NONSTATIONARY GABOR FRAMES - EXISTENCE AND CONSTRUCTION

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Abstract. Nonstationary Gabor frames were recently introduced in [2] and represent a natural generalization of classical Gabor frames by allowing for adaptivity of windows and lattice in either time or frequency. In this paper we show a general existence result for this family of frames. We also construct nonstationary Gabor frames with non-compactly supported windows from a related painless nonorthogonal expansion and give a perturbation result. Finally, the theoretical results are illustrated by two examples of practical relevance.

1. Introduction

The principal idea of Gabor frames was introduced in [17] with the aim to represent signals in a time-frequency localized manner. Since the work of Gabor himself, a lot of research has been done on the topic of atomic time-frequency representation. While it turned out that the original model proposed by Gabor does not yield stable representations in the sense of frames [5, 9, 13], the existence of Gabor frames was first established in the so called painless case, [9], which requires the use of compactly supported analysis windows. The existence of Gabor frames in more general situations was proved later [22, 25] and the proof often uses an argument invoking the invertibility of diagonally dominant matrices.

Various irregular and adaptive versions of Gabor frames have been introduced over the years, cf. [1, 6, 7, 10, 14, 23]. In these approaches, the irregularity usually concerns either the sampling set, which is allowed to deviate from a lattice, or the window, which is allowed to be modified. In [1], varying windows as well as irregular sampling points are allowed, however, existence of a local frame is assumed, from which a global frame is constructed. Nonstationary Gabor frames give up the strict regularity of the classical Gabor setting, but, as opposed to irregular frames, maintain enough structure to guarantee efficient implementation and, possibly approximate, reconstruction.

From an application point of view, the introduction of nonstationarity in the design of time-frequency dictionaries is often desirable. In analogy to the classical, regular case [9], painless nonstationary Gabor frames were introduced in [2], where the principal idea is described and illustrated in detail. The construction of nonstationary Gabor frames is similar to the construction of windowed modified cosine transforms and other lapped transforms [20, 26].
that allow for adaptivity of the window length. However, nonstationary Gabor transforms allow for redundancy, which is often a desirable property in signal processing. For example, an efficient and perfectly invertible constant-Q transform was recently introduced using nonstationary Gabor transforms, [24]. In this case, redundancy is needed since the orthogonal versions lead to dyadic wavelet transforms, which are inappropriate for most real-life applications, in particular audio signal processing.

Similar to the regular case, redundancy allows for good localization of both the analysis and synthesis windows, and their respective Fourier transforms. Finally, redundancy often allows for more sparse representations in adaptive processing.

While the painless case of nonstationary Gabor frames, involving either compactly supported or band limited analysis windows, was exhaustively presented in [2], the present contribution addresses the case of more general windows. In Theorem 3.3 we derive the existence of nonstationary Gabor frames directly from a generalized Walnut representation: under mild uniform decay conditions on all windows involved, we show an existence result of nonstationary Gabor frames in parallel to the result given in [25] for regular Gabor frames, also compare [18, Theorem 6.5.1]. Note that the existence of a more general class of nonstationary Gabor frames, the quilted Gabor frames, [11], was recently proved in the general context of spline type spaces in the remarkable paper [21].

In the construction of nonstationary Gabor frames we pursue two basic approaches.

1. In Proposition 3.5 we construct nonstationary Gabor frames by using knowledge about a related painless frame. We call the resulting systems "almost painless nonstationary Gabor frames".

2. Using tools from the theory of perturbation of frames, we construct nonstationary frames from an existing frame. This result is presented in Proposition 3.6.

This paper is organized as follows. In the next section, we introduce notation and state some auxiliary results. In Section 3 we first define nonstationary Gabor frames and recall known results for the painless case. In Section 3.2 a Walnut representation and a corresponding bound of the frame operator in the general setting is derived and Section 3.3 provides the existence of nonstationary Gabor frames. In Section 3.4 and 3.5 the construction of explicit nonstationary Gabor frames is given. In Section 4 we provide examples based on our theory and we conclude with a summary and some perspectives in Section 5.

2. Notation and Preliminaries

Given a non-zero function $g \in L^2(\mathbb{R})$, let $g_{k,l}(t) = M_{bl}T_{ak}g(t) := e^{2\pi ibl t}g(t - ak)$. $M_{bl}$ is a modulation operator, or frequency shift, and $T_{ak}$ is a time shift.
The set \( G(g, a, b) = \{ g_{k,l} : k, l \in \mathbb{Z} \} \) is called a Gabor system for any real, positive \( a, b \), \([13]\) Definition 5.2.1. \( G(g, a, b) \) is a Gabor frame for \( L^2(\mathbb{R}) \), if there exist constants \( 0 < A \leq B < \infty \) such that for every \( f \in L^2(\mathbb{R}) \) we have

\[
A \| f \|^2 \leq \sum_{k,l \in \mathbb{Z}} | \langle f, g_{k,l} \rangle |^2 \leq B \| f \|^2 .
\]

To every Gabor system, we associate the analysis operator \( C_g \) given by

\[
(C_g f)_{k,l} = \langle f, g_{k,l} \rangle ,
\]

and the synthesis operator \( U_g = C^*_g \), given by

\[
U_g c = \sum_{k,l \in \mathbb{Z}} c_{k,l} g_{k,l}
\]

for \( c \in \ell^2 \). The inequality (1) is equivalent to the invertibility of the frame operator \( S \) of \( G(g, a, b) \), where

\[
S f = \sum_{k,l \in \mathbb{Z}} \langle f, g_{k,l} \rangle g_{k,l} .
\]

The analysis operator is closely connected to the sampled short-time Fourier transform (STFT). For a fixed window \( g \in L^2(\mathbb{R}) \), the STFT of \( f \in L^2(\mathbb{R}) \) is

\[
V_g f(x, \omega) = \int_{\mathbb{R}} f(t) e^{-2\pi i \omega t} g(t-x) dt = \langle f, M_\omega T_x g \rangle .
\]

Setting \((x, \omega) = (ak, bl)\), leads to

\[
V_g f(ak, bl) = (C_g f)_{k,l}.
\]

When working with irregular grids, we assume that the sampling points form a separated set: a set of sampling points \( \{ a_k : k \in \mathbb{Z} \} \) is called \( \delta \)-separated, if \( |a_k - a_m| > \delta \) for \( a_k, a_m \), whenever \( k \neq m \). \( \chi_I \) will denote the characteristic function of the interval \( I \).

A convenient class of window functions for time-frequency analysis on \( L^2(\mathbb{R}) \) is the Wiener space.

**Definition 1.** A function \( g \in L^\infty(\mathbb{R}) \) belongs to the Wiener space \( W(L^\infty, \ell^1) \) if

\[
\| g \|_{W(L^\infty, \ell^1)} := \sum_{k \in \mathbb{Z}} \text{ess sup}_{t \in Q} |g(t+k)| < \infty , \quad Q = [0,1] .
\]

For \( g \in W(L^\infty, \ell^1) \) and \( \delta > 0 \) we have \([13]\)

\[
\text{ess sup}_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |g(t - \delta k)| \leq (1 + \delta^{-1}) \| g \|_{W(L^\infty, \ell^1)} .
\]

In dealing with polynomially decaying windows, we will repeatedly use the following lemma.

