Charting the Complexity Landscape of Virtual Network Embeddings

IFIP Networking 2018

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Introduction: Virtual Network Embeddings and Their Complexity

Virtualization: Resource Allocation Opportunities

‘Classic’ Cloud Computing

- User requests virtual machines
- No guarantee on network performance

Goal: Virtual Networks (since \(\approx 2006\))

- Communication requirements given
- Network performance will be guaranteed

User requests virtual machines

bunch of VMs

requests
Introduction: Virtual Network Embeddings and Their Complexity

Virtualization: Resource Allocation Opportunities

‘Classic’ Cloud Computing

Goal: Virtual Networks (since ≈ 2006)

Novel Service Abstractions

- Virtual Networks – overlays (≈ 2006)
- Virtual Clusters – batch processing (≈ 2011)
- Service Chain – stitch functions (≈ 2013)
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Virtualization: Resource Allocation Opportunities

‘Classic’ Cloud Computing

Goal: Virtual Networks (since ≈ 2006)

Novel Service Abstractions

- Virtual Network Embedding Problem
- Virtual Clusters Embedding Problem
- Service Chain Embedding Problem

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Virtual Network Embeddings at a Glance
Introduction: Virtual Network Embeddings and Their Complexity

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Virtual Network Embeddings at a Glance

Embedding Restrictions

Capacity
- V Node
- E Edge

Additional
- N Node placement
- R Routing
- L Latencies
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Novel Service Abstractions

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Virtual Network Embeddings at a Glance – Node Capacities

Embedding Restrictions

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Novel Service Abstractions

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Virtual Network Embeddings at a Glance – Edge Capacities

Embedding Restrictions

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Virtual Network Embeddings at a Glance – Node Placement

Embedding Restrictions

- Capacity
  - V Node
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Novel Service Abstractions

- Virtual Network Embedding Problem
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Virtual Network Embeddings at a Glance – Routing Restrictions

Embedding Restrictions

Capacity
V Node
E Edge

Additional
N Node placement
R Routing
L Latencies
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**Novel Service Abstractions**
- Virtual Network Embedding Problem
- Virtual Clusters Embedding Problem
- Service Chain Embedding Problem

**Virtual Network Embeddings at a Glance – Latencies**

**Embedding Restrictions**
- **Capacity**
  - V Node
  - E Edge
- **Additional**
  - N Node placement
  - R Routing
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Novel Service Abstractions

- Virtual Network Embedding Problem
- Virtual Clusters Embedding Problem
- Service Chain Embedding Problem

Taxonomy of VNEP Variants

VNEP $\langle \mathcal{C} | \mathcal{A} \rangle$: capacity restrictions $\mathcal{C}$ & additional restrictions $\mathcal{A}$

Embedding Restrictions

Capacity
- $\mathcal{V}$ Node
- $\mathcal{E}$ Edge

Additional
- $\mathcal{N}$ Node placement
- $\mathcal{R}$ Routing
- $\mathcal{L}$ Latencies

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Novel Service Abstractions

Virtual Network

Virtual Cluster

Service Chain

Taxonomy of VNEP Variants

VNEP $\langle C | A \rangle$: capacity restrictions $C$ & additional restrictions $A$

For example...

VNEP $\langle \text{VE} | - \rangle$: node and edge capacities
VNEP $\langle \text{V} | \text{R} \rangle$: node capacities and routing restrictions
VNEP $\langle - | \text{NL} \rangle$: node placement and latency restrictions

Embedding Restrictions

Capacity

- $\text{V}$ Node
- $\text{E}$ Edge

Additional

- $\text{N}$ Node placement
- $\text{R}$ Routing
- $\text{L}$ Latencies
### Related Work

<table>
<thead>
<tr>
<th>Theoretical Results: <strong>Few</strong></th>
<th>Practical Results: <strong>Many</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Andersen [2002] Considered $\langle \text{VE} \mid - \rangle$ and argued for $\mathcal{NP}$-hardness</td>
<td></td>
</tr>
<tr>
<td>Amaldi et al. [2016] Considered $\langle \text{VE} \mid \text{N} \rangle$ under <em>profit objective</em>, proved $\mathcal{NP}$-hardness and derived inapproximability result.</td>
<td></td>
</tr>
<tr>
<td>Generally More than 100 papers on VNEP alone, for example . . .</td>
<td></td>
</tr>
<tr>
<td>Chowdhury et al. [2009] Developed algorithms for variant $\langle \text{VE} \mid \text{N} \rangle$ and hoped to obtain <em>approximations</em>.</td>
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VNEP is of **crucial** importance, yet is **hardly understood**!
Introduction: Virtual Network Embeddings and Their Complexity

VNEP is of **crucial** importance, yet is **hardly understood**!

