HERMITE-SOBOLEV SPACES AND THE FEICHTINGER’S ALGEBRA

BY

S. THANGAVELU

ABSTRACT. We show that for any $\delta > 0$ the Hermite-Sobolev spaces $W^{n+\delta,1}_H$ are properly contained in the Feichtinger’s algebra $S_0$. This is done by using Hardy’s inequality for Hermite expansions and a result of A.J.E.M. Janssen.

1. Introduction

Feichtinger’s algebra $S_0$ introduced in [1] has been extensively studied in the contexts of abstract harmonic analysis as well as in time-frequency analysis. It has several equivalent definitions, the simplest being the characterisation given in terms of the Fourier-Wigner transform. For the convenience of the readers we recall relevant definitions here. Let $\pi$ be the Schrödinger representation of the Heisenberg group with parameter $\lambda = 1$ which is realised on $L^2(\mathbb{R}^n)$ and explicitly given by

$$\pi(z, t) \varphi(\xi) = e^{it}e^{i(x\xi + \frac{1}{2}x^2)} \varphi(\xi + y)$$

where $z = x + iy \in \mathbb{C}^n$, $\varphi \in L^2(\mathbb{R}^n)$. The Fourier-Wigner transform of two functions $f, g \in L^2(\mathbb{R}^n)$ is defined by $V(f, g)(z) = (2\pi)^{-\frac{n}{2}}(\pi(z, 0)f, g)$. It is well known that $V(f, g) \in L^2(\mathbb{C}^n)$ with $\|V(f, g)\|_2 = \|f\|_2\|g\|_2$. For properties of the Fourier-Wigner transform we refer to [3] and [14]. Fixing $g(x) = \Phi_0(x)$, the normalised

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Gaussian, the Feichtinger’s space $S_0$ is defined to be the space of all $f \in L^2$ for which $V(f, \Phi_0) \in L^1(\mathbb{C}^n)$. We refer to [2] and [4] for more about this algebra.

A new characterisation of $S_0$ in dimension one was obtained by Janssen [5] in 2005 in terms of the Hermite coefficients of $f$. If $h_k$ stand for the normalised Hermite functions on $\mathbb{R}$ then, he has shown that any $f$ for which $\sum_{k=0}^{\infty} |(f, h_k)|(2k + 1)^{\frac{1}{4}}$ belongs to $S_0$. This theorem of Janssen suggests that there should be a relation between Hermite-Sobolev spaces and Feichtinger’s algebra. In this note we substantiate this claim by proving the following result. For $1 \leq p \leq \infty$, $s \in \mathbb{R}$ let $W_{H}^{s,p}$ be the Hermite-Sobolev spaces defined as the space of tempered distributions $f$ for which $H^s f \in L^p(\mathbb{R}^n)$. Here $H = -\Delta + |x|^2$ is the Hermite operator on $\mathbb{R}^n$ and $H^s f$ is defined in terms of spectral theory.

**Theorem 1.1.** For $1 \leq p \leq 2$, $s > 2n \left(\frac{1}{p} - \frac{1}{2}\right)$ we have $\|V(f, \Phi_0)\|_p \leq C_p \|H^s f\|_p$. Consequently, for any $\delta > 0$ the Hermite-Sobolev space $W_{H}^{n+\delta,1}$ is properly contained in $S_0$.

We prove this theorem by combining a higher dimensional version of Janssen’s theorem with Hardy-type inequalities for Fourier-Hermite coefficients of integrable functions. The latter inequalities have been established by Kanjin [7] for the Hermite expansions on $\mathbb{R}$. Here we prove an $n-$dimensional version and use to get our main result. When $f$ belongs to the Hardy space $H^1(\mathbb{R}^n), n \geq 2$ we already have an improved version of Hardy’s inequality from the work [10] using which we can obtain the following.

**Theorem 1.2.** Suppose $n \geq 2$ and $f \in H^1(\mathbb{R}^n)$. Then $H^{-n} f$ belongs to the Feichtinger algebra $S_0$ and we have $\|H^{-n} f\|_{S_0} \leq C \|f\|_{H^1}$.