**Lemma 2.1.** For \( p > 1 \) the following estimates hold:

(a) Let \( \delta > 0 \), then

\[
\sum_{k=1}^{\infty} (1 + \delta k)^{-p} \leq (1 + \delta)^{-p}(\delta^{-1} + p)(p - 1)^{-1} .
\]
(b) Let \( \{a_k : k \in \mathbb{Z}\} \subset \mathbb{R} \) be a \( \delta \)-separated set. Then
\[
\text{ess sup}_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} (1 + |t - a_k|)^{-p} \leq 2 \left( 1 + (1 + \delta)^{-p}(\delta^{-1} + p)(p - 1)^{-1} \right).
\]

**Proof.** To show (a) we write
\[
\sum_{k=1}^{\infty} (1 + \delta k)^{-p} = (1 + \delta)^{-p} + \sum_{k=2}^{\infty} (1 + \delta k)^{-p} = (1 + \delta)^{-p} + \sum_{k=2}^{\infty} \int_{[0,1]+k} (1 + \delta k)^{-p} \, dt.
\]
for \( t \in [k, k+1] \), we have \( \delta t \leq \delta(k+1) \) which implies that \( 1 + \delta(t-1) \leq 1 + \delta k \). Therefore,
\[
\sum_{k=2}^{\infty} \int_{[0,1]+k} (1 + \delta k)^{-p} \, dt \leq \sum_{k=2}^{\infty} \int_{[0,1]+k} (1 + \delta(t-1))^{-p} \, dt = \int_2^{\infty} (1 + \delta(t-1))^{-p} \, dt = (1 + \delta)^{-p+1} \delta^{-1} (p-1)^{-1},
\]
and the estimate follows.

To prove (b), fix \( t \in \mathbb{R} \). Since \( |a_k - a_l| > \delta \) for \( k \neq l \), each interval of length \( \delta \) contains at most one point \( t - a_k \), \( k \in \mathbb{Z} \). Therefore we may write \( t - a_k = \delta n_k + x_k \) for unique \( n_k \in \mathbb{Z} \) and \( x_k \in [0, \delta) \), and by the choice of \( \delta \), we have \( n_k \neq n_l \) for \( k \neq l \). We assume, without loss of generality, that \( n_k = 0 \) for \( k = 0 \), and we find that
\[
\sum_{k \in \mathbb{Z}} (1 + |t - a_k|)^{-p} = \sum_{k \in \mathbb{Z}} (1 + |\delta n_k + x_k|)^{-p}
\]
\[
\leq 1 + \sum_{k \in \mathbb{Z} : n_k > 0} (1 + \delta n_k + x_k)^{-p} + \sum_{k \in \mathbb{Z} : n_k > 0} (1 + \delta n_k - x_k)^{-p}
\]
\[
\leq 1 + \sum_{k \in \mathbb{Z} : n_k > 0} (1 + \delta n_k)^{-p} + \sum_{k \in \mathbb{Z} : n_k > 0} (1 + \delta n_k - \delta)^{-p}
\]
\[
\leq 1 + \sum_{k=1}^{\infty} (1 + \delta k)^{-p} + \sum_{k=1}^{\infty} (1 + \delta(k-1))^{-p}
\]
\[
= 2 \left( 1 + \sum_{k=1}^{\infty} (1 + \delta k)^{-p} \right) \leq 2 \left( 1 + (1 + \delta)^{-p}(\delta^{-1} + p)(p - 1)^{-1} \right).
\]
The last expression is independent of \( t \), and the claim follows. \( \square \)

**Remark 1.** When the set \( A = \{a_k : k \in \mathbb{Z}\} \subset \mathbb{R} \) is relatively \( \delta \)-separated, meaning
\[
\text{rel}(A) := \max_{t \in \mathbb{R}} \# \{ A \cap ([0, \delta] + t) \} < \infty,
\]
then the estimate (b) in Lemma 2.1 becomes
\[
\text{ess sup}_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} (1 + |t - a_k|)^{-p} \leq 2 \text{rel}(A) \left( 1 + (1 + \delta)^{-p}(\delta^{-1} + p)(p - 1)^{-1} \right).
\]
Notice, that for a separated set, \( \text{rel}(A) = 1 \).
Nonstationary Gabor systems provide a generalization of the classical Gabor systems of time-frequency-shifted versions of a single window function.

**Definition 2.** Let \( g = \{g_k \in L^2(\mathbb{R}) : k \in \mathbb{Z}\} \) be a set of window functions and let \( b = \{b_k : k \in \mathbb{Z}\} \) be a corresponding sequence of frequency-shift parameters. Set \( g_{k,l} = M_{b_k} g_k \). Then, the set

\[
G(g, b) = \{g_{k,l} : k, l \in \mathbb{Z}\}
\]

is called a *nonstationary Gabor system*.

Note that, conceptually, we assume that the windows \( g_k \) are centered at points \( \{a_k : k \in \mathbb{Z}\} \), in direct generalization of the regular case, where \( g_k(t) = g(t - ak) \) for some time-shift parameter \( a \). In this sense, we have a two-fold generalization: the sampling points can be irregular and the windows can change for every sampling point. We are interested in conditions under which a nonstationary Gabor system forms a frame. We first recall the case of nonstationary Gabor frames with compactly supported windows, see [2] and [http://www.univie.ac.at/nonstatgab/](http://www.univie.ac.at/nonstatgab/) for further information.

### 3.1. Compactly supported windows: the painless case.

Based on the support length of the windows \( g_k \), we can easily determine frequency-shifts parameters \( b_k \), for which we obtain a frame. The following result is the nonstationary version of the result given in [9].

**Proposition 3.1 ([2]).** Let \( g_k \in L^2(\mathbb{R}) \) be compactly supported functions with \( |\text{supp } g_k| \leq 1/b_k \). Then \( G(g, b) \) is a frame for \( L^2(\mathbb{R}) \) if there exist constants \( A > 0 \) and \( B < \infty \) such that

\[
A \leq G_0(t) = \sum_{k \in \mathbb{Z}} b_k^{-1} |g_k(t)|^2 \leq B \quad \text{a.e.}.
\]

The dual atoms are then \( \gamma_{k,l}(t) = M_{b_k} G_0^{-1}(t) g_k(t) \).

**Remark 2.** An analogous theorem holds for bandlimited functions \( g_k \).

The above theorem follows from the fact that the frame operator associated to the collection of atoms described in the theorem can be written as

\[
S f(t) = \sum_{k \in \mathbb{Z}} b_k^{-1} |g_k(t)|^2 f(t) \quad \text{a.e.}.
\]

The diagonality of the frame operator in the painless case is derived from a generalized Walnut representation for the frame operator \( S \) of nonstationary Gabor frames. In the next section we will see that this representation immediately implies diagonality of \( S \) under the assumptions of Proposition 3.1.
3.2. A Walnut representation for nonstationary Gabor Frames.

