sequence. The triplicate mutual description is the "fixed point" of this recursion. With this example, we begin to see the subtlety of the concept of an eigenform, and how it may apply to diverse human situations. For indeed imagine the plight of three individual human beings Alice, Bob and Carol who each take on the task to describe another, with Bob describing Alice, Carol describing Bob and Alice describing Carol. In the mutual round of their descriptions they may converge on a mutual agreement as do the triplet of audio-active sequences (in the limit). Yet it may take some coaxing to bring forth the agreement and some creativity as well. More complex social situations will be beyond calculation, and yet the principles of the interaction, the possibility of eigenforms will apply. The concept is powerful and important to consider, particularly when one is faced with the incalculable nature of complex interaction.

VII. Generation of Objects

The true question about an object is: How is it generated?
The false question about an object is: What is its classification?

Take a mathematical case in point. Let **R** be the set of all sets that are not members of themselves. (Russell's famous paradoxical set.) We symbolize **R** as follows:
Let **AB** denote the condition that **B** is a member of **A**.
Define **R** by the equation

RX = ~XX

which says *X is a member of R means that it is not the case that X is a member of X.* From this we reach the paradox at once: Substitute R for X. You get

RR = ~RR

R is a member of R means that it is not the case that R is a member of R.

Something curious has happened. We attempt to classify **R** by finding if it was or was not a member of itself and we are led into a round robin that oscillates between membership and non-membership. Classification creates trouble.

Ask how **R** is generated.

We start with some sets we know. For example the empty set is not a member of itself, neither is the set of all cats. So a first approximation to R could be

R1 = {{}, Cats},

where **Cats** denotes the set of all cats (**Cats** is not a cat.)
Now we note that **R1** is also not a member of itself. So we have to add **R1** to get a better approximation **R2.**

R2 = { {} , Cats, {{}, Cats}}.

But **R2** is also not a member of itself and so we would have to add **R2** and keep on with this as well as throwing in other sets come along and are normal. A set is *normal* if it is not a member of itself.

This process will never end! Suppose **S** is any collection of sets that are not members of themselves. Then **S** itself is not a member of itself. **S** is barred from self-membership by the condition that members of **S** must not be members of themselves.
But this means that **S** CANNOT be the set of ALL sets that are not members of themselves, since **S** itself is a set that is not a member of itself, and **S** is not in **S**!

So we have found that any attempt to create the set of ALL normal sets is doomed to failure. It is always possible to make new normal sets from the ones we already have. This is good news. Normal sets are creative in this way. By taking a look at how the Russell set could be generated, we find that the Russell set will never be completed. We find that it is possible to have sets that continue to evolve in time and that time is a necessary concept in mathematics. Only in the course of time can we conceive of a Russell set. The set **R** and the eigenform **RR** exist only as instructions for action. They do not exist as completed entities.

VIII. Number and Multiplicity

Warren McCulloch wrote "What is a number that a man may know it, and what is a man that he may know a number?" (McCulloch, 1965).

It is a mutual shaping that produces these numbers, these ideal objects made from thin air and notches on a stick, and sheep in the field and stars in the sky, and all the discrete appearances of object and relation that fill our time and space.

I create distinct numbers by the conventions of sets and frames.

0 = {}

1 = {{}}

2 = { {},{{}} }

3 = { {}, {{}}, { {}, {{}} } }

...

An infinite sequence of distinct objects, each carrying its own multiplicity. Are each of these also to be seen as eigenforms? Should not each and every one of my constructions be seen as a form in the limit of recursions? Why do natural objects appear full born and just there, when I am sure that they are all limits of recursive process? Am I daft to consider it? Is not the very texture of reflective thought a limit of such a process? Must we go to infinity to find the eigenforms. I do not want to fall into infinite repetition. Not again.

IX. Russell's Numbers

Turn now to Bertrand Russell's definition of number:

"The number of a class is the class of all classes in one-to-one correspondence with that class."

Recall that two classes (sets) **S** and **T** are said to be in one-to-one correspondence if there is a mapping **f:S----->T** that is one-to-one and onto, i.e. every **t** in **T** is of the form **f(s)** for some **s** in **S**, and **f(s) = f(s')** only when **s** and **s'** are equal in **S.** We could say that **S** is a operator or rule(r) that can be used to measure the size of **T**, and that **S** and **T** are in one-to-one correspondence exactly when the ruler fits **T** perfectly. In this sense the number **1 = {{}}** is a measuring rod for singletons, the number **2 = {{},{{}}}** a measuring rod for pairs.

