Pseudodifferential operators on ultra-modulation spaces

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Abstract

The boundedness of pseudodifferential operators on modulation spaces defined by the means of almost exponential weights is studied. The results are applied to symbol class with almost exponential bounds including polynomial and ultra-polynomial symbols. The Weyl correspondence is used and it is noted that the results can be transferred to the operators with appropriate anti-Wick symbols. It is proved that a class of elliptic pseudodifferential operators can be almost diagonalized by the elements of Wilson bases, and estimates for their eigenvalues are given. Furthermore, it is shown that the same can be done by using Gabor frames.

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1. Introduction

Modulation spaces introduced by Feichtinger [9], have important applications in time frequency analysis (see [10,13,14,16,17,26,31]). They are usually defined by the means of polynomial weights. The use of almost exponential weights in the definition of modulation spaces leads to ultra-modulation spaces and even more to abstract spaces of ultradistributions [26]. We refer to [11] for the characterization of modulation spaces by the means of Gabor expansions, and to [12] for the description of the space of tempered distributions by the means of modulation spaces.

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Pseudodifferential operators (PDO) on modulation spaces defined via polynomial weights are fairly well studied [16,17,19,21,31–34]. In this paper we consider PDOs on ultra-modulation spaces defined via almost exponential (or subexponential) weights and thus defined on a class of ultradifferentiable functions. Consequently the results are formulated in the context of a certain Gevrey–Beurling type spaces; appropriate modifications would lead to the corresponding Gevrey–Roumieu class.

PDOs acting in the Gevrey–Roumieu class of spaces have been analyzed by many authors. For example, ultradifferential operators in [20], analytic PDOs in [3,35,36], and hypoelliptic PDOs in [1,18,23,24,29,39]. In this paper we consider the so-called Weyl correspondence, i.e. PDOs $\sigma(x,D)$ defined as the Weyl transforms [38] of their symbols $\sigma(x,\xi)$ as

$$
\sigma(x,D)f(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sigma\left(\frac{x+y}{2},\xi\right)e^{2\pi i x \cdot (x-y)}f(y)\,dy\,d\xi, \quad f \in \mathcal{G}^{(\gamma)}(\mathbb{R}^d)
$$

(see Section 2 for the definition of $\mathcal{G}^{(\gamma)}(\mathbb{R}^d)$). In order to emphasize this fact we call $\sigma(x,\xi)$ the Weyl symbol of the operator $\sigma(x,D)$. Our class of symbols is accommodated to ultra-modulation spaces and it is wide enough to contain symbols which do not belong to known classes considered in the above-mentioned papers. On the other hand, it contains polynomial symbols, and the symbol of a Schrödinger operator $-\Delta + V$, with an increasing potential $V$, see Section 5. Note, an anti-Wick symbol $\sigma$ determines Weyl symbol $\sigma \ast (2^{2d}e^{-|x|^2})$ [1]. It belongs to our class of symbols even if $\sigma$ is a tempered ultradistribution, see Section 4. Also, a Weyl symbol determines an anti-Wick symbol up to a smoothing operator [1]. This opens up the possibility of analyzing general Weyl symbols through the analysis of our class up to a smoothing operator. Anti-Wick operators in the context of modulation spaces are studied in [2,6].

Our general idea was to use the technique of PDOs for the analysis and synthesis of modulation spaces defined by almost exponential weights, and to study various classes of operators acting on them. This involves the use of elliptic ultradifferential operators and, consequently, more complex technical difficulties in comparison to the ones of [31,32] where, for the similar purpose, elliptic differential operators are used.