Let us now consider a nonstationary Gabor system $G(g,b)$, with all windows $g_k$ in $W(L^\infty,\ell^1)$. The frame operator associated to $G(g,b)$ reads

\begin{equation}
Sf = \sum_{k,l \in \mathbb{Z}} \langle f, M_{lb_k}g_k \rangle M_{lb_k}g_k.
\end{equation}

**Proposition 3.2.** The frame operator $S$ in (2) admits a Walnut representation

\begin{equation}
Sf = \sum_{k,l \in \mathbb{Z}} G_{k,l} \cdot T_{lb_k}^{-1} f,
\end{equation}

where $G_{k,l}(t) = b_k^{-1} g_k(t - lb_k^{-1})g_k(t)$, for $f \in L^2(\mathbb{R})$. Moreover, its operator norm can be bounded by

\begin{equation}
|\langle Sf, h \rangle| \leq \left( \text{ess sup}_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |g_k(t)| \right) \left( \sup_{k \in \mathbb{Z}} (1 + b_k^{-1}) \|g_k\|_{W(L^\infty,\ell^1)} \right) \|f\|_2 \|h\|_2
\end{equation}

for all $f, h \in L^2(\mathbb{R})$.

**Proof.** First assume that $f, h \in L^2(\mathbb{R})$ are compactly supported. Since $\langle f, M_{lb_k}g_k \rangle = (\hat{f}\hat{g}_k)(lb_k)$ we can write $S$ as

\begin{equation}
Sf(t) = \sum_{k,l \in \mathbb{Z}} (\hat{f}\hat{g}_k)(lb_k)M_{lb_k}g_k(t) = \sum_{k \in \mathbb{Z}} m_k(t)g_k(t),
\end{equation}

where $m_k(t) = \sum_{l \in \mathbb{Z}} (\hat{f}\hat{g}_k)(lb_k)e^{2\pi ilb_k t}$, for every $k \in \mathbb{Z}$. The functions $m_k$ are $b_k^{-1}$ periodic and by the Poisson formula can be written as

\begin{equation}
m_k(t) = b_k^{-1} \sum_{l \in \mathbb{Z}} (\hat{f}\hat{g}_k)(t - lb_k^{-1}).
\end{equation}

Therefore, substituting (6) in (5) yields the Walnut representation.

We next prove the boundedness (4). Using Cauchy-Schwartz inequality for integrals and series, and changing the order of summation and integral due to the compact support of $f, h$
we obtain

\[ |\langle Sf, h \rangle| = \left| \left\langle \sum_{k,l \in \mathbb{Z}} b_k^{-1} g_k(\cdot - lb_k^{-1})g_k(\cdot) f(\cdot - lb_k^{-1}), h \right\rangle \right| \]

\[ \leq \sum_{k,l \in \mathbb{Z}} b_k^{-1} \int_{\mathbb{R}} |g_k(t - lb_k^{-1})| |g_k(t)||f(t - lb_k^{-1})||h(t)| \, dt \]

\[ \leq \sum_{k,l \in \mathbb{Z}} b_k^{-1} \left[ \int_{\mathbb{R}} |g_k(t - lb_k^{-1})| |g_k(t)||f(t - lb_k^{-1})|^2 \, dt \right]^{1/2} \left[ \int_{\mathbb{R}} |g_k(t - lb_k^{-1})| |h(t)|^2 \, dt \right]^{1/2} \]

\[ \leq \left[ \sum_{k,l \in \mathbb{Z}} b_k^{-1} \int_{\mathbb{R}} |g_k(t)||g_k(t + lb_k^{-1})||f(t)|^2 \, dt \right]^{1/2} \left[ \sum_{k,l \in \mathbb{Z}} b_k^{-1} \int_{\mathbb{R}} |g_k(t)||g_k(t - lb_k^{-1})||h(t)|^2 \, dt \right]^{1/2} \]

\[ = \left[ \int_{\mathbb{R}} |f(t)|^2 \sum_{k,l \in \mathbb{Z}} b_k^{-1} |g_k(t)||g_k(t - lb_k^{-1})| \, dt \right]^{1/2} \left[ \int_{\mathbb{R}} |h(t)|^2 \sum_{k,l \in \mathbb{Z}} b_k^{-1} |g_k(t)||g_k(t - lb_k^{-1})| \, dt \right]^{1/2}. \]

Each term in the last expression can be estimated as

\[ \int_{\mathbb{R}} |f(t)|^2 \sum_{k,l \in \mathbb{Z}} b_k^{-1} |g_k(t)||g_k(t - lb_k^{-1})| \, dt \leq \]

\[ \leq \sum_{k \in \mathbb{Z}} b_k^{-1} \left( \text{ess sup}_{t \in \mathbb{R}} \sum_{l \in \mathbb{Z}} |g_k(t - lb_k^{-1})| \right) \int_{\mathbb{R}} |f(t)|^2 |g_k(t)| \, dt \]

\[ \leq \left( \sup_{k \in \mathbb{Z}} b_k^{-1} \text{ess sup}_{t \in \mathbb{R}} \sum_{l \in \mathbb{Z}} |g_k(t - lb_k^{-1})| \right) \left( \text{ess sup}_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |g_k(t)| \right) \|f\|_2^2. \]

By the sampling theorem for Wiener spaces, [13] Proposition 11.1.4], we have

\[ \text{ess sup}_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |g_k(t - lb_k^{-1})| \leq (1 + b_k)\|g_k\|_{W(L_\infty, \ell_1)}. \]

Substituting (3) and (9) into (7) yields (4). By the density of compactly supported functions in $L^2(\mathbb{R})$, the estimate holds for all of $L^2(\mathbb{R})$. \[ \square \]

**Remark 3.** In the standard setting of Gabor frames, i.e. $g_k(t) = g(t - ak)$ for fixed $a > 0$, and $b_k = b$ for all $k \in \mathbb{Z}$, the above bound reduces to the well know bound

\[ \langle Sf, h \rangle \leq (1 + a^{-1})(1 + b^{-1})\|g\|_{W(L_\infty, \ell_1)}^2 \|f\|_2 \|h\|_2 \]

**Remark 4.** (1) From (7) in the proof of Proposition 3.2, it follows that the frame operator $S$ is also bounded by

\[ |\langle Sf, h \rangle| \leq \left( \text{ess sup}_{t \in \mathbb{R}} \sum_{k,l \in \mathbb{Z}} b_k^{-1} |g_k(t - lb_k^{-1})||g_k(t)| \right) \|f\|_2 \|h\|_2. \]
Note that in the painless case of Theorem 3.1, the frame operator reduces to the multiplication operator $Sf = \sum_{k \in \mathbb{Z}} G_{k,0} \cdot f = G_0 \cdot f$.

3.3. **Existence of nonstationary Gabor frames.** For windows $g_k$ that are neither compactly supported nor bandlimited, we are interested in the existence of frames of the form $G(g, b)$ and in the construction of the involved parameters. The following theorem derives a sufficient condition for the existence of nonstationary Gabor frames and shows that this condition can be satisfied.

In this and the subsequent sections, $[b_1, b_2], [p_1, p_2], [C_1, C_2]$ are compact intervals of positive real numbers.