**Our Contributions**

1. \( \mathcal{NP} \)-completeness under restrictions \( \langle \text{VE} \mid - \rangle, \langle E \mid N \rangle, \langle V \mid R \rangle, \langle - \mid NR \rangle, \langle - \mid NL \rangle \).
2. Relaxed model: \( \mathcal{NP} \)-completeness of computing approximate embeddings.
3. Restricted input: \( \mathcal{NP} \)-completeness pertains when restricting request topologies.

**Practical Implications** (unless \( P = \mathcal{NP} \))

There cannot exist a polynomial-time algorithm . . .

1. always yielding a solution to the VNEP under any of the above restrictions,
2. which does not violate capacities or latencies by less than some amount,
3. even when virtual networks are acyclic, planar, and degree-bounded.
Introduction: Virtual Network Embeddings and Their Complexity

VNEP is of crucial importance, yet is hardly understood!

Our Contributions

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## Introduction: Virtual Network Embeddings and Their Complexity

VNEP is of **crucial** importance, yet is **hardly understood**!

### Our Contributions

1. **$NP$-completeness** under restrictions $\langle VE | - \rangle$, $\langle E | N \rangle$, $\langle V | R \rangle$, $\langle - | NR \rangle$, $\langle - | NL \rangle$.
2. **Relaxed model:** $NP$-completeness of computing **approximate embeddings**.
3. **Restricted input:** $NP$-completeness pertains when restricting request topologies.

### Practical Implications (unless $P = NP$)

There cannot exist a polynomial-time algorithm . . .

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Definition of the Virtual Network Embedding Problem
Definition of the Virtual Network Embedding Problem

Input

Substrate \( G_S = (V_S, E_S) \)
Request \( G_r = (V_r, E_r) \)
Restrictions ... 

Feasible Embedding

A feasible embedding is a mapping of \( G_r \) to \( G_S \) respecting all restrictions.

Feasible embedding meets all requirements

Virtual Network

Substrate (Physical Network)

Embedding
Definition of the Virtual Network Embedding Problem

**Input**
- **Substrate** \( G_S = (V_S, E_S) \)
- **Request** \( G_r = (V_r, E_r) \)
- **Restrictions** ...

**Feasible Embedding**
A *feasible* embedding is a mapping of \( G_r \) to \( G_S \) respecting all restrictions.

**Feasible embedding meets all requirements**

**Virtual Network Embedding Problem (Decision Variant)**
Decide whether a feasible embedding of request \( G_r \) on substrate \( G_S \) exists.
Output: Yes / No.
Methodology
Reminder: 3-SAT and \(NP\)-Completeness

3-SAT-Formula \(\phi\)

\(\phi = \bigwedge_{C_i \in C_\phi} C_i\) with \(C_i \in C_\phi\) being disjunctions of at most 3 (possible negated) literals.

Example 3-SAT formula \(\phi\) over literals \(L_\phi = \{x_1, x_2, x_3, x_4\}\)

\[
\phi = \left( x_1 \lor x_2 \lor x_3 \right) \land \left( \bar{x}_1 \lor x_2 \lor x_4 \right) \land \left( x_2 \lor \bar{x}_3 \lor x_4 \right)
\]
Reminder: 3-SAT and $\mathcal{NP}$-Completeness

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Definition of 3-SAT

Decide whether satisfying assignment $a : L_{\phi} \rightarrow \{F, T\}$ exists for formula $\phi$. Output: Yes/No.
Reminder: 3-SAT and $\mathcal{NP}$-Completeness

3-SAT-Formula $\phi$

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Decide whether satisfying assignment $a : L_\phi \rightarrow \{F, T\}$ exists for formula $\phi$. Output: Yes/No.

Theorem: Karp [1972]

3-SAT is $\mathcal{NP}$-complete.
Reminder: 3-SAT and $\mathcal{NP}$-Completeness

**Definition of 3-SAT**

Decide whether satisfying assignment $a : \mathcal{L}_\phi \rightarrow \{F, T\}$ exists for formula $\phi$. Output: Yes/No.

**Theorem: Karp [1972]**

3-SAT is $\mathcal{NP}$-complete.