We can write **n(X) = X** if it is possible to establish a one-to-one correspondence between **n** and **X.** As far as number is concerned **3({a,b,c}) = {a,b,c}.** In this way each number becomes an operator and the numerosity of a collection is the property of that collection to be an eigenform for some number **n.**

In Russell's definition of number we see the codification of that relationship between a person and his world that brings forth objects through the actions of the person in relationship to the stabilities of the world. These actions give rise to the higher eigenforms that we call numbers. The stabilities themselves, we tend to relegate to the physical realm, yet they include the stabilities of cognition -- the constancy of objects and relationships in the visual field, the balance of body, dexterity, gesture and imagined surrounding world.

We take these for granted, and it is usually thought that the miracle of recognition of an object arises in some simple way from the assumed existence of the object and the action of our perceiving systems. What is to be appreciated is that this is a fine tuning to the point where the action of the perceiver and the perception of the object are indistinguishable. Such tuning requires an intermixing of the perceiver and the perceived that goes beyond description. In the mathematical levels, such as number, part of the process is slowed down to the point where we can begin to apprehend the process. There is a stability in the comparison, in the one-to-one correspondence that is a process happening at once in the present time. It is this stability, this eigenform that constitutes my knowledge of two hands as two groups of fingers. Each such process depends upon linked and ongoing eigenbehaviors and yet is seen as simple by the perceiving mind.

X. Church and Curry

Church and Curry (Barendregt, 1981) showed (in the 1930's, long before Heinz wrote his essays) how to make eigenforms without apparent excursion to infinity. Their formalism is usually called the "lambda calculus."

Here is how it works.

We wish to find the eigenform for **F**. We want to find a **J** so that **F(J) = J.** Church and Curry admonish us to create an operator **G** with the property that

$$G(X) = F(X(X))$$

for any **X**. That is, when **G** operates on **X, G** makes a duplicate of **X** and allows **X** to act on its duplicate. Now comes the kicker.
Let **G** act on herself and look!

$$G(G) = F(G(G))$$

So **G(G)**, without further ado, is a fixed point for **F**. We have solved the problem without the customary ritual excursion to infinity.

I like to call the construction of the intermediate operator **G**, the "gremlin" (See (Kauffman, 1995, 2001).) Gremlins seem innocent enough. They duplicate entities that meet, and set up an operation of the duplicate on the duplicand. But when you let a gremlin meet a gremlin then strange things happen. It is a bit like the story of the sorcerer's apprentice, except that here the sorcerer is the mathematician or computer scientist who controls context, and the gremlins are like the self-duplicating brooms in the story. The gremlins can go wild without some control. In computer science the gremlins are programs with loops in them. If you do not put restrictions on the loops, things can get very chaotic!

Once an appropriate gremlin is in place, clocks will tick and numbers will count. Here is a simple model.

Please agree that **@ = @*** so that **@** is a fixed point for the token ***.**

Then

@ = @* = @ = @*** = @**** = @***** = ...**

A counting process ensues immediately from the equation

@=@*.

That equation is the result of the gremlin **XG = XX*** and **@ = GG.**

Counting is the emanation of self-reference.
Counting is the unfolding of an eigenform.

Here we see how close are the concepts of process (counting) and fixed point (eigenform). It is through the eigenform that counting occurs, but we still need a higher level of observation to make use of this capacity. It is still necessary to be able to read, start and stop the clock. This points to the fact that a formal eigenform must be placed in a context in order for it to have human meaning. The struggle on the mathematical side (or computer science side) is to control recursions, bending them to desired ends (See (Kauffman, 2002) for a discussion of these concepts in the context of proofs generated by humans and by machines.) The struggle on the human side is to cognise a world sensibly and communicate well and effectively with others. For each of us, there is a continual manufacture of eigenforms (tokens for eigenbehaviour) but such tokens will not pass as the currency of communication unless we achieve mutuality as well. One can say that mutuality itself is a higher eigenform. Achieving mutual understanding will be recognized when we have it or begin to have it. As with all eigenforms, the abstract version exists. Realization can happen in the course of time.

XI. Differentiation Creates Number

What does this Church Curry method have to do with the invariance of the exponential function under differentiation?

D(exp(t)) = exp(t) where **D=d/dt**?

In fact,

exp(t) = a[0] + a[1] + a[2] + ...

where

a[n] = (t^n)/n!

so that

Da[0] = 0,
Da[n+1] = a[n]

from which it follows that

D(exp(t)) = exp(t).