**Theorem 3.3.** Let $g = \{g_k \in W(L^\infty, \ell^1) : k \in \mathbb{Z}\}$ be a set of windows such that

i) for some positive constants $A_0, B_0$

\begin{equation}
0 < A_0 \leq \sum_{k \in \mathbb{Z}} |g_k(t)|^2 \leq B_0 < \infty \; \text{a.e. ;}
\end{equation}

ii) for all $k \in \mathbb{Z}$, the windows decay polynomially around a $\delta$-separated set $\{a_k : k \in \mathbb{Z}\}$ of time-sampling points $a_k$

\begin{equation}
|g_k(t)| \leq C_k (1 + |t - a_k|)^{-p_k},
\end{equation}

where $p_k \in [p_1, p_2] \subset \mathbb{R}$, $2 < p_1$ and $C_k \in [C_1, C_2]$.

Then there exists a sequence $\{b_k^0\}_{k \in \mathbb{Z}}$, such that for $b_k \leq b_k^0, k \in \mathbb{Z}$, the nonstationary Gabor system $G(g, b)$ forms a frame for $L^2(\mathbb{R})$.

**Proof.** Let $f \in L^2(\mathbb{R})$. Applying (3), we write

$$
\langle Sf, f \rangle = \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} b_k^{-1} |g_k(t)|^2 |f(t)|^2 \, dt + \int_{\mathbb{R}} \sum_{l \in \mathbb{Z} \setminus \{0\}} \sum_{k \in \mathbb{Z}} b_k^{-1} g_k(t)g_k(t - lb_k^{-1}) \overline{f(t - lb_k^{-1})\overline{f(t)}} \, dt
$$

Using similar arguments as in the derivation of (7), we obtain

$$
\left| \int_{\mathbb{R}} \sum_{l \in \mathbb{Z} \setminus \{0\}} \sum_{k \in \mathbb{Z}} b_k^{-1} g_k(t)g_k(t - lb_k^{-1}) \overline{f(t - lb_k^{-1})\overline{f(t)}} \, dt \right| \leq \left( \esssup_{t \in \mathbb{R}} \sum_{l \in \mathbb{Z} \setminus \{0\}} \sum_{k \in \mathbb{Z}} b_k^{-1} |g_k(t)||g_k(t - lb_k^{-1})| \right) \|f\|^2_2
$$

$$
\leq \max_{k \in \mathbb{Z}} \{b_k^{-1}\} \sum_{l \in \mathbb{Z} \setminus \{0\}} \left( \esssup_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |g_k(t)||g_k(t - lb_k^{-1})| \right) \|f\|^2_2.
$$
Therefore, lower and upper frame bounds are obtained from

\[
\langle Sf, f \rangle \|f\|_2^{-2} \geq \min_{k \in \mathbb{Z}} \{b_k^{-1}\} \left( \text{ess inf}_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |g_k(t)|^2 - \frac{\max_{k \in \mathbb{Z}} \{b_k^{-1}\}}{\min_{k \in \mathbb{Z}} \{b_k^{-1}\}} R \right)
\]

\[
\langle Sf, f \rangle \|f\|_2^{-2} \leq \max_{k \in \mathbb{Z}} \{b_k^{-1}\} \left( \text{ess sup}_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |g_k(t)|^2 + R \right),
\]

We need to construct a sequence of \(b_k, k \in \mathbb{Z}\), such that for all \(f \in L^2(\mathbb{R})\), (12) is bounded away from zero.

Let \(\epsilon < C_1\) and consider the sequence of frequency shifts \(b_k = (\frac{\epsilon}{C_k})^{1/p_k}\). Then \(\min_{k \in \mathbb{Z}} \{b_k^{-1}\} \geq (C_1\epsilon^{-1})^{1/p}\), \(\max_{k \in \mathbb{Z}} \{b_k^{-1}\} \leq (C_2\epsilon^{-1})^{1/p}\) and

\[
\frac{\max_{k \in \mathbb{Z}} \{b_k^{-1}\}}{\min_{k \in \mathbb{Z}} \{b_k^{-1}\}} \leq C_2^{1/p_1} C_1^{-1/p_2} \epsilon^{1/p_2 - 1/p_1}.
\]

Since \((1 + |x + y|)^{-p} \leq (1 + |x|)^p(1 + |y|)^{-p}\) for \(x, y \in \mathbb{R}\) and \(p \geq 0\), using (11), we have, for some \(p_1 - 2 > \mu > 0\):

\[
|g_k(t)||g_k(t - lb_k^{-1})| \leq C_k^2(1 + |t - a_k|)^{-p_k}(1 + |t - a_k - lb_k^{-1}|)^{-p_k + (1+\mu)}
\]

\[
\leq C_k^2(1 + |t - a_k|)^{-(1+\mu)}(1 + |t - a_k - lb_k^{-1}|)^{-p_k + (1+\mu)}
\]

\[
\leq C_k^2(1 + |t - a_k|)^{-(1+\mu)} |l|^{-p_k + (1+\mu)} b_k^{p_k - (1+\mu)}
\]

\[
= C_k^{1+(1+\mu)/p_k}(1 + |t - a_k|)^{-(1+\mu)} |l|^{-p_k + (1+\mu)} \epsilon^{1-(1+\mu)/p_k}.
\]

Hence,

\[
\sum_{k \in \mathbb{Z}} |g_k(t)||g_k(t - lb_k^{-1})| \leq \sum_{k \in \mathbb{Z}} E_k (1 + |t - a_k|)^{-1+(1+\mu)} |l|^{-p_k + (1+\mu)} \epsilon^{1-(1+\mu)/p_k}
\]

\[
\leq \max_{k \in \mathbb{Z}} E_k |l|^{-p_1 + (1+\mu)} \epsilon^{1-(1+\mu)/p_1} \sum_{k \in \mathbb{Z}} (1 + |t - a_k|)^{-(1+\mu)}
\]

\[
\leq \max_{k \in \mathbb{Z}} E_k |l|^{-p_1 + (1+\mu)} \epsilon^{1-(1+\mu)/p_1} 2 \left(1 + (1 + \delta)^{-1+(1+\mu)(\delta^{-1} + 1 + \mu)^{-1}}\right),
\]

where the last estimate follows from Lemma 2.1 (b). Summing the expression (14) over \(l \in \mathbb{Z} \setminus \{0\}\) using Lemma 2.1 (a), we see that \(R\), as a function of \(\epsilon\), behaves like \(\epsilon^{1-(1+\mu)/p}\), i.e. \(R \approx \epsilon^{1-(1+\mu)/p_1}\), and \(R\) tends to 0 for \(\epsilon \to 0\). Moreover,

\[
\frac{\max_{k \in \mathbb{Z}} \{b_k^{-1}\}}{\min_{k \in \mathbb{Z}} \{b_k^{-1}\}} R \approx \epsilon^{1-(2+\mu)/p_1 + 1/p_2}
\]

can be made arbitrarily small by choosing \(\epsilon\) small since \(1 - (2 + \mu)/p_1 + 1/p_2 > 0\). Therefore, if \(\epsilon_0\) is such that for \(b_k^0 := (\frac{\epsilon_0}{C_k})^{1/p_k}\), \(\max_{k \in \mathbb{Z}} \{b_k^{-1}\} R < A_0\), then \(\{M_{b_k}g_k\}_{k \in \mathbb{Z}}\) is a frame for all \(b_k \leq b_k^0\). 