The paper is organized as follows. In Section 2 some notions and facts for the later use are listed. Ultra-modulation spaces and their relation to modulation spaces defined via polynomial weights are considered in Section 3. In Section 4 a class of pseudodifferential operators is defined and their boundedness on ultra-modulation spaces is proved. The corresponding class of symbols is compared to some well-known classes of symbols, and it is shown that the Weyl symbol $\sigma \ast (2^{2d}e^{-|x|^2})$, where $\sigma$ is a tempered ultradistribution, belongs to the introduced class of symbols. In Section 5 a class of elliptic pseudodifferential operators is introduced and their approximate diagonalization by the means of Wilson bases is shown. Their boundedness on ultra-modulation spaces is proved in the same section. In Section 6 the results obtained in Section 5 are applied to the spectral asymptotics of elliptic...
pseudodifferential operators. Spectral properties of integral and pseudodifferential operators are studied in [18]. It is known that certain local trigonometric bases and Gabor frames could be used for the analysis of pseudodifferential operators acting on modulation spaces defined by polynomial weights [7,18,28]. In Section 7 it is shown that the results of Sections 5 and 6 can be obtained using Gabor frames instead of Wilson bases.

2. Preliminaries

We write $D^\alpha = D_1^{\alpha_1} \cdots D_d^{\alpha_d} = \left( \frac{1}{2\pi i} \frac{\partial}{\partial x_1} \right)^{\alpha_1} \cdots \left( \frac{1}{2\pi i} \frac{\partial}{\partial x_d} \right)^{\alpha_d}$, where $\alpha = \alpha_1 ! \cdots \alpha_d !$, $\alpha \in \mathbb{N}_0^d$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, $d \in \mathbb{N}$. If $\beta$ is another multi index such that $\beta_j \leq \alpha_j$, $j \in \{1, 2, \ldots, d\}$, then $(\beta) = (\beta_1) \cdots (\beta_d)$. The letter $C$ denotes a positive constant, not necessarily the same at every occurrence. The symbol $\gamma$ is reserved for a real number in $(0, 1)$, unless otherwise indicated. The translation and modulation operators on a space of test functions are given by $T_x f(\cdot) = f(\cdot - x)$, $x \in \mathbb{R}^d$, and $M_x \psi(\cdot) = e^{2\pi i x \cdot \cdot} \psi(\cdot)$, $\xi \in \mathbb{R}^d$, respectively. These operators are extended to the dual space via duality. The dual pairing is denoted by $\langle \cdot, \cdot \rangle$. For functions $\varphi, \psi \in \mathcal{S}$ ($\mathcal{S}$ is the space of rapidly decreasing functions), $\langle \varphi, \psi \rangle = \int \varphi \bar{\psi} \, dx$. As usual, $\hat{\psi}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \cdot} \psi(x) \, dx$ is the Fourier transform of $\psi \in L^2(\mathbb{R}^d)$. We denote the norm in $L^2$ by $\| \cdot \|$, and $\| \cdot \|_\infty$ denotes the $L^\infty$ norm. Recall [25], the space $\mathcal{S}(\gamma)$, is defined by $\mathcal{S}(\gamma) = \text{proj lim}_{h \to 0} \mathcal{S}(\gamma)$, where $\mathcal{S}(\gamma)$, $h \geq 0$, is the space of smooth functions $f$ on $\mathbb{R}^d$ such that

$$\sup_{x, \beta \in \mathbb{N}_0^d} h^{|\beta|/\beta_1!} \| x^\beta D^\beta f(x) \|_\infty < \infty. \quad (2)$$

It is a Banach space of Gelfand–Shilov type [15] and the Fourier transform is an isomorphism of $\mathcal{S}(\gamma)$ onto itself. This fact is essential for applications in time frequency analysis. The inclusion $\mathcal{S}(\gamma) \hookrightarrow \mathcal{S}$ is dense and continuous and the dual $\mathcal{S}'(\gamma)$ is called the space of Gevrey–Beurling tempered ultradistributions. Recall, $M(\rho) = \sup \ln \frac{p}{\rho}$, $\rho > 0$, is the associated function to the sequence $(\rho^{1/\gamma})_{\rho \in \mathbb{N}_0}$ and $M(\rho) \sim \rho^\gamma$, $\rho \to \infty$. We will use the following estimates [20]:

$$\left( \exists C > 0 \right) \left( e^{2M(L)} \leq |p L(\zeta)| \leq C e^{\frac{4}{1-\gamma^2}M(L/\zeta)} , \quad \zeta \in C \right), \quad (3)$$