**A Decision Problem is $\mathcal{NP}$-complete if . . .**

. . . it lies in $\mathcal{NP}$ and *all* other decision problems in $\mathcal{NP}$ can be reduced to it.
Methodology: Proving $\mathcal{NP}$-completeness

1. VNEP lies in $\mathcal{NP}$ (answer can be checked in polynomial time).
2. Reduction from 3-SAT to VNEP.
Methodology: Proving $NP$-completeness

Proving $NP$-completeness of the VNEP

1. VNEP lies in $NP$ (answer can be checked in polynomial time).
2. Reduction from 3-SAT to VNEP.

Outline of Reduction Framework

3-SAT instance $\phi$ $\mapsto$ VNEP instance $(G_r(\phi), G_S(\phi), \text{mapping restrictions})$

$\phi$ satisfiable? $\iff$ feasible embedding of $G_r(\phi)$ on $G_S(\phi)$ under restrictions?
Methodology: Proving $NP$-completeness

Proving $NP$-completeness of the VNEP

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3-SAT instance $\phi$ $\overset{\longrightarrow}{\mapsto}$ VNEP instance $(G_r(\phi), G_S(\phi), \text{mapping restrictions})$

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Our Reduction Framework

Proving \( \mathcal{NP} \)-completeness of the VNEP

1. VNEP lies in \( \mathcal{NP} \) (answer can be checked in polynomial time).
2. Reduction from 3-SAT to VNEP.

Outline of Reduction Framework

3-SAT instance \( \phi \) \( \rightarrow \) VNEP instance \((G_r(\phi), G_S(\phi), \text{mapping restrictions})\)

\( \phi \) satisfiable? \( \iff \) feasible embedding of \( G_r(\phi) \) on \( G_S(\phi) \) under restrictions?
Our Reduction Framework

Input: 3-SAT formula

\[ \phi = (x_1 \lor x_2 \lor x_3) \land (\bar{x}_1 \lor x_2 \lor x_4) \land (x_2 \lor \bar{x}_3 \lor x_4) \]

Request \( G_r(\phi) \)

- \( V_r(\phi) = \{v_i | C_i \in C_\phi\} \)
- \( E_r(\phi) = \{ (v_i, v_j) | C_i \text{ introduces literal used by } C_j \} \)

Substrate \( G_S(\phi) \)

- one node per clause and per satisfying assignment
- edges as for the requests, if assignments do not contradict

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\( \phi: (x_1 \lor x_2 \lor x_3) \land (\bar{x}_1 \lor x_2 \lor x_4) \land (x_2 \lor \bar{x}_3 \lor x_4) \)

\( G_{r(\phi)}: \)

\( G_{S(\phi)}: \)

v1

\( x_1, x_2, x_3 : TTT \)
\( x_1, x_2, x_3 : TTF \)
\( x_1, x_2, x_3 : TFT \)
\( x_1, x_2, x_3 : TFF \)
\( x_1, x_2, x_3 : FTT \)
\( x_1, x_2, x_3 : FFT \)
\( x_1, x_2, x_3 : FFF \)

v2

\( x_1, x_2, x_4 : TTT \)
\( x_1, x_2, x_4 : TTF \)
\( x_1, x_2, x_4 : TFT \)
\( x_1, x_2, x_4 : TFF \)
\( x_1, x_2, x_4 : FTT \)
\( x_1, x_2, x_4 : FFT \)
\( x_1, x_2, x_4 : FFF \)

v3

v1 \rightarrow v2 \rightarrow v3
3-SAT instance $\phi \rightarrow$ VNEP instance ($G_r(\phi), G_S(\phi), \text{mapping restrictions}$)

$\phi$ satisfiable? $\iff$ feasible embedding of $G_r(\phi)$ on $G_S(\phi)$ under restrictions?
Our Reduction Framework

### Base Lemma

Formula $\phi$ is satisfiable if and only if there exists a mapping of $G_r(\phi)$ on $G_s(\phi)$, s.t.

1. each virtual node $v_i$ is mapped to a ‘satisfying assignment node’ of the $i$-th clause, and
2. all virtual edges are mapped on exactly one substrate edge.

### Visualization of Conditions (1) and (2)

![Visualization Diagram]

$v_1 \rightarrow v_2 \rightarrow v_3$

$x_1, x_2, x_3 : TTT$  $x_1, x_2, x_4 : TTT$  $x_2, x_3, x_4 : TTT$

$x_1, x_2, x_3 : TTF$  $x_1, x_2, x_4 : TTF$  $x_2, x_3, x_4 : TTF$

$x_1, x_2, x_3 : TFT$  $x_1, x_2, x_4 : TFT$  $x_2, x_3, x_4 : TFT$

...  ...  ...