\(\square\)
For completeness, we state the equivalent result for analysis windows \( g_k \) with polynomial decay in the frequency domain.

**Corollary 3.4.** Let \( g = \{g_k \in L^2(\mathbb{R}) : \hat{g}_k \in W(L^\infty, \ell^1), k \in \mathbb{Z}\} \) be a set of windows such that

i) for some positive constants \( A_0, B_0 \)

\[
0 < A_0 \leq \sum_{k \in \mathbb{Z}} |\hat{g}_k(t)|^2 \leq B_0 < \infty \quad \text{a.e. ;}
\]

(15)

ii) for all \( k \in \mathbb{Z} \), the windows decay polynomially around a \( \delta \)-separated set \( \{b_k : k \in \mathbb{Z}\} \) of frequency-sampling points \( b_k \):

\[
|\hat{g}_k(t)| \leq C_k (1 + |t - b_k|)^{-p_k},
\]

where \( p_k \) and \( C_k \) are chosen as in Theorem 3.3.

Then there exists a sequence \( \{a_0^k\}_{k \in \mathbb{Z}} \), such that for \( a_k \leq a_0^k, k \in \mathbb{Z} \), the nonstationary Gabor system \( \{T_{a_k}g_k : k, l \in \mathbb{Z}\} \) forms a frame for \( L^2(\mathbb{R}) \).

### 3.4. Almost painless nonstationary Gabor frames.

Theorem 3.3 shows that for windows with sufficient uniform decay, nonstationary Gabor frames can always be constructed by choosing sufficient density in the frequency samples. In this section we construct a painless nonstationary Gabor frame from a given nonstationary Gabor frame and use the information gained from this construction to check the frame property of the original, non-painless frame. This is a situation of practical relevance, since we may often be interested in using windows that decay fast and are negligible outside a support of interest.

In the present and subsequent section, we will work with the following constants, that depend on the separation of the sampling points, the decay of the windows and the frequency-sampling parameters:

\[
E_1 = 1 + \frac{\delta^{-1} + p_1}{(1 + \delta)^{p_1}(p_1 - 1)} \quad \text{and} \quad E_2 = 1 + \frac{b_2 + p_2}{(1 + b_2^{-1})^{p_1}(p_1 - 1)}
\]

(17)

**Proposition 3.5.** Let \( g = \{g_k \in W(L^\infty, \ell^1) : k \in \mathbb{Z}\} \) be a set of windows that are essentially bounded away from zero on the intervals \( I_k = [a_k - (2b_k)^{-1}, a_k + (2b_k)^{-1}] \) where \( \{a_k : k \in \mathbb{Z}\} \) forms a separated set and \( b_k \in [b_1, b_2] \) for all \( k \in \mathbb{Z} \). Further let \( 0 < \tilde{A} \leq \sum_{k \in \mathbb{Z}} b_k^{-1} |g_k^\circ(t)|^2 \leq \tilde{B} < \infty \), where \( g_k^\circ = g_k \chi_{I_k} \) and \( g_k^\circ = g_k - g_k^\circ \). Assume that

\[
|g_k^\circ(t)| \leq \begin{cases} C_k(1 + t - a_k - (2b_k)^{-1})^{-p_k}, & t > a_k + (2b_k)^{-1}; \\ 0, & t \in I_k; \\ C_k(1 - t + a_k - (2b_k)^{-1})^{-p_k}, & t < a_k - (2b_k)^{-1}, \end{cases}
\]

(18)
where \( C_k \in [C_1, C_2] \) and \( p_k \in [p_1, p_2] \) with \( p_1 > 1 \). If

\[
4 \cdot C_2 \cdot \frac{1}{\delta b_k^2} \cdot E_1 \cdot (C_2 \cdot E_2 + \max_{k \in \mathbb{Z}} \|g_k\|_\infty < \tilde{A},
\]

then \( \mathcal{G}(g, b) \) forms a nonstationary Gabor frame for \( L^2(\mathbb{R}) \).

Proof. Let \( f \in L^2(\mathbb{R}) \). Using similar arguments as in the proof of Proposition 3.2 we obtain

\[
\langle Sf, f \rangle = \sum_{k \in \mathbb{Z}} b_k^{-1} \int_{\mathbb{R}} |g_k^0(t)|^2 |f(t)|^2 \, dt + R \geq \text{ess inf}_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} b_k^{-1} |g_k^0(t)|^2 \|f\|_2^2 + R,
\]

where

\[
|R| \leq \text{ess sup}_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |g_k^0(t)| \cdot \sup_{k \in \mathbb{Z}} \text{ess sup}_{t \in \mathbb{R}} |g_k^0(t - lb_k^{-1})| \|f\|_2^2 + 2 \cdot \text{ess sup}_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |g_k^0(t)| \cdot \sup_{k \in \mathbb{Z}} \text{ess sup}_{t \in \mathbb{R}} b_k^{-1} |g_k^0(t - lb_k^{-1})| \|f\|_2^2.
\]

Since \( g_k^0 \) are compactly supported with support size \( b_k^{-1} \), the shifts \( g_k^0(t - lb_k^{-1}) \) do not overlap, hence \( \text{ess sup}_{t \in \mathbb{R}} \sum_{l \in \mathbb{Z}} b_k^{-1} |g_k^0(t - lb_k^{-1})| \leq b_k^{-1} \|g_k\|_\infty \).

Now, the expression \( \sum_{l \in \mathbb{Z}} |g_k^0(t - lb_k^{-1})| \) is \( b_k^{-1} \)-periodic, therefore

\[
\text{ess sup}_{t \in \mathbb{R}} \sum_{l \in \mathbb{Z}} |g_k^0(t - lb_k^{-1})| = \text{ess sup}_{t \in I_k} \sum_{l \in \mathbb{Z}} |g_k^0(t - lb_k^{-1})|.
\]

Let \( t \in I_k \), then, by [18]

\[
\sum_{l \in \mathbb{Z}} |g_k^0(t - lb_k^{-1})| \leq C_k \left[ \sum_{l > 0} (1 + a_k - (2b_k)^{-1} - t + lb_k^{-1})^{-p_k} \right.
\]

\[
+ \sum_{l < 0} (1 - a_k - (2b_k)^{-1} + t - lb_k^{-1})^{-p_k} \right]
\]

\[
\leq C_k \left[ \sum_{l = 1}^{\infty} (1 + (l - 1)b_k^{-1})^{-p_k} + \sum_{l = -\infty}^{-1} (1 - (l + 1)b_k^{-1})^{-p_k} \right]
\]

\[
= C_k 2 \sum_{l = 0}^{\infty} (1 + b_k^{-1})^{-p_k} \leq 2C_k (1 + (1 + b_k^{-1})^{-p_k} (b_k + p_k)(p_k - 1)^{-1}),
\]

where the last estimate follows from Lemma 2.1(a).