$$\left( \exists C_0 > 0 \right) \left( |a_p| \leq C_1 \left( \frac{4}{1 - \gamma^2} \right)^{p/\gamma} \frac{L_p}{p^{1/\gamma}} \right), \quad p \in \mathbb{N}_0 \quad (4)$$

where $L > 0$ and $P_L(\zeta) = \prod_{p=1}^{\infty} \left( 1 + \frac{L \zeta}{p^{1/\gamma}} \right) = \sum_{p=0}^{\infty} a_p \zeta^p$, $\zeta \in C$, $\Re \zeta \geq 0$.
We recall that a Wilson basis $\psi_{k,n}$, $k \in \mathbb{N}$, $n \in \mathbb{Z}$, of exponential decay is generated by a suitable real-valued and even function $\varphi \in \mathcal{S}(\mathbb{R})$:

$$\psi_{1,n}(\xi) = e^{-2\pi i n t} \varphi(-\xi), \quad n \in \mathbb{Z},$$

$$\psi_{2l+n,n}(\xi) = \frac{e^{-2\pi i (n+\xi/2)}}{\sqrt{2}} \left( \varphi(-\xi - l) + (-1)^{l+n} \varphi(-\xi + l) \right), \quad l \in \mathbb{N}, \quad \kappa \in \{0,1\}$$

(5)

As in [31], the $d$-dimensional Wilson basis is given by the tensor product

$$\psi_{k,n}(x) = \psi_{k_1,n_1}(x_1) \otimes \psi_{k_2,n_2}(x_2) \otimes \cdots \otimes \psi_{k_d,n_d}(x_d),$$

$x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, $k = (k_1, k_2, \ldots, k_d) \in \mathbb{N}^d$, $n = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$. Functions $\psi_{k,n}$, $k \in \mathbb{N}^d$, $n \in \mathbb{Z}^d$, are real valued and satisfy

$$|\psi_{k,n}(x)| \leq Ce^{-a|\xi|}, \quad |\psi_{k,n}(\xi)| \leq Ce^{-b|\xi|}, \quad x, \xi \in \mathbb{R}^d, \quad k \in \mathbb{N}^d, \quad n \in \mathbb{Z}^d,$$

for some constants $a, b, C > 0$, depending on $k$ and $n$.

For $f, g \in L^2$ the (cross) Wigner transform $W(f, g)$ is given by

$$W(f, g)(x, \xi) = \int_{\mathbb{R}^d} e^{-2\pi i p \cdot x} f(x + p/2) \overline{g(x - p/2)} \, dp, \quad x, \xi \in \mathbb{R}^d.$$

The connection to the Weyl symbol $\sigma(x, \xi)$ of a $\Psi$DO $\sigma(x, D)$ is given by

$$\langle \hat{g}, \sigma(x, D)f \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sigma(x, \xi) W(f, g)(x, \xi) \, dx \, d\xi$$

(see Section 4 for more on this). For the Wigner transform we have the relation

$$W(f, g)(x, \xi) = W(f, \hat{g})(\xi, -x).$$

(6)

The Wigner transform maps $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^{2d})$. Consequently, for every $\varphi \in \mathcal{S}_k(\mathbb{R}^d)$ and $h \geq 0$, we have

$$\sup_{x, \xi \in \mathbb{R}^d} \sup_{\alpha, \beta \in \mathbb{N}^d} \frac{h^{a+b}}{|\alpha|^{1/2} |\beta|^{1/2}} |D^\alpha_x D^\beta_\xi W(\varphi, \varphi)(x, \xi) e^{h(x^2 + |\xi|^2)}| < \infty.$$

(7)

3. Modulation and ultra-modulation spaces

Modulation spaces consist of functions or distributions whose Short-time Fourier transform satisfies some prescribed decay at infinity as well as some integrability conditions. The decay is controlled by a weight function, i.e. a nonnegative locally
integrable function on $\mathbb{R}^{2d}$. Recall, a weight $v$ is submultiplicative if $v(z_1 + z_2) \leq v(z_1)v(z_2)$, $z_1, z_2 \in \mathbb{R}^{2d}$, and a weight $m$ is moderate with respect to submultiplicative weight $v$ if

$$m(x + y, \xi + \eta) \leq Cv(x, \xi)m(y, \eta), \quad x, y, \xi, \eta \in \mathbb{R}^{2d}.$$ 

Weights $m_1$ and $m_2$ are equivalent if $C_1m_1 \leq m_2 \leq C_2m_1$ for some positive constants $C_1$ and $C_2$. Every submultiplicative weight is equivalent to a continuous weight. We start with a special class of moderate weights.