Note that this is like modeling D[X] as the operator that REMOVES a box from around X. Strange thoughts ensue such as

$$D\,\boxed{} \;=$$

$$D\,\boxed{\boxed{}} \;=\; \boxed{}$$

$$D\,\boxed{\boxed{\boxed{}}} \;=\; \boxed{\boxed{}}$$

$$D\,\boxed{\boxed{\boxed{\boxed{}}}} \;=\; \boxed{\boxed{\boxed{}}}$$

$$J = \boxed{\boxed{\boxed{\boxed{\cdots}}}}$$

$$DJ = J$$

The strangeness here is the meaning of the infinite concatenation of the operator D. For if D* = ...DDD, then D*(J) = "Nothing" in at least one interpretation of these symbols.

Lets return to the construction of numbers.

0={}

1={{}}

2={{},{{}}}

It is a recursive process where

0={}

and

n +1 = {0,1,2,3,...,n} = { Dn, n}

where DS is the list obtained from a set S by removing its outer bracket.

Note that **D{}** is nothing, **D{{}} = {}. D{{},{{}}} = {},{{}}** and so on.

For the limit singleton **W={W}** we have **DW = W.** We now have the basic equation

N + 1 = {DN, N}.

In this form, the counting process resists the production of fixed points. For example, if we let

I = {0,1,2,3,...}

be the first ordered countable infinity of integers, then

$I + 1 = \{DI, I\} = \{0,1,2,3,..., I\}$

is a new set distinct from I, and I + 2 is distinct from I +1. The counting process continues without end.

XII. The Object of Set Theory

Lets look at objects from the point of view of a set theoretician. If A and B are objects, then we can form a *new object* C = {A,B}, the set consisting of A and B. This seems harmless enough. After all, if Chicago and New York are objects, then the set of large coastal cities in the United States should also be an object, albeit of a different type. We give up something with these mathematical objects. We do not assume that they have specific spatial locations. After all, what is the spatial location of the set {Chicago, New York}?

Take New York. This is a good big object to talk about. It is a place. It has a location. It has contents, all the people in it, all the goods and people and ideas and music running through it.
And we will leave all that and just take the set theoretic point of view and look at the singleton set {New York}. Now {New York} is not New York. Not by a long shot! New York is a hustling bustling metropolis on the East Coast of the United States of America. New York has millions of inhabitants and buildings, and New York is

constantly changing. On the other hand, the singleton {New York} has exactly one member. It never changes. It is always the set whose member is New York. On top of this, once we have admitted the singleton object {New York} into existence, we are compelled to allow the singleton of its singleton to come on the stage with its only member the singleton of New York: {{New York}}. There is a infinity of singleton objects derived from New York waiting in the wings:

{New York}
{{New York}}
{{{New York}}}
{{{{New York}}}
...
{{{{{{{{{{{{{{{{{New York}}}}}}}}}}}}}}}}}
...
...
{{{{{{{{{{{{{{{{{{{{{{{...}}}}}}}}}}}}}}}}}}}}}}} = W

The limiting New York singleton W has New York infinitely down in the nest of parentheses. New York has disappeared, and all that is left of the Cheshire Cat is its grin. All that is left in the limit W is the fact that W is invariant under the act of forming the singleton. We have that W is its own singleton!

W = {W}

Adding one more level of parentheses makes no difference.

W is at a level where the level and the metalevel are one. W is both object and subject of the set theoretic discourse. And if you think that W has nothing to do with New York you are wrong! W is the very identity of New York. W is the ultimate singleton associated with New York. W is the essence of New York. And at the same time W is entirely content free and has nothing to do with New York. W is just an infinite nest of brackets. An uninterpreted bit of self-reference in the void. Will you have it both ways? You could locate W anywhere. Why not New York?

XIII. Singletons and Eigenforms

The example of New York illustrates the extreme eigenform associated with any object in the set theoretic universe. We can iterate the operation of framing X to form the singleton {X} ad infinitum, and lose X in the infinite depths of the recursion. We lose X and regain the ubiquitous and self-referential W = {W}. This could bring one to be suspicious of the concept of singleton set. After all, why should New York or any proper object in the world be surrounded by an infinite halo of singletons? When I eat an apple, must I devour {apple}, {{apple}}, {{{apple}}}, ... as well? Quine and others have suggested that we take a different approach to framing so that singletons do not appear. One way to achieve this is to legislate that {S} = S for any S that is non-empty.

