ILP formulations for finding optimal locations for charging stations in an electric car sharing network

Georg Brandstätter\textsuperscript{1}  Markus Leitner\textsuperscript{1}  
Ivana Ljubic\textsuperscript{2}  Mario Ruthmair\textsuperscript{1}  

\textsuperscript{1}University of Vienna, Austria  
\textsuperscript{2}ESSEC Business School of Paris, France

January 21, 2016
Introduction

Context

- **e4-share**: Models for Ecological, Economical, Efficient, Electric Car-Sharing
- Study and solve optimization problems arising in planning and operating car sharing system using electric vehicles

Electric Vehicles

- **more efficient** and **less polluting** (in urban settings)
- **shorter range** and thus frequent recharging necessary

This work

- ILP formulations to find **optimal locations for charging stations**
- cars are picked up from / returned to these stations
- start and end station need not coincide.
Problem description
Problem description – Stations

Given a street network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$
Problem description – Stations

Given a street network $G = (\mathcal{V}, \mathcal{E})$ and a set of potential locations of charging stations $S \subseteq \mathcal{V}$, where each station $i$ has

- a cost $F_i$ for constructing it,
- a maximum capacity for charging slots $C_i$, each of which costs $q_i$, 

![Street network diagram with charging stations marked]
Problem description – Stations

Given a street network $G = (V, E)$ and a set of potential locations of charging stations $S \subseteq V$, where each station $i$ has

- a cost $F_i$ for constructing it,
- a maximum capacity for charging slots $C_i$, each of which costs $q_i$,
- a neighborhood $N(i)$ in which people will walk from/to the station,
Problem description – Stations

Given a **street network** $G = (\mathcal{V}, \mathcal{E})$ and a set of potential **locations of charging stations** $S \subseteq \mathcal{V}$, where each station $i$ has

- a **cost** $F_i$ for constructing it,
- a **maximum capacity** for charging slots $C_i$, each of which costs $q_i$,
- a **neighborhood** $\mathcal{N}(i)$ in which people will walk from/to the station,

we select a **subset of stations** to be constructed, as well as their **size**, subject to a **budget constraint**.

![Diagram showing a street network with potential locations of charging stations marked.](image-url)
Given a set $K$ of requested trips, where each trip has

- **origin** $o_k$ and **destination** $d_k$,
- **start** $s_k$ and **end** $e_k$ time,
- a **profit** $p_k$ and
- an (over-)estimated **battery usage** $b_k$,

we select a **set of trips** we want to accept to **maximize** the operator’s profit.
Each accepted trip is assigned to

- a **start** station where the car is picked up,
Each accepted trip is assigned to
- a **start** station where the car is picked up,
- an **end** station where the car is dropped off, and
Problem description – Trip assignment

Each accepted trip is assigned to

- a **start** station where the car is picked up,
- an **end** station where the car is dropped off, and
- a car with **sufficient battery level** parked at the start station.
ILP Model
Assumptions and Definitions

- homogeneous fleet of cars $H$, each costing $\zeta$
- parked cars are recharged at fixed rate $\rho$
- planning horizon $T = \{0, \ldots, T_{\text{max}}\}$
- $N(v)$: stations within walking distance from $v$
- $\Delta_k = e_k - b_k$: the duration of trip $k$
Assumptions and Definitions

- homogeneous fleet of cars $H$, each costing $\zeta$
- parked cars are recharged at fixed rate $\rho$
- planning horizon $T = \{0, \ldots, T_{\text{max}}\}$
- $N(v)$: stations within walking distance from $v$
- $\Delta_k = e_k - b_k$: the duration of trip $k$

Decision variables

- $y_i \in \{0, 1\}$: whether station $i$ is opened or not
- $z_i \in \{0, \ldots, C_i\}$: station $i$’s assigned capacity
- $a_h \in \{0, 1\}$: whether car $h$ is bought
- $x_k \in \{0, 1\}$: whether trip $k$ is accepted
- $x_k^h \in \{0, 1\}$: whether car $h$ performs trip $k$
ILP model

\[
\begin{align*}
\text{max} & \quad \sum_{k \in K} p_k x_k \\
\text{s.t.} & \quad \sum_{i \in S} (F_i y_i + q_i z_i) + \sum_{h \in H} \zeta a_h \leq W \\
& \quad y_i \leq z_i \leq C_i y_i \quad \forall i \in S \\
& \quad \sum_{h \in H} x_h^k = x_k \quad \forall k \in K \\
& \quad \sum_{k \in K : s_k \leq t, e_k > t} x_h^k \leq a_h \quad \forall t \in T, h \in H
\end{align*}
\]
ILP model

max \sum_{k \in K} p_k x_k \tag{1} \\
\text{s.t.} \sum_{i \in S} (F_i y_i + q_i z_i) + \sum_{h \in H} \zeta a_h \leq W \tag{2} \\
y_i \leq z_i \leq C_i y_i \tag{3} \\
\sum_{h \in H} x^h_k = x_k \tag{4} \\
\sum_{k \in K} x^h_k \leq a_h \tag{5} \\
\forall t \in T, h \in H \\
\text{objective function: maximize profit of accepted trips}
ILP model

\[
\begin{align*}
\text{max} & \quad \sum_{k \in K} p_k x_k \\
\text{s.t.} & \quad \sum_{i \in S} (F_i y_i + q_i z_i) + \sum_{h \in H} \zeta a_h \leq W \\
& \quad y_i \leq z_i \leq C_i y_i \quad \forall i \in S \\
& \quad \sum_{h \in H} x^h_k = x_k \quad \forall k \in K \\
& \quad \sum_{k \in K : s_k \leq t, e_k > t} x^h_k \leq a_h \quad \forall t \in T, h \in H
\end{align*}
\]