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Our Reduction Framework

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Example

$\phi = (x_1 \lor x_2 \lor x_3) \land (\overline{x}_1 \lor x_2 \lor x_4) \land (x_2 \lor \overline{x}_3 \lor x_4)$

Assignment for $\phi$

- $x_1 \mapsto T$
- $x_2 \mapsto T$
- $x_3 \mapsto F$
- $x_4 \mapsto F$

Example

Request

Embedding satisfying conditions (1) and (2)
Our Reduction Framework

**Base Lemma**

Formula $\phi$ is satisfiable if and only if there exists a mapping of $G_r(\phi)$ on $G_S(\phi)$, s.t.

1. each virtual node $v_i$ is mapped to a ‘satisfying assignment node’ of the $i$-th clause, and
2. all virtual edges are mapped on exactly one substrate edge.

**Application of Base Lemma for VNEP Variant $\langle X \mid Y \rangle$**

VNEP $\langle X \mid Y \rangle$ is $\mathcal{NP}$-complete if we can enforce all feasible embeddings to satisfy (1) and (2).

3-SAT instance $\phi$ $\mapsto$ VNEP instance $(G_r(\phi), G_S(\phi), \text{under mapping restrictions})$

$\phi$ satisfiable? $\iff$ feasible embedding of $G_r(\phi)$ on $G_S(\phi)$ under restrictions?
Results

Base Lemma

Formula φ is satisfiable if and only if there exists a mapping of $G_r(\phi)$ on $G_s(\phi)$, s.t.

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VNEP $\langle - | \text{NR} \rangle$ is $\mathcal{NP}$-complete.

Node placement enforce (1)

Routing restrictions enforce (2)
Results

Base Lemma

Formula $\phi$ is satisfiable if and only if there exists a mapping of $G_r(\phi)$ on $G_S(\phi)$, s.t.

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2. all virtual edges are mapped on exactly one substrate edge.

VNEP $\langle - \mid \text{NL} \rangle$ is $NP$-complete.

Node placement enforce (1)

Placement and latency restrictions enforce (2)

maximal latency per virtual edge: 1

latency of substrate edges: 1
$\mathcal{NP}$-Completeness shown for $\langle - | \text{NR} \rangle$ and $\langle - | \text{NL} \rangle$

In the paper: $\langle \text{VE} | - \rangle$, $\langle E | N \rangle$, $\langle V | R \rangle$. 
Results

\( \mathcal{NP} \)-Completeness shown for \( \langle - \mid NR \rangle \) and \( \langle - \mid NL \rangle \)

In the paper: \( \langle VE \mid - \rangle \), \( \langle E \mid N \rangle \), \( \langle V \mid R \rangle \).

Implications of \( \mathcal{NP} \)-Completeness

- Finding a feasible embedding for the VNEP is \( \mathcal{NP} \)-complete.
- Finding an optimal feasible embedding subject to any objective is \( \mathcal{NP} \)-hard.
- There cannot exist polynomial-time approximation algorithms (unless \( P = NP \)).
$\mathcal{NP}$-Completeness of Computing Approximate Embeddings
NP-Completeness of Computing Approximate Embeddings

**Insight:** If the problem is too hard, relax the model.

How hard are the VNEP variants when we allow for capacity violations or latency violations?

**Allowing for Node Capacity Violations**

Relaxation: We allow for substrate node capacity violations by a factor $\alpha < 2$.

Result: $\langle \text{VE} \mid \cdot \rangle$ and $\langle \text{V} \mid \text{R} \rangle$ stay NP-complete and inapproximable (unless $\mathcal{P} = \mathcal{NP}$).

**Allowing for Latency Violations**

Relaxation: We allow for latency violations by a factor $\gamma < 2$.

Result: $\langle \cdot \mid \text{NL} \rangle$ stays NP-complete and inapproximable (unless $\mathcal{P} = \mathcal{NP}$).

**Allowing for Edge Capacity Violations** (proven in our technical report [Rost and Schmid, 2018])

Relaxation: We allow for substrate edge capacity violations by a factor $\beta < 2$.

Result: $\langle \text{VE} \mid \cdot \rangle$ and $\langle \text{E} \mid \text{N} \rangle$ stay inapproximable for $\beta \in \mathcal{O}(\log n / \log \log n)$, $n = |V_S|$, unless $\mathcal{NP} \subseteq \mathcal{BP}$-TIME$^a(\bigcup_{d \geq 1} n^d \log \log n)$.