**Definition 1.** Let $\gamma \in [0, 1)$. A strictly positive and continuous function $w_\gamma$ on $\mathbb{R}^d \times \mathbb{R}^d$ is called an exp-type weight if there exist $s \geq 0$ and $C > 0$ such that

$$w_\gamma(x + y, \xi + \eta) \leq Ce^{s(|x| + |\xi|)}w_\gamma(y, \eta), \quad x, y, \xi, \eta \in \mathbb{R}^d$$

and

$$w_\gamma(x, e_1 \cdot \xi_1, \ldots, e_d \cdot \xi_d) = w_\gamma(x, \xi), \quad e = (e_1, \ldots, e_d) \in \{-1, 1\}^d.$$ 

By the above definition, a weight $w_\gamma$ is an exp-type weight if it is moderate with respect to $e^{s(|x| + |\xi|)}$ for some $s \geq 0$ and fixed $\gamma \in [0, 1)$. A typical example of an exp-type weight is

$$w_\gamma(x, \xi) = e^{a_1|x|^2 + a_2|\xi|^2}, \quad x, \xi \in \mathbb{R}^d,$$

By the above definition, a weight $w_\gamma$ is an exp-type weight if it is moderate with respect to $e^{a_1|x|^2 + a_2|\xi|^2}$ for some $s \geq 0$ and fixed $\gamma \in [0, 1)$. A typical example of an exp-type weight is

$$(8) w_\gamma(x, \xi) = e^{a_1|x|^2 + a_2|\xi|^2}, \quad x, \xi \in \mathbb{R}^d, \quad s_1, s_2 \geq 0.$$ 

More generally, $(1 + |x| + |\xi|)^\gamma e^{a_1|x|^2 + a_2|\xi|^2}$ is an exp-type weight for any $a, b, c \geq 0$ and $\gamma_1, \gamma_2 \in [0, 1)$. If $s = 0$ then $w_\gamma(x, \xi)$ is bounded, and if $\gamma = 0$ then $w_\gamma(x, \xi)$ is a constant. The definition also implies

$$\frac{1}{C}w_\gamma(0, 0)e^{-s(|x|^2 + |\xi|^2)} \leq w_\gamma(-x, -\xi), \quad x, \xi \in \mathbb{R}^d.$$ 

Note also that an exp-type weight $w_\gamma$ satisfies Beurling–Domar’s non-quasianalicity condition $\sum_{n=1}^{\infty} n^{-2} \log w_\gamma(nx, n\xi) < \infty, x, \xi \in \mathbb{R}^d$.

**Definition 2.** Let there be given $\gamma \in [0, 1), 0 \not= g \in \mathcal{G}(\gamma)$ (if $\gamma = 0$, then $0 \not= g \in \mathcal{G}$), an exp-type weight $w_\gamma$, $t \in \mathbb{R}$, and $1 \leq p, q < \infty$. Then

$$M^{w_\gamma}_{p,q} = \{f \in \mathcal{F}(\gamma) : \|f\|_{M^{w_\gamma}_{p,q}} < \infty\},$$

$$\|f\|_{M^{w_\gamma}_{p,q}} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |\mathcal{T}_f^\gamma M_t^g(f)|^{p/\gamma} d\xi \right)^{\gamma/p} d\xi \right)^{1/q}$$

is called the ultra-modulation space.
$M_{p,q}^{w}$ is a Banach space [16, Theorem 11.3.5]. If $t = 0$, we write $M_{p,q}^{w} = M_{p,q}^{w,t}$, for short. The above definition is independent of the choice of $g$, $0 \neq g \in \mathcal{S}'^{(0)}$, in the sense that different functions define the same ultra-modulation space and equivalent norms [9].