budget constraint
ILP model

\begin{align*}
\text{max} & \quad \sum_{k \in K} p_k x_k \quad (1) \\
\text{s.t.} & \quad \sum_{i \in S} (F_i y_i + q_i z_i) + \sum_{h \in H} \zeta a_h \leq W \quad (2) \\
& \quad y_i \leq z_i \leq C_i y_i \quad \forall i \in S \quad (3) \\
& \quad \sum_{h \in H} x_k^h = x_k \quad \forall k \in K \quad (4) \\
& \quad \sum_{k \in K : s_k \leq t, e_k > t} x_k^h \leq a_h \quad \forall t \in T, h \in H \quad (5)
\end{align*}

stations may not exceed their maximum capacity
ILP model

\[
\text{max} \quad \sum_{k \in K} p_k x_k
\]

s.t.

\[
\sum_{i \in S} (F_i y_i + q_i z_i) + \sum_{h \in H} \zeta a_h \leq W
\]

\[
y_i \leq z_i \leq C_i y_i \quad \forall i \in S
\]

\[
\sum_{h \in H} x_k^h = x_k \quad \forall k \in K
\]

\[
\sum_{k \in K: s_k \leq t, e_k > t} x_k^h \leq a_h \quad \forall t \in T, h \in H
\]

every opened station has at least one charging slot
ILP model

\[
\begin{align*}
\text{max} & \quad \sum_{k \in K} p_k x_k \\
\text{s.t.} & \quad \sum_{i \in S} (F_i y_i + q_i z_i) + \sum_{h \in H} \zeta a_h \leq W \\
& \quad y_i \leq z_i \leq C_i y_i \\
& \quad \sum_{h \in H} x^h_k = x_k \\
& \quad \sum_{k \in K: s_k \leq t, e_k > t} x^h_k \leq a_h \\
& \quad \forall i \in S \\
& \quad \forall k \in K \\
& \quad \forall t \in T, h \in H
\end{align*}
\]

assign every accepted trip to a car
ILP model

\[
\begin{align*}
\text{max} & \quad \sum_{k \in K} p_k x_k \\
\text{s.t.} & \quad \sum_{i \in S} (F_i y_i + q_i z_i) + \sum_{h \in H} \zeta a_h \leq W \\
& \quad y_i \leq z_i \leq C_i y_i \\
& \quad \sum_{h \in H} x^h_k = x_k \\
& \quad \sum_{k \in K: s_k \leq t, e_k > t} x^h_k \leq a_h \\
& \quad \forall i \in S \\
& \quad \forall k \in K \\
& \quad \forall t \in T, h \in H
\end{align*}
\]

a car may perform at most one trip at any time
ILP model – what’s still missing?

So far, the model does **not** ensure that

- cars move along a **consistent path** throughout the network, that
- **stations’ capacities** are never exceeded, or that
- a **car’s battery** level never gets below zero.

We will present **two models** to enforce the first two missing aspects ("location feasibility")

- **flow model** on a time-expanded location graph
- **no-flow model**

and **three models** that enforce battery feasibility

- **flow model** on a time-expanded battery graph
- **continuous** battery tracking
- battery-infeasible **path cuts**
Location feasibility
To model the location of each car at each point in time, we use a time-expanded location graph $G = (V, A)$. 

\[
\begin{array}{cccc}
\text{s}_1 & \text{s}_2 & \text{s}_3 & \text{s}_4 \\
\hline
\text{t = 0} & \bullet & \bullet & \bullet & \bullet \\
\text{t = 1} & \bullet & \bullet & \bullet & \bullet \\
\text{t = 2} & \bullet & \bullet & \bullet & \bullet \\
\text{t = 3} & \bullet & \bullet & \bullet & \bullet \\
\text{t = 4} & \bullet & \bullet & \bullet & \bullet \\
\end{array}
\]
To model the location of each car at each point in time, we use a time-expanded location graph $G = (V, A)$.

**root arcs** $A_I$ and **sink arcs** $A_C$ for initialization and collection.
To model the location of each car at each point in time, we use a time-expanded location graph $G = (V, A)$.

**root arcs** $A_I$ and **sink arcs** $A_C$
for initialization and collection

**waiting arcs** $A_W$
for parked cars

\[
\begin{array}{c}
\text{s_1} \\
\text{s_2} \\
\text{s_3} \\
\text{s_4}
\end{array}
\]
Location feasibility – Location graph

To model the location of each car at each point in time, we use a **time-expanded location graph** $G = (V, A)$.