$^a\text{BP}$-TIME: Bounded-Error Probabilistic Polynomial-Time
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### Allowing for Latency Violations

**Relaxation:** We allow for latency violations by a factor $\gamma < 2$.

**Result:** $\langle - \mid \mathbf{NL} \rangle$ stays $\mathcal{NP}$-complete and inapproximable (unless $\mathcal{P} = \mathcal{NP}$).

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\textit{aBPTIME: Bounded-Error Probabilistic Polynomial-Time}
### NP-Completeness of Computing Approximate Embeddings

**Insight:** If the problem is too hard, relax the model.

How hard are the VNEP variants when we allow for capacity violations or latency violations?

#### Allowing for Node Capacity Violations

**Relaxation:** We allow for substrate node capacity violations by a factor $\alpha < 2$.

**Result:** $\langle \text{VE} \mid - \rangle$ and $\langle V \mid R \rangle$ stay $\mathcal{NP}$-complete and inapproximable (unless $P = \mathcal{NP}$).

#### Allowing for Latency Violations

**Relaxation:** We allow for latency violations by a factor $\gamma < 2$.

**Result:** $\langle - \mid \text{NL} \rangle$ stays $\mathcal{NP}$-complete and inapproximable (unless $P = \mathcal{NP}$).

#### Allowing for Edge Capacity Violations (proven in our technical report [Rost and Schmid, 2018])

**Relaxation:** We allow for substrate edge capacity violations by a factor $\beta < 2$.

**Result:** $\langle \text{VE} \mid - \rangle$ and $\langle E \mid N \rangle$ stay inapproximable for $\beta \in \mathcal{O}(\log n / \log \log n)$, $n = |V_S|$, unless $\mathcal{NP} \subseteq \mathcal{BP}$-$\text{TIME}^a(\bigcup_{d \geq 1} n^d \log \log n)$.

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*a\mathcal{BP}$-$\text{TIME}:$ Bounded-Error Probabilistic Polynomial-Time
\( \mathcal{NP} \)-Completeness when Restricting Graph Classes
NP-Completeness for Restricted Graph Classes

Insight: If the problem is too hard, restrict the model inputs.

How hard are the VNEP variants when we restrict the graph classes for the substrate and the requests?

Restriction on Directed Acyclic Graphs

By construction, the graphs $G_r(\phi)$ and $G_S(\phi)$ are directed acyclic graphs (DAGs). Accordingly, the hardness results pertain when restricting the input graphs to be DAGs.

Restriction of Requests to Planar Degree-Bounded Graphs

Restriction: The request graph must be a planar and degree-bounded.

Result: All previous results pertain based on a reduction from a special planar 3-SAT variant.
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Conclusion
Our Contributions

1. $\mathcal{NP}$-completeness under restrictions $\langle \text{VE} \mid - \rangle$, $\langle \text{E} \mid \text{N} \rangle$, $\langle \text{V} \mid \text{R} \rangle$, $\langle - \mid \text{NR} \rangle$, $\langle - \mid \text{NL} \rangle$.

2. Relaxed model: $\mathcal{NP}$-completeness of computing approximate embeddings.

3. Restricted input: $\mathcal{NP}$-completeness pertains when restricting request topologies.
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Outlook

Further Results  Proof \(\mathcal{NP}\)-completeness for \(\langle \text{V} \mid \text{RL} \rangle\), consider uniform capacities, ... 

   Improvements Improvement of lower bounds for approximate embeddings.

Gained Insights  The VNEP is really hard.

   • Justifies using heuristics and exponential-time algorithms.
   • Approximations require model relaxations/restrictions.
Our Contributions

1. \( \mathcal{NP} \)-completeness under restrictions \( \langle VE \mid - \rangle, \langle E \mid N \rangle, \langle V \mid R \rangle, \langle - \mid NR \rangle, \langle - \mid NL \rangle \).
2. Relaxed model: \( \mathcal{NP} \)-completeness of computing approximate embeddings.
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2. **Relaxed model:** NP-completeness of computing approximate embeddings.

3. **Restricted input:** NP-completeness pertains when restricting request topologies.

### Outlook

**Further Results**
Proof NP-completeness for $\langle \text{V} | \text{RL} \rangle$, consider *uniform* capacities, ... 

**Improvements**
Improvement of lower bounds for approximate embeddings.

**Gained Insights**
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*More about VNEP approximations in my talk tomorrow.*
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1. \( \mathcal{NP} \)-completeness under restrictions \( \langle \text{VE} \mid - \rangle, \langle \text{E} \mid \text{N} \rangle, \langle \text{V} \mid \text{R} \rangle, \langle - \mid \text{NR} \rangle, \langle - \mid \text{NL} \rangle \).
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Thank you! Questions?

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