The operator $H^{-n}$ is an integral operator with an explicit positive kernel. The theorem says that this operator maps $H^1(\mathbb{R}^n)$ into $S_0$ continuously. By interpolation it follows that we can take $\delta = 0$ in Theorem 1.1 when $n \geq 2$ and $1 < p \leq 2$. 
2. Hermite-Sobolev spaces and Hardy’s inequalities

We use standard notations for Hermite and special Hermite functions. We refer to the monographs [3], [14] for results concerning them. Let \( \Phi_\alpha, \alpha \in \mathbb{N}^n \) be the normalised Hermite functions on \( \mathbb{R}^n \) defined by taking tensor products of the one dimensional Hermite functions

\[
h_k(x) = (\sqrt{\pi}2^k k!)^{-\frac{1}{2}}e^{\frac{1}{2}x^2}(-1)^k\frac{d^k}{dx^k}(e^{-x^2}).
\]

They are eigenfunctions of \( H \) and form an orthonormal basis for \( L^2(\mathbb{R}^n) \). For any tempered distribution \( f \) the Hermite coefficients \( (f, \Phi_\alpha) \) are defined. The Hermite-Sobolev space \( W^{s,p}_H \) consists of all tempered distributions for which the distribution \( H^s f \) is given by an \( L^p \) function. These spaces have been studied in [9] and [12]. When \( p = 2 \) they are Hilbert spaces which are invariant under the Fourier transform a property shared by Feichtinger algebra.

Hardy’s inequality was originally obtained for the Fourier coefficients of functions belonging to Hardy spaces on the unit disc. Later, such inequalities have been studied for various orthogonal expansions such as Jacobi, Hermite and Laguerre, see [6],[8] and [10]. The most relevant result for us is the recent work of Kanjin [7] where he has established the following theorem.

**Theorem 2.1.** For every \( \epsilon > 0 \) there is a constant \( C_\epsilon \) such that

\[
\sum_{k=0}^{\infty} |(f, h_k)|(2k + 1)^{-\frac{1}{4}} \leq C_\epsilon \|f\|_1
\]

for all \( f \in L^1(\mathbb{R}) \).

The above result is the best possible as Kanjin has shown that there exists \( f \in L^1(\mathbb{R}) \) for which \( \sum_{k=0}^{\infty} |(f, h_k)|(2k + 1)^{-\frac{1}{4}} \) diverges. The above result can be put in the form

\[
\sum_{k=0}^{\infty} |(f, h_k)|(2k + 1)^{\frac{1}{4}} \leq C_\epsilon \|H^s f\|_1
\]
for any \( s > 1 \). This together with Janssen’s result shows that \( W_{H}^{s,1} \) is contained in \( S_0 \) as soon as \( s > 1 \). In order to prove this in the higher dimensional case we need a version of Kanjin’s theorem for \( \mathbb{R}^n \).

**Theorem 2.2.** For any \( s > \frac{3}{4}n \) there is a constant \( C_s \) such that

\[
\sum_{\alpha \in \mathbb{N}^n} |(f, \Phi_\alpha)| (2|\alpha| + n)^{-s} \leq C_s \|f\|_1
\]

for all \( f \in L^1(\mathbb{R}^n) \).

**Proof:** As in Kanjin [7] it is enough to show that

\[
\sum_{\alpha \in \mathbb{N}^n} |\Phi_\alpha(x)|(2|\alpha| + n)^{-s} \leq C_s
\]

uniformly in \( x \in \mathbb{R}^n \). For any \( \delta > \frac{n}{2} \) the series

\[
\sum_{\alpha \in \mathbb{N}^n} (2|\alpha| + n)^{-2\delta}
\]

converges and hence by Cauchy-Schwarz applied to the above we are left with proving the uniform bound

\[
\sum_{\alpha \in \mathbb{N}^n} |\Phi_\alpha(x)|^2 (2|\alpha| + n)^{-2s+2\delta} \leq C_s.
\]

Note that this can be written as

\[
\int_0^\infty t^{2s-2\delta-1} K_t(x)dt
\]

where

\[
K_t(x) = \sum_{\alpha \in \mathbb{N}^n} e^{-(2|\alpha|+n)t|\Phi_\alpha(x)|^2}.
\]

In view of Mehler’s formula (see [13]) for Hermite functions we have

\[
K_t(x) = c_n (\sinh(2t))^{-\frac{n}{2}} e^{-t|\tanh t||x|^2}.
\]

Once we have this it is easy to see that \( \int_0^\infty t^{2s-2\delta-1} K_t(x)dt \) is uniformly bounded as soon as \( s > \frac{3}{4}n \). This completes the proof.

We can now show that the Hermite-Sobolev space \( W_{H}^{s,1} \) can be continuously embedded in the Feichtinger algebra.