- **Root arcs** $A_I$ and **sink arcs** $A_C$ for initialization and collection
- **Waiting arcs** $A_W$ for parked cars
- **Trip arcs** $A_T$ for cars used for trips
Location feasibility – Location graph

Additional variables

- Flow variable $f^h_a \in \{0, 1\}$: whether car $h$ moves along arc $a$

\[
\sum_{h \in H} \sum_{a \in \delta^+(i_t) \setminus A_T} f^h_a \leq z_i \quad \forall i \in S, \ t \in T \tag{6}
\]

\[
f^h[\delta^-(i_t)] \leq y_i \quad \forall h \in H, \ i \in S, \ t \in T \tag{7}
\]

\[
f^h[\delta^+(r_s)] = a_h \quad \forall h \in H \tag{8}
\]

\[
f^h[\delta^-(i_t)] - f^h[\delta^+(i_t)] = 0 \quad \forall h \in H, \ i \in S, \ t \in T \tag{9}
\]

\[
\sum_{a \in A^k_T} f^h_a = x^h_k \quad \forall h \in H, \ k \in K \tag{10}
\]
Location feasibility – Location graph

## Additional variables

- **Flow variable** $f_h^a \in \{0, 1\}$: whether car $h$ moves along arc $a$

\[
\sum_{h \in H} \sum_{a \in \delta^+(i_t) \setminus A_T} f_h^a \leq z_i \\
\forall i \in S, \ t \in T 
\]  

\[
f_h^{\delta^-(i_t)} \leq y_i \\
\forall h \in H, \ i \in S, \ t \in T 
\]  

\[
f_h^{\delta^+(r_s)} = a_h \\
\forall h \in H 
\]  

\[
f_h^{\delta^-(i_t)} - f_h^{\delta^+(i_t)} = 0 \\
\forall h \in H, \ i \in S, \ t \in T 
\]  

\[
\sum_{a \in A_k^T} f_h^a = x_k^h \\
\forall h \in H, \ k \in K 
\]  

- never exceed a station’s capacity
Additional variables

- Flow variable $f^h_a \in \{0, 1\}$: whether car $h$ moves along arc $a$

\[
\sum_{h \in H} \sum_{a \in \delta^+(i_t) \setminus A_T} f^h_a \leq z_i \quad \forall i \in S, \ t \in T \quad (6)
\]

\[
f^h[\delta^-(i_t)] \leq y_i \quad \forall h \in H, \ i \in S, \ t \in T \quad (7)
\]

\[
f^h[\delta^+(r_s)] = a_h \quad \forall h \in H \quad (8)
\]

\[
f^h[\delta^-(i_t)] - f^h[\delta^+(i_t)] = 0 \quad \forall h \in H, \ i \in S, \ t \in T \quad (9)
\]

\[
\sum_{a \in A^k_T} f^h_a = x^h_k \quad \forall h \in H, \ k \in K \quad (10)
\]

only opened stations may be used
Location feasibility – Location graph

Additional variables

- Flow variable $f_a^h \in \{0, 1\}$: whether car $h$ moves along arc $a$

\[
\sum_{h \in H} \sum_{a \in \delta^+(i_t) \setminus A_T} f_a^h \leq z_i \quad \forall i \in S, \ t \in T \tag{6}
\]

\[
f^h[\delta^-(i_t)] \leq y_i \quad \forall h \in H, \ i \in S, \ t \in T \tag{7}
\]

\[
f^h[\delta^+(r_s)] = a_h \quad \forall h \in H \tag{8}
\]

\[
f^h[\delta^-(i_t)] - f^h[\delta^+(i_t)] = 0 \quad \forall h \in H, \ i \in S, \ t \in T \tag{9}
\]

\[
\sum_{a \in A^k_T} f_a^h = x_k^h \quad \forall h \in H, \ k \in K \tag{10}
\]

every bought car leaves the root
Flow variable $f^h_a \in \{0, 1\}$: whether car $h$ moves along arc $a$.

\[
\sum_{h \in H} \sum_{a \in \delta^+(i_t) \setminus A_T} f^h_a \leq z_i \quad \forall i \in S, \ t \in T \tag{6}
\]

\[
f^h[\delta^-(i_t)] \leq y_i \quad \forall h \in H, \ i \in S, \ t \in T \tag{7}
\]

\[
f^h[\delta^+(r_s)] = a_h \quad \forall h \in H \tag{8}
\]

\[
f^h[\delta^-(i_t)] - f^h[\delta^+(i_t)] = 0 \quad \forall h \in H, \ i \in S, \ t \in T \tag{9}
\]

\[
\sum_{a \in A^k_T} f^h_a = x^h_k \quad \forall h \in H, \ k \in K \tag{10}
\]
Location feasibility – Location graph

Additional variables

- Flow variable $f^h_a \in \{0, 1\}$: whether car $h$ moves along arc $a$

\[
\sum_{h \in H} \sum_{a \in \delta^+(i_t) \setminus A_T} f^h_a \leq z_i \quad \forall i \in S, t \in T \tag{6}
\]

\[
f^h[\delta^-(i_t)] \leq y_i \quad \forall h \in H, i \in S, t \in T \tag{7}
\]

\[
f^h[\delta^+(r_s)] = a_h \quad \forall h \in H \tag{8}
\]

\[
f^h[\delta^-(i_t)] - f^h[\delta^+(i_t)] = 0 \quad \forall h \in H, i \in S, t \in T \tag{9}
\]

\[
\sum_{a \in A^k_T} f^h_a = x^h_k \quad \forall h \in H, k \in K \tag{10}
\]

if a car performs a trip, it must move along one of its trip arcs
Location feasibility – No-flow model

### Additional variables

- $\tilde{x}_k^i \in \{0, 1\}$: whether trip $k$ starts at station $i$
- $\hat{x}_k^i \in \{0, 1\}$: whether trip $k$ ends at station $i$

\[
\sum_{i \in N(o_k)} \tilde{x}_k^i = x_k \quad \forall k \in K 
\]

\[
\sum_{i \in N(d_k)} \hat{x}_k^i = x_k \quad \forall k \in K 
\]

\[
\tilde{x}_k^i \leq y_i \quad \forall k \in K, i \in N(o_k) 
\]

\[
\hat{x}_k^i \leq y_i \quad \forall k \in K, i \in N(d_k) 
\]
Location feasibility – No-flow model