Let there be given $\lambda, \tau \in \mathbb{R}$, an exp-type weight $w_{\gamma}(x, \xi)$, and let

$$\tilde{w}_{\gamma}(x, \xi) = w_{\gamma}(x, \xi)e^{-\lambda|x|^2 - \tau|\xi|^2}, \quad x, \xi \in \mathbb{R}^d. \quad (10)$$

The function $\tilde{w}_{\gamma}$ is also exp-type weight, and for $\lambda, \tau \geq 0$ and $\gamma \in [0, 1)$ we have $M_{p,q}^{w} \subseteq M_{p,q}^{w,e}$, which we use in the proof of Corollary 1. This fact is a consequence of a more general property of moderate weights, see [16, Lemma 11.1.1].

Modulation spaces $M_{p,q}^{w}$, where $1 \leq p, q < \infty$, and $w$ being an $s$-moderate weight, $s \geq 0$, i.e. $w(x + y, \xi + \eta) \leq C(1 + |x| + |\xi|)^s(w(x, \eta)_s + w(y, \eta))$, for some $C > 0$ and all $x, y, \xi, \eta \in \mathbb{R}^d$, are studied in [14,16,31]. Obviously, every $s$-moderate weight defined in such a way is also an exp-type weight. Particularly, for $w(x, \xi) = (1 + |x| + |\xi|)^s$, $x, \xi \in \mathbb{R}^d$, $s \geq 0$ and $p = q = 2$, properties of pseudodifferential operators whose Weyl symbols belong $M_{p,q}^{w}$ are given in [17], see also [19,33,34] for some generalizations. The composition of pseudodifferential operators with the symbols in $M_{p,q}^{w}$ are studied in [21], where $w(x, \xi) = (1 + |x|^2 + |\xi|^2)^{s/2}$, $x, \xi \in \mathbb{R}^d$, $s \geq 0$. Note that $M_{p,q}^{w} = M_{p,q}^{w,2}$, since $(1 + |x| + |\xi|)^s$ and $(1 + |x|^2 + |\xi|^2)^{s/2}$ are equivalent weights. Moreover, they are equivalent to the weight $(1 + |x|^2 + |\xi|^2)^{s/2}$ invariant under rotations. To emphasize the difference between modulation spaces defined by polynomial weights and ultra-modulation spaces defined by exp-type weights, note that $e^{e^{(|x|^2 + |\xi|^2)^{s/2}}}$ is not an exp-type weight, although $e^{e^{(|x|^2 + |\xi|^2)^{s/2}}}$ is. For modulation spaces defined by the weight $e^{e^{(|x|^2 + |\xi|^2)^{s/2}}}$ we refer the reader to [16].

Note $\mathcal{S} = \text{proj lim}_{\gamma \to \infty} M_{p,q}^{w,2} [11,12]$. However, we are able to study the spaces of ultradistributions by the means of ultra-modulation spaces, since $\mathcal{S}'^{(0)} = \text{proj lim}_{\gamma \to \infty} M_{p,q}^{w}$, where $w_{\gamma}(x, \xi) = e^{e^{(|x|^2 + |\xi|^2)^{s/2}}}$, $\gamma \in (0, 1)$, see [26].

The following theorem is proved in [26]. We refer to [10, pp. 367; 31, pp. 266] for modulation spaces defined via polynomial weights.

**Theorem 1.** Let $M_{p,q}^{w}$ be an ultra-modulation space defined by (9), and \{\psi_{k,n}, k \in \mathbb{N}^d, n \in \mathbb{Z}^d\} any Wilson basis of exponential decay. Then

(a) The Wilson basis is an unconditional basis for $M_{p,q}^{w}$.