**Theorem 2.3.** For any \( s > n \), \( W_{H}^{s,1} \) is properly contained in \( S_0 \) and the inclusion is continuous.
Proof: We look at the special Hermite expansion of $V(f, \Phi_0)$ which reads as

$$V(f, \Phi_0)(z) = \sum_{\alpha \in \mathbb{N}^n} (f, \Phi_\alpha) \Phi_{0,\alpha}(z)$$

where $\Phi_{\alpha,\beta} = V(\Phi_\alpha, \Phi_\beta)$ are the special Hermite functions. It is enough to show that the above series is absolutely summable in $L^1(\mathbb{C}^n)$ which amounts to show the convergence of

$$\sum_{\alpha \in \mathbb{N}^n} |(f, \Phi_\alpha)||\Phi_{0,\alpha}|_1.$$

The special Hermite functions $\Phi_{0,\alpha}$ are explicitly known:

$$\Phi_{0,\alpha}(z) = (2\pi)^{-\frac{n}{2}}(\alpha!)^{-\frac{1}{2}} \frac{i^{\|\alpha\|}}{\sqrt{2}} e^{-\frac{1}{4}|z|^2},$$

see e.g. [13] Theorem 1.3.5. Calculating the $L^1(\mathbb{C}^n)$ norm of $\Phi_{0,\alpha}$ and estimating it as in Janssen we get

$$\|V(f, \Phi_0)\|_1 \leq C \sum_{\alpha \in \mathbb{N}^n} |(f, \Phi_\alpha)|(2|\alpha| + n)^\frac{n}{2}.$$

Writing $f = H^{-s}H^sf$ and using Hardy’s inequality we see that

$$\|V(f, \Phi_0)\|_1 \leq C \sum_{\alpha \in \mathbb{N}^n} |(H^sf, \Phi_\alpha)|(2|\alpha| + n)^{-s+\frac{n}{2}}$$

which is bounded by $\|H^sf\|_1$ provided $s > n$. To see that the inclusion is proper, recall that $S_0$ is invariant under the Fourier transform. Hence all we need to show is that $W_H^{s,1}$ is not invariant under the Fourier transform. To check this choose $g \in L^1$ for which $\hat{g}$ is not in $L^1$ and take $f = H^{-s}g$. Then clearly $f \in W_H^{s,1}$ but $\hat{f} = H^{-s}\hat{g}$ is not in $W_H^{s,1}$. This completes the proof.

Finally, Theorem 1.1 stated in the introduction is proved using analytic interpolation. We consider the analytic family of operators $T_sf(z) = V(H^{-1-\delta+s}f, \Phi_0)$ where $\delta > 0$. When $Re(s) = 0, T_s$ maps $L^1$ into itself by the above theorem and when $Re(s) = 1$ it maps $L^2$ into itself as $H^t$ is unitary on $L^2$ for any real $t$. By appealing to Stein’s analytic interpolation theorem (see e.g. [11]) we obtain the result.

In [10] the authors have established the following result which can be used in place of Theorem 2.2 to prove Theorem 1.2.
Theorem 2.4. Let \( n \geq 2 \). Then we have the inequality
\[
\sum_{\alpha \in \mathbb{N}^{n}} |(f, \Phi_{\alpha})| (2|\alpha| + n)^{-\frac{4}{n}} \leq C \|f\|_{H^{1}}
\]
for all \( f \in H^{1}(\mathbb{R}^{n}) \).

Since the inequality in the theorem can be written as
\[
\sum_{\alpha \in \mathbb{N}^{n}} |(H^{-n}f, \Phi_{\alpha})| (2|\alpha| + n)^{\frac{2}{n}} \leq C \|f\|_{H^{1}}
\]
applying Janssen’s theorem we immediately get Theorem 1.2.

We conclude this section with a couple of remarks. As mentioned in the introduction the operator \( H^{-n} \) is a nice integral operator. The kernel \( k_{n}(x, y) \) of this operator is given by
\[
k_{n}(x, y) = \frac{1}{\Gamma(n)} \int_{0}^{\infty} t^{n-1} K_{t}(x, y) dt
\]
where \( K_{t}(x, y) \) is the kernel of the semigroup \( e^{-tH} \) which is given explicitly by
\[
K_{t}(x, y) = c_{n}(\sinh(2t))^{-\frac{4}{n}} e^{-\frac{1}{2}(\coth(2t)(|x|^{2}+|y|^{2})+(cosech(2t))x\cdot y)}.
\]
Therefore, it is clear that \( k_{n}(x, y) \) is a nice positive kernel.

It is not known if Theorem 2.4 is true when \( n = 1 \) or not. Analytic interpolation with \( H^{1}(\mathbb{R}^{n}) \) in place of \( L^{1}(\mathbb{R}^{n}) \) (as the domain of \( T_{s} \)) shows that we can take \( \delta = 0 \) in Theorem 1.1 when \( n \geq 2 \) and \( p > 1 \).

References


Department of Mathematics, Indian Institute of Science, Bangalore 560 012, India. e-mail: veluma@math.iisc.ernet.in