### Additional variables

- \( \tilde{x}_k^i \in \{0, 1\} \): whether trip \( k \) starts at station \( i \)
- \( \hat{x}_k^i \in \{0, 1\} \): whether trip \( k \) ends at station \( i \)

1. \[ \sum_{i \in N(o_k)} \tilde{x}_k^i = x_k \quad \forall k \in K \] (11)
2. \[ \sum_{i \in N(d_k)} \hat{x}_k^i = x_k \quad \forall k \in K \] (12)
3. \[ \tilde{x}_k^i \leq y_i \quad \forall k \in K, i \in N(o_k) \] (13)
4. \[ \hat{x}_k^i \leq y_i \quad \forall k \in K, i \in N(d_k) \] (14)

Assign a start and end station to each accepted trip
Location feasibility – No-flow model

Additional variables

- $\tilde{x}_k^i \in \{0, 1\}$: whether trip $k$ starts at station $i$
- $\hat{x}_k^i \in \{0, 1\}$: whether trip $k$ ends at station $i$

\begin{align*}
\sum_{i \in N(o_k)} \tilde{x}_k^i &= x_k \quad \forall k \in K \tag{11} \\
\sum_{i \in N(d_k)} \hat{x}_k^i &= x_k \quad \forall k \in K \tag{12} \\
\tilde{x}_k^i &\leq y_i \quad \forall k \in K, i \in N(o_k) \tag{13} \\
\hat{x}_k^i &\leq y_i \quad \forall k \in K, i \in N(d_k) \tag{14}
\end{align*}

only use opened stations as start/end stations
Location feasibility – No-flow model

Additional variables

- \( a_h^i \in \{0, 1\} \): whether car \( h \) starts at station \( i \)

\[
\sum_{i \in S} a_h^i = a_h \quad \forall h \in H \tag{15}
\]

\[
a_h^i \leq y_i \quad \forall i \in S, h \in H \tag{16}
\]

\[
0 \leq \sum_{h \in H} a_h^i - \sum_{k \in K: i \in N(o_k), s_k \leq t} \hat{x}_k^i + \sum_{k \in K: i \in N(d_k), e_k \leq t} \hat{x}_k^i \leq z_i \quad \forall i \in S, t \in T \tag{17}
\]
Location feasibility – No-flow model

Additional variables

- \( a^i_h \in \{0, 1\} \): whether car \( h \) starts at station \( i \)

\[
\sum_{i \in S} a^i_h = a_h \quad \forall h \in H \tag{15}
\]

\[
a^i_h \leq y_i \quad \forall i \in S, h \in H \tag{16}
\]

\[
0 \leq \sum_{h \in H} a^i_h - \sum_{k \in K: i \in N(o_k), s_k \leq t} \tilde{x}^i_k + \sum_{k \in K: i \in N(d_k), e_k \leq t} \hat{x}^i_k \leq z_i \quad \forall i \in S, t \in T \tag{17}
\]

assign a start station to each bought car
Location feasibility – No-flow model

Additional variables

- \( a^i_h \in \{0, 1\} \): whether car \( h \) starts at station \( i \)

\[
\begin{align*}
\sum_{i \in S} a^i_h &= a_h & \forall h \in H \\
\boxed{a^i_h} &\leq y_i & \forall i \in S, h \in H \\
0 &\leq \sum_{h \in H} a^i_h - \sum_{k \in K : i \in N(o_k), s_k \leq t} \tilde{x}^i_k + \sum_{k \in K : i \in N(d_k), e_k \leq t} \hat{x}^i_k \leq z_i & \forall i \in S, t \in T
\end{align*}
\]

only use opened stations as start stations for cars
Location feasibility – No-flow model

Additional variables

- \( a^i_h \in \{0, 1\} \): whether car \( h \) starts at station \( i \)

\[
\sum_{i \in S} a^i_h = a_h \quad \forall h \in H \tag{15}
\]

\[
a^i_h \leq y_i \quad \forall i \in S, h \in H \tag{16}
\]

\[
0 \leq \sum_{h \in H} a^i_h - \sum_{k \in K : i \in N(o_k), s_k \leq t} \tilde{x}^i_k + \sum_{k \in K : i \in N(d_k), e_k \leq t} \hat{x}^i_k \leq z_i \quad \forall i \in S, t \in T \tag{17}
\]

number of cars parked at station \( i \) at time \( t \)
Location feasibility – No-flow model

Additional variables

- $a^i_h \in \{0, 1\}$: whether car $h$ starts at station $i$

\[
\sum_{i \in S} a^i_h = a_h \quad \forall h \in H \quad (15)
\]

\[
a^i_h \leq y_i \quad \forall i \in S, h \in H \quad (16)
\]

\[
0 \leq \sum_{h \in H} a^i_h - \sum_{k \in K \mid i \in N(o_k), s_k \leq t} \hat{x}^i_k + \sum_{k \in K \mid i \in N(d_k), e_k \leq t} \hat{x}^i_k \leq z_i \quad \forall i \in S, t \in T \quad (17)
\]

ensure that capacity is never exceeded
Location feasibility – No-flow model

Additional variables

- $a^i_h \in \{0, 1\}$: whether car $h$ starts at station $i$

\[
\sum_{i \in S} a^i_h = a_h \quad \forall h \in H \tag{15}
\]