(b) Every element $f \in M_{p,q}^{w}$ has a unique expansion

$$f = \sum_{k \in \mathbb{N}^d, n \in \mathbb{Z}^d} c_{k,n} \psi_{k,n} \quad \text{where} \quad c_{k,n} = \langle \psi_{k,n}, f \rangle, \quad k \in \mathbb{N}^d, n \in \mathbb{Z}^d.$$
There exist positive constants $C_1, C_2$ such that

$$C_1 \| f \|_{M^{p,q}_{\alpha,\beta}} \leq \left[ \sum_{k \in \mathbb{N}^d} \left( \sum_{n \in \mathbb{Z}^d} |c_{k,n}|^p \left( \frac{n!}{2^k} k! \right)^{q/p} \right)^{1/p} \right]^{1/q} \leq C_2 \| f \|_{M^{p,q}_{\alpha,\beta}} \quad f \in M^{p,q}_{\alpha,\beta}.$$

4. A class of pseudodifferential operators

As indicated in the introduction, $\gamma$ denotes a real number in $(0, 1)$. Let there be given $L_1, L_2 \geq 0$, and $\lambda, \tau \in \mathbb{R}$. We define the symbol class $S^{\lambda,\tau}_{L_1, L_2}(\mathbb{R}^{2d}) = S^{\lambda,\tau}_{L_1, L_2}$ as a set of $\sigma \in C^{\infty}(\mathbb{R}^{2d})$ satisfying

$$\left| \frac{L_1|\alpha|}{x^{1/2}} \frac{L_2|\beta|}{\beta^{1/2}} \partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi) \right| \leq C e^{d|x|^\gamma + |\xi|^\gamma}, \quad \alpha, \beta \in \mathbb{N}_0^d, \quad x, \xi \in \mathbb{R}^d$$

for some positive constant $C = C_3$ depending on $L_1, L_2, \lambda, \tau$, and $\gamma$. The infimum of such constants $C_3$ will be denoted by $\|\sigma\|_{S^{\lambda,\tau}_{L_1, L_2}}$.

We consider the Weyl correspondence (1) between a pseudodifferential operator $\sigma(x, D)$ and a symbol $\sigma(x, \xi) \in S^{\lambda,\tau}_{L_1, L_2}$ and call $\sigma(x, \xi)$ the Weyl symbol of $\sigma(x, D)$.

Let us compare the class $S^{\lambda,\tau}_{L_1, L_2}$ to the class $S^{4\lambda,\tau}_{L_1, L_2}$ containing polynomial symbols (such as the symbol of the Schrödinger operator for the harmonic oscillator $[31]$). $S^{4\lambda,\tau}_{L_1, L_2}$ contains polynomial symbols as well as ultrapolynomial symbols (for $\lambda, \tau \geq 0$) which do not belong to $S^{\lambda,\tau}$. Proof of the following proposition is given in [27].

**Proposition 1.** (a) Let there be given a sequence $\{a_n\}_{n \in \mathbb{N}_0}$ such that $|a_n| \leq C_n^{\lambda,\tau}$, for some positive constants $C$ and $k$, and all $n \in \mathbb{N}_0$. Then $\sigma(x, \xi) = \sum_{n=0}^{\infty} a_n (1 + |x|^2 +$
\(|\xi|^{2}\gamma/2, x, \xi \in \mathbb{R}^{d}\), belongs to \(S^{1,\gamma}_{L_1, L_2}\) for all \(L_1, L_2 > 0\) and \(\lambda, \tau \geq (k^2/2\gamma^2)(1 + d(L_1^2 + L_2^2))^{\gamma/2}/\gamma\).

(b) Let there be given a sequence of smooth functions on \(\mathbb{R}^{d}\), \(\{a_n(x)\}_{n \in \mathbb{N}_0}\), such that

\[
|D^\alpha a_n(x)| \leq C \frac{1}{L_1^{|\alpha|}} \frac{k^{|\alpha|}}{n^{1/\gamma}}, \quad x \in \mathbb{R}^{d}, \quad \alpha, n \in \mathbb{N}_0,
\]

for some positive constants \(k, L_1\) and \(C\).