Think about this proposal. We would have {{}} = {} and there would be no way to produce a set with one element! One *could* say that there are special objects in the theory, lets call them a,b,c, ... such that each object is its own singleton:

{a}=a, {b} = b, {c} = c, ...

Then, at least the singletons for these "real" objects collapse back to them. This approach raises many questions. What are the special objects. Certainly no mathematician would want the empty set to be among them, since we wish to discriminate between the empty set and the set whose member is the empty set. Searching for these special objects is something like searching for elementary particles. Where are they? Could it be that I am a special object? Lets see. Is it the case that **I = {I}.** Why yes indeed! I can frame (think about) myself and I am still myself! In fact if we interpret the emergence of a frame as an act of reflection (thought), then the special objects appear as elements of I-ness, as signs for themselves in the sense of Charles Sanders Peirce (Kauffman, 2001).

But set theory goes its own way, and would weave these special objects together into hierarchies that embody singletons once again. Take two specials **a** and **b.** Form their union **{a,b}.** Is this special? Why not? Why not allow that if any element of a set **S** is special, then **S** is also special? This still allows room for classical mathematics. We can always form sets like {{},{{}}} that are not special. Note that **a = {a}** does not imply that **a** is equal to an infinite nest of parentheses. The infinite nest **W = {{{{{{{...}}}}}}}** is but one of many special objects that are their own singletons.

By including special objects into set theory and these rules for their composition, we have created a model for a set theoretic world that contains a parable (a parallel parable!) of the mental and the physical. The purely mental world is the class of sets generated from the empty set. The purely physical world is the class of sets generated from the special objects. The interface of mental and physical occurs as they touch in the limit of nested parentheses. **W = {{{{{{{...}}}}}}}** is an amphibian living in both worlds. **W** is the eigenform that crosses the boundary from the mental to the physical.

XIV. The Form of Names

The simplicity of a thought, the apparent clarity of distinction is mirrored in the sort of eigenforms that come from the Church-Curry realm as described in the last section. Consider a linguistic example: Each person has a name (at least one). In the course of time we are introduced to people and come to know their names. We know that name not as an item to look up about the person (and this applies to certain objects as well) but as a direct property of the person. That is, if I meet Heinz he appears to me as Heinz, not as this person with certain characteristics, whose name I can find in my social database if I care to do so. It is like this only when we are first introduced. At the point of introduction there is this person and there is his name separate from him. Once learned, *the name is shifted* and occurs in space right along with the person. Heinz and his name are in the same cognitive space which is also in the same place as the apparent physical space. We can observe this shifting process in the course of learning a name. We can also observe how physical and cognitive spaces are superimposed. The many classical optical illusions illustrate these matters vividly.

Now we have Heinz with his name inseparable from his presence, and this is true even if he is not physically present, for the shift has occurred and will not be undone. But we also have his name Heinz separate from him, and able to be pinned upon another. And we have his name not quite separate from him, but rather this Heinz is the name of the name we have attached to him! This is Heinz's metaname. How do we distinguish among all these different names for Heinz? We use the same symbols for them, yet they are different. Lets choose a way to indicate the differences.
We start with the reference.

Heinz -----> Cybernetic Magician

(The arrow will indicate that the entity on the left is the name of the entity on the right.)

We get to know him and shift the reference.

#Heinz -----> Cybernetic Magician Heinz

Now the name is in the cognitive space of Heinz, and the metaname #Heinz refers to that conjunction. We shall call this **the indicative shift.**

name -----> object
#name-----> object name

The indicative shift occurs, constantly weaving the apparent external reality with the linguistic reality.

Self-reference occurs when one calls up (names) the metanaming operator.

At first the metanaming operator is not marked and no name has been chosen for it. But then its name is chosen (as #).
We have

 # ----->

That is, # refers at first to the singular place where there is an absence of naming, a void in the realm of distinctions.

Then the shift occurs. We have the reference of the meta-naming operator to itself (as the operator enters a space formerly void!).

 # -----> #

Suppose that the meta naming operator has another name, say M. Then we have

M -----> #

which shifts to a self-reference at the second articulated level of meta naming.

#M -----> #M

These are the eigenforms of self consciousness in the realm of names.

Heinz said (von Foerster, 1981):

"I am the observed link between myself and observing myself."