\[
a^i_h \leq y_i \quad \forall i \in S, h \in H \tag{16}
\]

\[
0 \leq \sum_{h \in H} a^i_h - \sum_{k \in K : i \in N(o_k), s_k \leq t} \hat{x}_k^i + \sum_{k \in K : i \in N(d_k), e_k \leq t} \hat{x}_k^i \leq z_i \quad \forall i \in S, t \in T \tag{17}
\]

ensure that no more cars leave a station than are available there
Location feasibility – No-flow model

first step to ensure connectivity: a trip \( k \) may only be assigned to a car if that car is potentially in \( N(o_k) \)

\[
x^h_k \leq \sum_{i \in N(o_k)} a^i_h + \sum_{k' \in K: e_{k'} \leq s_k, \ N(o_{k'}) \cap S \setminus N(o_k) \neq \emptyset, \ N(d_{k'}) \cap N(o_k) \neq \emptyset} x^h_{k'} - \sum_{k' \in K: s_{k'} \leq s_k, \ N(o_{k'}) \subseteq N(o_k), \ N(d_{k'}) \subseteq S \setminus N(o_k)} x^h_{k'} \quad \forall k \in K, h \in H
\]
first step to ensure connectivity: a trip $k$ may only be assigned to a car if that car is potentially in $N(o_k)$

$$x^h_k \leq \sum_{i \in N(o_k)} a^i_h + \sum_{k' \in K : e_{k'} \leq s_k, \ N(o_{k'}) \cap S \setminus N(o_k) \neq \emptyset, \ N(d_{k'}) \cap N(o_k) \neq \emptyset} x^h_{k'} - \sum_{k' \in K : s_{k'} \leq s_k, \ N(o_{k'}) \subseteq N(o_k), \ N(d_{k'}) \subseteq S \setminus N(o_k)} x^h_{k'} \quad \forall k \in K, h \in H$$

whether car $i$ starts in $N(o_k)$
first step to ensure connectivity: a trip $k$ may only be assigned to a car if that car is potentially in $N(o_k)$

$$x^h_k \leq \sum_{i \in N(o_k)} a^i_h + \sum_{k' \in K : e_{k'} \leq s_k, \ N(o_{k'}) \cap S \setminus N(o_k) \neq \emptyset, \ N(d_{k'}) \cap N(o_k) \neq \emptyset} x^h_{k'} - \sum_{k' \in K : s_{k'} \leq s_k, \ N(o_{k'}) \subseteq N(o_k), \ N(d_{k'}) \subseteq S \setminus N(o_k)} x^h_{k'} \quad \forall k \in K, h \in H$$

how often car $i$ (potentially) enters $N(o_k)$ via a trip
first step to ensure connectivity: a trip $k$ may only be assigned to a car if that car is potentially in $N(o_k)$

$x^h_k \leq \sum_{i \in N(o_k)} a^i_h + \sum_{k' \in K : e_{k'} \leq s_k, N(o_{k'}) \cap S \setminus N(o_k) \neq \emptyset, N(d_{k'}) \cap N(o_k) \neq \emptyset} x^h_{k'} - \sum_{k' \in K : s_{k'} \leq s_k, N(o_{k'}) \subseteq N(o_k), N(d_{k'}) \subseteq S \setminus N(o_k)} x^h_{k'} \quad \forall k \in K, h \in H$

how often car $i$ (potentially) enters $N(o_k)$ in total
Location feasibility – No-flow model

first step to ensure connectivity: a trip $k$ may only be assigned to a car if that car is potentially in $N(o_k)$

$$x^h_k \leq \sum_{i \in N(o_k)} a^i_h + \sum_{k' \in K : e_{k'} \leq s_k, \ N(o_{k'}) \cap S \setminus N(o_k) \neq \emptyset, \ N(d_{k'}) \cap N(o_k) \neq \emptyset} x^h_{k'} - \sum_{k' \in K : s_{k'} \leq s_k, \ N(o_{k'}) \subseteq N(o_k), \ N(d_{k'}) \subseteq S \setminus N(o_k)} x^h_{k'} \quad \forall k \in K, h \in H$$

how often car $i$ leaves $N(o_k)$
first step to ensure connectivity: a trip $k$ may only be assigned to a car if that car is potentially in $N(o_k)$

$$x_k^h \leq \sum_{i \in N(o_k)} a_i^h + \sum_{k' \in K : e_{k'} \leq s_k, N(o_{k'}) \cap S \setminus N(o_k) \neq \emptyset} x_{k'}^h - \sum_{k' \in K : s_{k'} \leq s_k, N(o_{k'}) \subseteq N(o_k), N(d_{k'}) \subseteq S \setminus N(o_k)} x_{k'}^h \quad \forall k \in K, h \in H$$

If this whole expression is

- $\geq 1$: car $i$ might be in $N(o_k)$
- $\leq 0$: car $i$ cannot be in $N(o_k)$
first step to ensure connectivity: a trip $k$ may only be assigned to a car if that car is potentially in $N(o_k)$

$$ x^h_k \leq \sum_{i \in N(o_k)} a^i_h + \sum_{k' \in K: e_{k'} \leq s_k, \ N(o_{k'}) \cap S \setminus N(o_k) \neq \emptyset, \ N(d_{k'}) \cap N(o_k) \neq \emptyset} x^h_{k'} - \sum_{k' \in K: s_{k'} \leq s_k, \ N(o_{k'}) \subseteq N(o_k), \ N(d_{k'}) \subseteq S \setminus N(o_k)} x^h_{k'} \quad \forall k \in K, \ h \in H $$