We now derive a bound for \( \text{ess sup}_{t \in \mathbb{R}} \sum_{l \in \mathbb{Z}} |g_k^0(t)| \). First observe, that for any \( t \in \mathbb{R} \) and all \( k \), we have

\[
|g_k^0(t)| \leq C_k \left[ (1 + |t - a_k - (2b_k)^{-1}|)^{-p_k} + |t - a_k + (2b_k)^{-1}|^{-p_k} \right]
\]

\[
\leq C_2 \left[ (1 + |t - a_k - (2b_k)^{-1}|)^{-p_2} + (1 + |t - a_k + (2b_k)^{-1}|)^{-p_2} \right].
\]
Since the frequency shifts $b_k$ are from a finite interval and the set \{\(a_k : k \in \mathbb{Z}\) is \(\delta\)-separated, the sets \(\Gamma^+ = \{a_k + (2b_k)^{-1} : k \in \mathbb{Z}\}\) and \(\Gamma^- = \{a_k - (2b_k)^{-1} : k \in \mathbb{Z}\}\) are relatively \(\delta\)-separated with \(\text{rel}(\Gamma) = \text{rel}(\Gamma^+) = \text{rel}(\Gamma^-) = [(2b_1\delta)^{-1}]\). Therefore,

\[
\sum_{k \in \mathbb{Z}} |g_k(t)| \leq C_2 \sum_{k \in \mathbb{Z}} (1 + |t - a_k - (2b_k)^{-1}|)^{-p_1} + C_2 \sum_{k \in \mathbb{Z}} (1 + |t - a_k + (2b_k)^{-1}|)^{-p_1} \\
\leq 4 C_2 \text{rel}(\Gamma) (1 + (1 + \delta)^{-p_1}(\delta^{-1} + p_1)(p_1 - 1)^{-1}),
\]

where the estimate follows from Remark 1. Collecting the obtained estimates, we find that

\[
|R| \leq \left(4 \cdot C_2 \cdot \frac{1}{\delta b_1} \cdot E_1 \cdot (C_2 \cdot E_2 + \max_{k \in \mathbb{Z}} \|g_k\|_{\infty})\right) \|f\|_2^2
\]

and, by assumption \(\text{(19)}\) of the theorem, the frame operator \(S\) is bounded below:

\[
\langle Sf, f \rangle > \inf_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} b_k^{-1} |g_k(t)|^2 \|f\|_2^2 + R > \tilde{A} \|f\|_2^2 - |R| > 0.
\]

For the upper frame bound, consider the sum \(\sum_{k \in \mathbb{Z}} b_k^{-1} |g_k(t)|^2\): since only finitely many intervals \(I_k\) overlap at a given point \(t\), this sum is finite for every \(t \in \mathbb{R}\). Then

\[
\langle Sf, f \rangle \leq \sup_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} b_k^{-1} |g_k(t)|^2 \|f\|_2^2 + R \leq \sup_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} b_k^{-1} |g_k(t)|^2 \|f\|_2^2 + |R| < \infty.
\]

\[\square\]

3.5. A perturbation result for nonstationary Gabor frames. We now consider nonstationary Gabor frames obtained by perturbation of a known frame. This result is in the spirit of similar results for regular Gabor frames \(\text{[3, 4, 8]}\).

**Proposition 3.6.** Let \(\{a_k : k \in \mathbb{Z}\}\) be a \(\delta\)-separated set and \(\mathcal{G}(h, b)\) a nonstationary Gabor frame with frame constants \(A_h, B_h\), frame operator \(S_h\) and frequency-shift parameters \(b_k \in [b_1, b_2]\). Let \(g_k \in L^2(\mathbb{R})\) be a set of windows such that

\[
|g_k(t) - h_k(t)| \leq C(1 + |t - a_k|)^{-p_k} \text{ a.e. ,}
\]

for all \(k \in \mathbb{Z}\) and \(t \in \mathbb{R}\), where \(p_k \in [p_1, p_2]\) with \(p_1 > 1\), and \(C < \sqrt{A_h} \lambda^{-1}\) with

\[
\lambda = 4b_1^{-1} \cdot E_1 \cdot E_2
\]

Then \(\mathcal{G}(g, b)\) is a frame for \(L^2(\mathbb{R})\) with frame bounds \(A = A_h(1 - \sqrt{C^2 \lambda A_h^{-1}})^2\) and \(B = B_h(1 + \sqrt{C^2 \lambda B_h^{-1}})^2\).

**Proof.** We apply \(\text{[8] Proposition 4.1.}\), which states that, if \(\{h_i\}_{i \in \mathbb{Z}}\) is a frame and, for some set of windows \(\{g_i\}_{i \in \mathbb{Z}}\) and all \(f \in L^2(\mathbb{R})\), \(\sum_{i \in \mathbb{Z}} |\langle f, h_i - g_i \rangle|^2 \leq A_h \|f\|_2^2\), then the sequence \(\{g_i\}_{i \in \mathbb{Z}}\) is also a frame. Here, \(A_h\) is a lower frame bound of \(\{h_i\}_{i \in \mathbb{Z}}\).
In our setting, for \( G(\mathbf{g}, \mathbf{b}) \) to be a frame for \( L^2(\mathbb{R}) \), it suffices to show that \( \sum_{k,l \in \mathbb{Z}} |\langle f, g_{k,l} - h_{k,l} \rangle|^2 \leq R \|f\|_2^2 \) for some \( R < A_h \).

Let \( \psi_k(t) := g_k - h_k \). Then it follows that

\[
\sum_{k,l \in \mathbb{Z}} |\langle f, \psi_k(t) \rangle|^2 \leq \left( \text{ess sup}_{t \in \mathbb{R}} \sum_{k,l \in \mathbb{Z}} |\psi_k(t)| \right) \left( \sup_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} |\psi_k(t - lb_k^{-1})| \right) \|f\|_2^2.
\]

We approximate each term of the above bound separately. By assumption (20) and Lemma 2.1 (b)

\[
\sum_{k \in \mathbb{Z}} |\psi_k(t)| \leq C \sum_{k \in \mathbb{Z}} (1 + |t - a_k|)^{-p_k} \leq C \sum_{k \in \mathbb{Z}} (1 + |t - a_k|)^{-p_1} \leq 2CE_1.
\]

On the other hand,

\[
\sum_{l \in \mathbb{Z}} |\psi_k(t - lb_k^{-1})| \leq C \sum_{l \in \mathbb{Z}} (1 + |t - a_k - lb_k^{-1}|)^{-p_k}.
\]

Note that \( \sum_{l \in \mathbb{Z}} (1 + |t - a_k - lb_k^{-1}|)^{-p_k} \) is \( b_k^{-1} \)-periodic, therefore

\[
\text{ess sup}_{t \in \mathbb{R}} \sum_{l \in \mathbb{Z}} (1 + |t - a_k - lb_k^{-1}|)^{-p_k} = \text{ess sup}_{t - a_k \in [0,b_k^{-1}]} \sum_{l \in \mathbb{Z}} (1 + |t - a_k - lb_k^{-1}|)^{-p_k}.
\]