Self-reference, the action of a domain upon itself, leading to cognition, is the beginning of the realm of eigenforms in Heinz's world. "I am I." is the shortest explicit loop. "I am." is the shortest prescription for eigenform.

Remark. The indicative shift provides the formalism for statements that refer to their own referents. Let

g -----> F#

denote a reference of **g** to the statement (form) **F#.** Shifting, we obtain

#g -----> F#g.

This is the form of **F** speaking about **#g**, where **#g** is the name of the statement **F#g**. Thus **F#g** is speaking about itself. For a longer discussion of this structure, see (Kauffman, 1994, 1995).

XV. Cantor's Diagonal Argument, Lawvere's Eigenform and Imaginary Values Beyond True and False

We cannot resist including here a magnificent application of fixed points as eigenforms that is due to the category theorist Lawvere (Lawvere, 1970). Lawvere found a new way to prove a famous theorem of Georg Cantor, founder of transfinite set theory. Cantor's theorem states that:

Cantor's Theorem. For every set **S**, there are more subsets of **S** than there are members of **S.**

Lawvere's Proof of Cantor's Theorem.
Every subset A of a set S can be

specified by a a mapping from S to the set of truth values {T,F}, specifying T if an element is in the subset A and specifying F if the element is not in the subset. We can write

<A>: S -----> {T,F}

for this function.

Now suppose that there are just as many subsets of S as there are elements of S. Then we could label each subset of S with an element of S. We would have subsets indexed by elements of S, writing **A[s]** for the subset that corresponds to the element **s**. We are going to prove that we always run out of labels! The crux of the matter is the fact that the element **s** either is or is not a member of **A[s].**

The hypothesis that every subset has a label means that every corresponding function **<A[s]>(t) = True or False** is really a function of two elements of **S**. We can define a function

G: S x S -----> {T,F}

(**S x S** denotes pairs **(s,t)** of elements of **S**.)
by the formula

G(s,t) = <A[s]>(t),

and this function will specify **T** or **F** according to whether **t** is a member of the subset with label **s**. Our assumption (which we want to show is absurd) is that there exists this function **G** from **S x S** to the set of truth values **{T, F}.**

Now Lawvere does the following. He shows that *the labeling hypothesis leads to the existence of a fixed point for every mapping from the truth set {T,F} to itself!* This is a contradiction since negation (**~T=F** and **~F=T**) has no fixed points. Thus the existence of fixed points is at the heart of Lawvere's proof of Cantor's Theorem.

How does Lawvere accomplish this feat?
Take any function **J** defined on the set of truth values to itself.

$$\mathbf{J: \{T,F\} \text{ -----> } \{T,F\}.}$$

Now look at the function

$$\mathbf{Q(t) = J(G(t,t)).}$$

This is a function from **S** to **{ T, F}**, and so by our assumption there must be an element **s** of **S** such that
Q(t) = G(s ,t) for all **t** in **S.** Thus we have

$$\mathbf{J(G(t,t)) = G(s, t)} \quad \text{for all } \mathbf{t}.$$

Hence

$$\mathbf{J(G(s,s)) = G(s,s).}$$

G(s,s) is fixed by the mapping **J.**
We have shown that every mapping from **{T,F}** to itself has a fixed point! This is the desired contradiction, and completes Lawvere's proof of Cantor's Theorem.//

If we had enlarged the truth set to

$$\mathbf{\{T,F,\ I\}}$$

where **~I=I** is an eigenform for negation, then **G(s,s)** would have value **I**. What does this mean? It means that the index of the set **A** corresponding to **G** would have an oscillating truth value. If that index **s** is in **A** then it is not in **A** and if the index is not in **A** then it is in **A**. We would be propelled into sets that vary in time. Not a bad idea, even if it is time to end this essay.

XVI. Epilogue

This paper has been a contemplation of the concept of recursion in the context of second order cybernetics. The simple idea of iterating an operation upon itself is seen to be a key to understanding the nature of objects and the relationship of an observer and the apparent world of the observer. In this view, the observer does not stand outside the world and "see" it. Rather, what is seen is a token, an eigenform, of the recursive participation of the observer in a world where there is no separation of the observer and the observed. The experience of separation can just as well be an experience of joining in that participation. Objects become our own creations and the world is the theatre of our actions upon it, which is us. It was Heinz von Foerster's ability to create himself, us, and our world as magic theatre that makes these views so vivid to us. The existence of clear formalisms for recreating these ideas will help them grow forward into the future. We imagine eternity, and in so doing create the flow of time.

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