If this whole expression is

- $\geq 1$: car $i$ might be in $N(o_k)$
- $\leq 0$: car $i$ cannot be in $N(o_k)$

This prevents many invalid trip assignments, and guarantees connectivity if $|N(o_k)| = |N(d_k)| = 1, \forall k \in K$. 
Location feasibility – No-flow model

However, this alone is not enough in general (cars are not guaranteed to be in \( N(o_k) \))
⇒ dynamically add additional constraints

If car \( h \) is assigned two consecutive trips \( k_1 \) and \( k_2 \) where \( k_2 \) doesn’t start at the station where \( k_1 \) ends, add the following constraint

\[
(1 - x^h_{k_1}) + (1 - x^h_{k_2}) + (1 - \hat{x}^{i_1}_{k_1}) + (1 - \tilde{x}^{i_2}_{k_2}) + \sum_{k \in K: s_k \geq e_{k_1}, e_k \leq s_{k_2}, o_k \in \bar{N}(i_1)} x^h_k \geq 1
\]

which ensure that
Location feasibility – No-flow model

However, this alone is not enough in general (cars are not guaranteed to be in \(N(o_k)\))
⇒ dynamically add additional constraints

If car \(h\) is assigned two consecutive trips \(k_1\) and \(k_2\) where \(k_2\) doesn’t start at the station where \(k_1\) ends, add the following constraint

\[
(1 - x^h_{k_1}) + (1 - x^h_{k_2}) + (1 - \hat{x}^i_{k_1}) + (1 - \hat{x}^i_{k_2}) + \sum_{k \in K: s_k \geq e_k, e_k \leq s_{k_2}, o_k \in \hat{N}(i_1)} x^h_k \geq 1
\]

which ensure that

- car \(h\) doesn’t do trip \(k_1\)
Location feasibility – No-flow model

However, this alone is not enough in general (cars are not guaranteed to be in \( N(o_k) \))
⇒ dynamically add additional constraints

If car \( h \) is assigned two consecutive trips \( k_1 \) and \( k_2 \) where \( k_2 \) doesn’t start at the station where \( k_1 \) ends, add the following constraint

\[
(1 - x^h_{k_1}) + (1 - x^h_{k_2}) + (1 - \hat{x}^i_{k_1}) + (1 - \tilde{x}^i_{k_2}) + \sum_{k \in K: s_k \geq e_k, e_k \leq s_{k_2}, o_k \in \bar{N}(i_1)} x^h_k \geq 1
\]

which ensure that
- car \( h \) doesn’t do trip \( k_1 \)
- car \( h \) doesn’t do trip \( k_2 \)
Location feasibility – No-flow model

However, this alone is not enough in general (cars are not guaranteed to be in $N(o_k)$)
$\Rightarrow$ dynamically add additional constraints

If car $h$ is assigned two consecutive trips $k_1$ and $k_2$ where $k_2$ doesn’t start at the station where $k_1$ ends, add the following constraint

$$(1 - x^h_{k_1}) + (1 - x^h_{k_2}) + (1 - \tilde{x}^i_{k_1}) + (1 - \tilde{x}^i_{k_2}) + \sum_{k \in K: s_k \geq e_k, e_k \leq s_{k_2}, o_k \in \bar{N}(i_1)} x^h_k \geq 1$$

which ensure that

- car $h$ doesn’t do trip $k_1$
- car $h$ doesn’t do trip $k_2$
- the end station of $k_1$ is changed
However, this alone is not enough in general (cars are not guaranteed to be in $N(o_k)$)
⇒ dynamically add additional constraints

If car $h$ is assigned two consecutive trips $k_1$ and $k_2$ where $k_2$ doesn’t start at the station where $k_1$ ends, add the following constraint

$$(1 - x^h_{k_1}) + (1 - x^h_{k_2}) + (1 - \hat{x}^i_{k_1}) + (1 - \hat{x}^n_{k_2}) + \sum_{k \in K: s_k \geq e_k, e_k \leq s_{k_2}, o_k \in \bar{N}(i_1)} x^h_k \geq 1$$

which ensure that

- car $h$ doesn’t do trip $k_1$
- car $h$ doesn’t do trip $k_2$
- the end station of $k_1$ is changed
- the start station of $k_2$ is changed
Location feasibility – No-flow model

However, this alone is not enough in general (cars are not guaranteed to be in $N(o_k)$)
⇒ dynamically add additional constraints

If car $h$ is assigned two consecutive trips $k_1$ and $k_2$ where $k_2$ doesn’t start at the station where $k_1$ ends, add the following constraint

$$(1 - x^h_{k_1}) + (1 - x^h_{k_2}) + (1 - \hat{x}^i_{k_1}) + (1 - \tilde{x}^i_{k_2}) + \sum_{k \in K : s_k \geq e_{k_1}, e_k \leq s_{k_2}, o_k \in \bar{N}(i_1)} x^h_k \geq 1$$

which ensure that
- car $h$ doesn’t do trip $k_1$
- car $h$ doesn’t do trip $k_2$
- the end station of $k_1$ is changed
- the start station of $k_2$ is changed
- car $h$ does at least one additional trip between $k_1$ and $k_2$
Battery feasibility
Battery feasibility – Battery graph