Hence, for \( t - a_k \in [0,b_k^{-1}] \) and by Lemma 2.1, we obtain

\[
\sum_{l \in \mathbb{Z}} (1 + |t - a_k - lb_k^{-1}|)^{-p_k} \leq 1 + \sum_{l=1}^{\infty} (1 + |t - a_k + lb_k^{-1}|)^{-p_k} + \sum_{l=1}^{\infty} (1 + t - a_k + lb_k^{-1})^{-p_k}
\]

\[
\leq 1 + \sum_{l=1}^{\infty} (1 + lb_k^{-1})^{-p_k} + \sum_{l=1}^{\infty} (1 + (l - 1)b_k^{-1})^{-p_k}
\]

\[
= 2 \sum_{l=0}^{\infty} (1 + lb_k^{-1})^{-p_k} = 2 \left( 1 + \sum_{l=1}^{\infty} (1 + lb_k^{-1})^{-p_k} \right) \leq 2E_2.
\]

Gathering all the estimates, we obtain

\[
\sum_{k,l \in \mathbb{Z}} |\langle f, \psi_k(t) \rangle|^2 \leq C^24b_1^{-1}E_1 \cdot E_2 \|f\|_2^2 = C^2\lambda \|f\|_2^2.
\]

By assumption \( C^2\lambda < A_h \), and the proof is complete. \( \square \)

Remark 5. (a) An example for the starting frame is a collection of windows \( h_k \) with compact support on the intervals \( I_k = [a_k - (2b_k)^{-1}, a_k + (2b_k)^{-1}] \) and shift parameters \( b_k \), cf. Proposition 3.1. Proposition 3.5 can be considered as a special case of this method, namely where windows \( g_k \) are created by ‘adding’ tails to \( h_k \) but not changing the windows \( h_k \). Then, \( h_k = g_k\chi_{I_k} \), and one can think of \( \{M_{lb_k}g_k : k \}, l \in \mathbb{Z} \} \) as a perturbation of \( \{M_{lb_k}g_k\chi_{I_k} : k \}, l \in \mathbb{Z} \} \) and vice versa, while only considering the decay of the tails of \( g_k \) in (20), which leads to smaller constants \( C_k \).

(b) Note that non-harmonic Gabor frames were discussed in [1], for compactly supported...
windows and under a frame condition for subspaces of compactly supported windows. Under suitable density conditions on the frequency sampling points, Theorem 3.3 and Proposition 3.6 can be generalized to non-uniform frequency sampling, thus yielding non-harmonic, non-compactly supported Gabor frames.

3.6. Nonstationary Gabor frames on modulation spaces. Modulation spaces, cf. [15, 16, 18], are considered as the appropriate function spaces for time-frequency analysis and in particular, for the study of Gabor frames. By their definition, modulation spaces require decay in both time and frequency. Under additional assumptions on the windows \( g_k \), the collection \( G(g, b) \) is a frame for all modulation spaces \( M^p \), \( 1 \leq p \leq \infty \).

Proposition 3.7. Let \( G(g, b) \) be a frame for \( L^2(\mathbb{R}) \) satisfying the uniform estimate

\[
|V_\phi g_k(x, \omega)| \leq C (1 + |x-a_k|)^{-r-2}(1 + |\omega|)^{-r-2}, ~ r > 2
\]

where \( \phi \) is a Gaussian window. Then the frame operator \( S \) is invertible simultaneously on all modulation spaces \( M^p \) for \( 1 \leq p \leq \infty \).

Proof. Notice, that

\[
|V_\phi g_{k,l}(x, \omega)| \leq C (1 + |(x, \omega) - (a_k, lb_k)|)^{-r-2},
\]

since the weights \( (1 + |(x, \omega)|)^r \) and \( (1 + |x| + |\omega|)^r \) are equivalent and \( 1 + |x| + |\omega| \leq (1 + |x|)(1 + |\omega|) \). A result on Gabor molecules [19] states that, if an \( L^2 \)-frame \( \{g_z : z \in Z\} \), where \( Z \) is some separable set in \( \mathbb{R}^2 \), is a frame for \( L^2(\mathbb{R}) \), satisfies the uniform estimate \( |V_\phi g_z(x, \omega)| \leq C (1 + |(x, \omega) - z|)^{-r-2} \), then the frame operator \( Sf = \sum_{z \in Z} \langle f, g_z \rangle g_z \) is invertible simultaneously on all \( M^p \) for each \( 1 \leq p \leq \infty \). The result hence follows from condition (22).

Remark 6. Since the windows \( g_k \in L^2(\mathbb{R}) \) satisfying condition (22) are elements of a compact set in \( S_0 \), cf. [12], we can bound the norm of the frame operator \( S \), acting from \( S_0 \) to \( S_0 \):

\[
\|S\|_{S_0 \to S_0} \leq \left( \sup_{k \in Z} \|g_k\|_{S_0} \right) \cdot \left( \text{ess sup}_{(x, \omega) \in \mathbb{R}^2} \sum_{k, l \in Z} |V_\phi g_{k,l}(x, \omega)| \right).
\]

Similar estimates hold on all modulation spaces. These estimates are important in the construction of dual frames.

4. Examples

We illustrate our theory on two examples. In the first example we construct a nonstationary Gabor frame using Proposition 3.5. The windows in this example are constructed in analogy to the windows used in scale frames, which were introduced in [2] to automatically
Then the tails of the points equal to $\tilde{\text{frame bound}}$ The set of translation parameters \( \{a_k : k \in \mathbb{Z}\} \) is separated with minimum distance between the points equal to $\delta = 1/4$.

Let $I_k = [a_k - 2^{-s_k-1}, a_k + 2^{-s_k-1}]$ and define new set of windows by $h_k(t) = g_k(t)\chi_{I_k}$. Then \( \{M_{lb_k}h_k : k, l \in \mathbb{Z}\} \) with $b_k = 2^{s_k}$ is a painless nonstationary Gabor frame with lower frame bound $\tilde{A} = 0.016$.

The set of translation parameters \( \{a_k : k \in \mathbb{Z}\} \) is separated with minimum distance between the points equal to $\delta = 1/4$.

Let $I_k = [a_k - 2^{-s_k-1}, a_k + 2^{-s_k-1}]$ and define new set of windows by $h_k(t) = g_k(t)\chi_{I_k}$. Then \( \{M_{lb_k}h_k : k, l \in \mathbb{Z}\} \) with $b_k = 2^{s_k}$ is a painless nonstationary Gabor frame with lower frame bound $\tilde{A} = 0.016$.

Then the tails of $g_k$, that is $g_k^*(t) = g_k(t) - h_k(t)$ decay like

$$|g_k^*(t)| \leq \begin{cases} \sqrt{2^{s_k}}g(1/2)(1 + t - a_k - 2^{-s_k-1})^{-21} & t > a_k + 2^{-s_k-1} \\ 0 & t \in I_k \\ \sqrt{2^{s_k}}g(1/2)(1 + a_k - 2^{-s_k-1} - t)^{-21} & t < a_k - 2^{-s_k-1}. \end{cases}$$

From Proposition 3.2 it follows that $4(\delta b_l^2)^{-1} \cdot C_2 \cdot E_1 \cdot (C_2 \cdot E_2 + \max_{k \in \mathbb{Z}} \{\|g_k\|_{\infty}\}) = 0.0086$ which is smaller than $\tilde{A}$, and \( \{M_{lb_k}T_{a_k}g_k : k, l \in \mathbb{Z}\} \) is a nonstationary Gabor frame.