Time-expanded battery graph $G_B = (V_B, A_B)$

<table>
<thead>
<tr>
<th></th>
<th>$t = 0$</th>
<th>$t = 1$</th>
<th>$t = 2$</th>
<th>$t = 3$</th>
<th>$t = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100%</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
</tr>
<tr>
<td>80%</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
</tr>
<tr>
<td>60%</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
</tr>
<tr>
<td>40%</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
</tr>
<tr>
<td>20%</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
</tr>
</tbody>
</table>
Battery feasibility – Battery graph

Time-expanded battery graph \( G_B = (V_B, A_B) \)

charging arcs
for parked cars

\[
\begin{array}{cccccc}
  t = 0 & t = 1 & t = 2 & t = 3 & t = 4 \\
  100\% & 80\% & 60\% & 40\% & 20\%
\end{array}
\]
Battery feasibility – Battery graph

Time-expanded battery graph $G_B = (V_B, A_B)$

- **charging arcs**
  - for parked cars

- **waiting arcs**
  - for fully charged cars

$t = 0$  $t = 1$  $t = 2$  $t = 3$  $t = 4$
Battery feasibility – Battery graph

Time-expanded battery graph $G_B = (V_B, A_B)$

- **charging arcs** for parked cars
- **waiting arcs** for fully charged cars
- **trip arcs** $A^k_B$ for cars used for trip $k$

$t = 0$  $t = 1$  $t = 2$  $t = 3$  $t = 4$
Additional variables

- Flow variable $g^h_a \in \{0, 1\}$

\[
g^h[\delta^+(b_0^{\text{max}})] = a_h \quad \forall h \in H \tag{18}
\]

\[
g^h[\delta^-(u_t)] - g^h[\delta^+(u_t)] = 0 \quad \forall h \in H, u_t \in V_B, 1 \leq t < T_{\text{max}} \tag{19}
\]

\[
\sum_{a \in A^k_B} g^h_a = x^h_k \quad \forall h \in H, k \in K \tag{20}
\]
Additional variables

- Flow variable $g^h_a \in \{0, 1\}$

\[
g^h[\delta^+(b^{max}_0)] = a_h \quad \forall h \in H \quad (18)
\]

\[
g^h[\delta^-(u_t)] - g^h[\delta^+(u_t)] = 0 \quad \forall h \in H, u_t \in V_B, 1 \leq t < T_{\text{max}} \quad (19)
\]

\[
\sum_{a \in A_B^k} g^h_a = x^h_k \quad \forall h \in H, k \in K \quad (20)
\]

all bought cars start at battery level $b^{max}$ at $t = 0$
Additional variables

- Flow variable $g^h_a \in \{0, 1\}$

\[
g^h[\delta^+ (b_0^{\text{max}})] = a_h
\]

\[
g^h[\delta^- (u_t)] - g^h[\delta^+ (u_t)] = 0
\]

\[
\sum_{a \in A^k_B} g^h_a = x^h_k
\]

\[
\forall h \in H \quad (18)
\]

\[
\forall h \in H, u_t \in V_B, 1 \leq t < T_{\text{max}} \quad (19)
\]

\[
\forall h \in H, k \in K \quad (20)
\]

flow conservation
Battery feasibility – Battery graph

Additional variables

- Flow variable $g^h_a \in \{0, 1\}$

\[
g^h[\delta^+(b_0^{\text{max}})] = a_h \quad \forall h \in H \tag{18}
\]
\[
g^h[\delta^-(u_t)] - g^h[\delta^+(u_t)] = 0 \quad \forall h \in H, u_t \in V_B, 1 \leq t < T_{\text{max}} \tag{19}
\]
\[
\sum_{a \in A^k_B} g^h_a = x^h_k \quad \forall h \in H, k \in K \tag{20}
\]

if a car performs a trip, it must go over one of its trip arcs
Battery feasibility – Continuous battery tracking

Additional variables

- Continuous variable $g_t^h \in [0, b^{\text{max}}]$: battery level of car $h$ at time $t$

\[
\begin{align*}
g_0^h &= b^{\text{max}} \\
g_{e_k}^h - g_{s_k}^h &\leq -b_k x_k^h + \Delta_k \rho (1 - x_k^h) \\
g_{t+1}^h - g_t^h &\leq \rho a_h
\end{align*}
\]

\[
\begin{align*}
\forall h \in H &&& (21) \\
\forall h \in H, k \in K &&& (22) \\
\forall h \in H, t \in T \setminus T_{\text{max}} &&& (23)
\end{align*}
\]
Battery feasibility – Continuous battery tracking

Additional variables

• Continuous variable $g^h_t \in [0, b^{\text{max}}]$: battery level of car $h$ at time $t$

\[
\begin{align*}
  g^h_0 &= b^{\text{max}} \quad \forall h \in H \\
  g^h_{e_k} - g^h_{s_k} &\leq -b_k x^h_k + \Delta_k \rho (1 - x^h_k) \quad \forall h \in H, k \in K \\
  g^{h}_{t+1} - g^{h}_t &\leq \rho a_h \quad \forall h \in H, t \in T \setminus T_{\text{max}}
\end{align*}
\]

all bought cars start at battery level $b^{\text{max}}$ at $t = 0$
Battery feasibility – Continuous battery tracking

Additional variables

- Continuous variable $g_t^h \in [0, b^\text{max}]$: battery level of car $h$ at time $t$

\[
g_0^h = b^\text{max} \quad \forall h \in H \tag{21}
\]

\[
g_{e_k}^h - g_{s_k}^h \leq -b_k x_k^h + \Delta_k \rho (1 - x_k^h) \quad \forall h \in H, k \in K \tag{22}
\]

\[
g_{t+1}^h - g_t^h \leq \rho a_h \quad \forall h \in H, t \in T \setminus T_{\text{max}} \tag{23}
\]

if a car performs a trip, its battery is depleted accordingly
Battery feasibility – Continuous battery tracking

**Additional variables**

- **Continuous variable** $g_t^h \in [0, b_{\text{max}}]$: battery level of car $h$ at time $t$

\[
g_0^h = b_{\text{max}} \quad \forall h \in H \tag{21}
\]

\[
ge_{e_k}^h - g_{s_k}^h \leq -b_k x_k^h + \Delta_k \rho (1 - x_k^h) \quad \forall h \in H, k \in K \tag{22}
\]

\[
g_{t+1}^h - g_t^h \leq \rho a_h \quad \forall h \in H, t \in T \setminus T_{\text{max}} \tag{23}
\]

cars are recharged by up to $\rho$ each time period
explicitly forbid all battery-infeasible paths

Whenever we find a path that is infeasible w.r.t. battery consumption, we add

$$\sum_{k \in K'} x^h_k \leq f_{K'} a_h \quad \forall K' \subseteq K, h \in H$$

(24)

to the model, where $f_{K'}$ is the maximum number of trips from $K'$ that can be performed by a single car.
Results
random instances with
- grid street network
- number of stations $|S| \in \{10, 25, 50\}$
  - random location
  - random cost
  - random maximum capacity
- number of trips $|K| \in \{10, 25, 50, 75, 100\}$
  - random start and end location
  - random start and end time
  - uniform profit $p_k = 1$

We evaluated several variants of our algorithm
- **FG**: flow model with battery graph
- **FC**: flow model with continuous battery tracking
- **N**: no-flow model with battery cuts
- **NC**: no-flow model with continuous battery tracking

Computations were done with **CPLEX**, **10800 s** time limit and **3 GB** memory limit.
Improvements
Heuristic

To improve the performance of our ILP solver, we want to provide it with a good initial solution. We want to find a set of car paths that

- covers many profitable trips, and
- is feasible w.r.t. our budget constraints

We can find such paths by repeatedly solving the resource-constrained longest path problem (RCLP) on a variant of the location graph, where each arc is assigned

- a length \( \ell_a \)
  - \( \ell_a = p_k \) for trip arcs
  - \( \ell_a = 0 \) otherwise
- a battery consumption \( b_a \)
  - \( b_a = -b_k \) for trip arcs
  - \( b_a = \rho \) for waiting arcs
  - \( b_a = 0 \) otherwise

Since the location graph is acyclic, this is equivalent to solving the resource-constrained shortest path problem (RCSP) on a variant where all arc lengths are negated.
We solve the RCLP with a **dynamic programming labeling algorithm**. A label $L$ consists of a profit $p_L$ and a battery level $b_L$, and dominates $L'$ if

$$p_L \geq p_{L'} \land b_L \geq b_{L'}$$

(25)

with at least one inequality being strict.

1. $\text{labels}(v) = \emptyset$
2. $\text{labels}(i_0) = \{(0, 100)\}, \forall i \in S$
3. **for** $t \in T, i \in S$ **do**
4.  **for** $l \in \text{labels}(i_t)$ **do**
5.     **for** $(i_t, j_{t'}) \in \delta^+(i_t)$ **do**
6.         **if** $l$ not dominated by any $l' \in \text{labels}(j_{t'})$ **then**
7.             add $l$ to $\text{labels}(j_{t'})$
8.         remove all dominated $l'$ from $\text{labels}(j_{t'})$
9. build car path from best label at sink
pathlist = ∅

while $W \geq \zeta$ do
    $W = W - \zeta$
    find new path with RCLP
    if $W < \text{path.cost}$ then
        try to remove trips from path to make it feasible
    if $W \geq \text{path.cost}$ then
        pathlist = pathlist $\cup \{\text{path}\}$
        $W = W - \text{path.cost}$
        $A_T = A_T \setminus \{a \mid a.trip \in \text{path.trips}\}$
        remove waiting arcs from vertices at maximum capacity
Symmetry breaking

Since our car fleet is homogeneous, our models have lots of symmetries. We can break these by adding constraints

\[ \sum_{k \in K} \alpha_k x_k^h \geq \sum_{k \in K} \alpha_k x_k^{h+1} \quad \forall h \in H \setminus \{H_{\text{max}}\} \]  

(26)

that impose an ordering on cars. The value of a car depends on the trips it performs, such as

- their number (i.e., \( \alpha_k = 1 \))
- their profit (i.e., \( \alpha_k = p_k \))
- their duration (i.e., \( \alpha_k = \Delta_k \))
Symmetry breaking

Since our car fleet is homogeneous, our models have lots of symmetries. We can break these by adding constraints

\[ \sum_{k \in K} \alpha_k x_k^h \geq \sum_{k \in K} \alpha_k x_k^{h+1} \quad \forall h \in H \setminus \{H_{\text{max}}\} \quad (26) \]

that impose an ordering on cars.

The value of a car depends on the trips it performs, such as

- their number (i.e., \( \alpha_k = 1 \))
- their profit (i.e., \( \alpha_k = p_k \))
- their duration (i.e., \( \alpha_k = \Delta_k \))

Unfortunately, preliminary results are not very encouraging.
Future work

- Model extensions
  - integrating uncertainty
  - allowing car relocation
- Instances based on real world data
- Computational enhancements
  - constraint separation for fractional solutions
- Alternative formulations
  - set covering formulation (branch-and-price)