Next, we consider a frame that arises as a perturbation of a painless nonstationary Gabor frame.

Example 4.2. Let $h$ be a raised cosine window, meaning $h(t) = 1 + \cos(2\pi t)$ for $t \in [-1/2, 1/2]$, and zero otherwise, and let \( \{s_k\}_{k \in \mathbb{Z}} \) be a sequence with values from the set \{-1, 0, 1\} and such that $|s_k - s_{k-1}| \in \{0, 1\}$. Then \( \{M_{lb_k}h_k : k, l \in \mathbb{Z}\} \), with $b_k = 2^{s_k}$, forms a painless nonstationary Gabor frame with lower frame bound $A_k = 1/2$, where

$$h_k(t) = T_{a_k}\sqrt{2^{s_k}}h(2^{s_k}t)$$
and the shift parameters are

\[ a_{k+1} = a_k + 2^{-s_k} \cdot \frac{5}{6} \quad \text{if} \quad s_k > s_{k+1}, \]

\[ a_{k+1} = a_k + 2^{-s_k+1} \cdot \frac{1}{3} \quad \text{if} \quad s_k = s_{k+1}, \]

\[ a_{k+1} = a_k + 2^{-s_k+1} \cdot \frac{5}{6} \quad \text{if} \quad s_k < s_{k+1}. \]

The points \( a_k \) form a separated set with minimum distance between the points equal \( \delta = 1/6 \).

Let \( \Omega = 0.02 \) and \( \phi \) be a bandlimited filter given by

\[ \hat{\phi}(\omega) = \frac{1}{2} (1 + \cos(2\pi\Omega^{-1}\omega)) \text{ on its support } [-\Omega/2, \Omega/2]. \]

We build new windows \( g_k \) by convolving \( \phi \) with \( h_k \). The windows \( g_k := \phi \ast h_k \) are no more compactly supported. Since \( |\phi(t)| \leq C_\phi (1 + |t|)^{-3} \), where \( C_\phi = \Omega \), it follows from Lemma A.1 that

\[
|g_k(t) - h_k(t)| \leq \|h_k\|_\infty \frac{C_\phi}{2} \begin{cases} 
(1 + t - 2^{-s_k-1})^{-2} - (1 + t + 2^{-s_k-1})^{-2} & t > 2^{-s_k-1} \\
2 - (1 + t + 2^{-s_k-1})^{-2} - (1 - t + 2^{-s_k-1})^{-2} & t \in I_k \\
(1 - t + 2^{-s_k-1})^{-2} - (1 - t + 2^{-s_k-1})^{-2} & t < -2^{-s_k-1}
\end{cases}
\]

The above relation can be bounded above by

\[
|g_k(t) - h_k(t)| \leq C_k (1 + |t|)^{-2}, \quad \text{where} \quad C_k = \|h_k\|_\infty C_\phi (1 + 2^{-s_k-1})^2/2.
\]

Therefore, for all \( k \in \mathbb{Z} \)

\[
|g_k(t) - h_k(t)| \leq C_{\text{max}} (1 + |t|)^{-2}, \quad C_{\text{max}} = \max_{k \in \mathbb{Z}} C_k = 0.0566.
\]

Since \( C_{\text{max}} = 0.0566 < \sqrt{A_k \lambda^{-1}} = 0.0572 \), with \( \lambda \) as defined in (21), \( \{M_{lb}g_k : k, l \in \mathbb{Z}\} \) is a Gabor frame by Proposition 3.6.

5. Conclusions and perspectives

In this contribution, the existence of nonstationary Gabor frames was proved under mild decay conditions on all windows involved. We further gave tools to construct nonstationary Gabor frames with general windows and illustrated the developed theory on two examples. The reconstruction of signals from their nonstationary Gabor coefficients will be addressed in a subsequent paper. Future work will also provide algorithms for efficient approximate reconstruction and corresponding error estimates.

\[ ^1 \text{M. Dörfler and E. Matusiak: Nonstationary Gabor Frames - Approximate duals and reconstruction, in preparation.} \]
6. Acknowledgement

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**Appendix A. Pointwise convolution estimate**

**Lemma A.1.** Let $h \in L^\infty(\mathbb{R})$ be a function compactly supported on $[-t_0, t_0]$ and continuous at point $t$. Let $\phi$ be such that $|\phi(t)| \leq C_\phi (1 + |t|)^{-p}$. Then

$$
|\phi * h(t) - h(t)| \leq \|h\|_\infty \frac{C_\phi}{p-1} \begin{cases} 
(1 + t - t_0)^{-p+1} - (1 + t + t_0)^{-p+1} & t > t_0 \\
2 - (1 + t + t_0)^{-p+1} - (1 - t - t_0)^{-p+1} & t \in [-t_0, t_0] \\
(1 - t - t_0)^{-p+1} - (1 - t + t_0)^{-p+1} & t < -t_0.
\end{cases}
$$

**Proof.** The pointwise value of $h$, can be written as

$$h(t) = \lim_{a \to 0} \int_{-t_0}^{t_0} h(s) \eta_a(t - s) \, ds,$$

where $\eta_a(t) = \sqrt{\frac{2}{\pi a}} \frac{a}{a^2 + t^2}$, due to the dominated convergence theorem. We have

$$
|\phi * h(t) - (\eta_a * h)(t)| = \left| \int_{-t_0}^{t_0} h(s)(\phi(t - s) \, ds - \eta_a(t - s)) \, ds \right|
\leq \|h\|_\infty \int_{-t_0}^{t_0} |\phi(t - s) - \eta_a(t - s)| \, ds \
\leq \|h\|_\infty \left( \int_{-t_0}^{t_0} |\phi(t - s)| \, ds + \sqrt{\frac{2}{\pi}} \left[ \arctg\left( \frac{t - t_0}{a} \right) - \arctg\left( -\frac{t - t_0}{a} \right) \right] \right).
$$

With $a \to 0$ the above expression yields

$$
|\phi * h(t) - h(t)| \leq \|h\|_\infty \int_{-t_0}^{t_0} |\phi(t - s)| \, ds
$$

To compute the remaining expression in the brackets, we consider three cases. First, let $t > t_0$, then

$$
\int_{-t_0}^{t_0} |\phi(t - s)| \, ds \leq C_\phi \int_{-t_0}^{t_0} (1 + t - s)^{-p} \, ds
= \frac{C_\phi}{p-1} \left[ (1 + t - t_0)^{-p+1} - (1 + t + t_0)^{-p+1} \right].
$$

Similarly, computing the integrals for $t < -t_0$ and for $t \in [-t_0, t_0]$ we arrive at the desired estimates.  \(\square\)
References


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