A class of symbols \(S^{\infty, \gamma}_{\Omega}(\mathbb{R}^{d})\), here denoted by \(\Sigma^{\infty, \gamma}_{\Omega}(\mathbb{R}^{d})\), and the corresponding operators acting on the space of ultradistributions of Roumieu type has been studied in [39]. We compare our symbol class to a subclass of \(\Sigma^{\infty, \gamma}_{\Omega}(\mathbb{R}^{d})\) defined by a global condition with respect to the variable \(x\). Let there be given \(\theta, \varrho\) and \(\delta\) such that \(\theta > 1, 0 < \delta < \varrho < 1, \theta \varrho > 1\). Following [39], \(\Sigma^{\infty, \gamma}_{\Omega}(\mathbb{R}^{2d})\) consists of all functions \(\sigma \in C^{\infty}(\mathbb{R}^{2d})\) satisfying the condition: there exist constants \(C > 0\) and \(B > 0\) such that for every \(\epsilon > 0\) there is a constant \(c_\epsilon > 0\) such that

\[
\sup_{x \in \mathbb{R}^{d}} |D^\alpha D^\beta \sigma(x, \xi)| \leq c_\epsilon C^{\gamma/2} \beta^{|\beta|} (1 + |\xi|)^{\gamma/2} e^{\varrho |\xi|/\gamma} \quad (11)
\]

for every \(\sigma, \beta \in \mathbb{N}_0^d\) and every \(|\xi| \geq B\beta^\varrho\).

Let \(\sigma(x, \xi) \in \Sigma^{\infty, \gamma}_{\Omega}(\mathbb{R}^{2d})\). Since \(\alpha^{|\alpha|} (1 + |\xi|)^{\gamma/2} e^{\varrho |\xi|/\gamma} \leq \beta^{|\beta|} (1 + |\xi|)^{\gamma/2} e^{\varrho |\xi|/\gamma},\) putting \(L_1 = L_2 = C^{-1}\), with the constant \(C\) given in (11) and \(\gamma = \gamma^{-1}\), we have

\[
\sup_{x \in \mathbb{R}^{d}} |D^\alpha D^\beta \sigma(x, \xi)| \leq c_\epsilon L_1^{-|\alpha|} L_2^{-|\beta|} (1 + |\xi|)^{\gamma/2} e^{\varrho |\xi|/\gamma} \leq c_\epsilon L_1^{-|\alpha|} L_2^{-|\beta|} \alpha^{|\alpha|} \beta^{|\beta|} (1 + |\xi|)^{\gamma/2} e^{\varrho |\xi|/\gamma},
\]

for any \(\tau > \varrho^{|\alpha|/\gamma}\), sufficiently large \(|\xi|\) and fixed \(\epsilon > 0\). Therefore, for \(q = 1, \delta = 0\) and \(\theta = \gamma^{-1}\),

\[
\Sigma^{\infty, 1/\gamma}_{\Omega}(\mathbb{R}^{2d}) = \bigcap_{\tau > 0} S^{\infty, 1/\gamma}_{L_1, L_2}(\mathbb{R}^{2d}),
\]

where \(\lambda \geq 0\), and \(L_1, L_2\) as above.

Remark 1. Our global condition on \(x\) variable can be transferred in an adequate way to symbols with the supremum taken over compact subsets of \(\Omega \subset \mathbb{R}^{d}\) (instead of the supremum taken over the whole \(\mathbb{R}^{d}\)). Note that in (11), the right-hand side of the
inequality does not depend on $x$, while in our case, where both $x$ and $\xi$ are treated symmetrically, it is bounded by a constant times $e^{\lambda|\xi|^s}$.

An interesting subclass of $S^{\lambda,\tau}_{L_1, L_2}(\mathbb{R}^{2d})$, $L_1, L_2, \lambda, \tau > 0$ is considered in [4] in the study of the solvability of a Cauchy problem. For given real numbers $\mu, \nu > 1$, and $\theta = \max\{\mu, \nu\}$, the class $I_{\mu, \nu, \theta}^{\infty}$ is defined by

$$I_{\mu, \nu, \theta}^{\infty}(\mathbb{R}^{2d}) = \text{proj} \lim_{\xi \to \infty} I_{\mu, \nu, \theta}^{\infty}(\mathbb{R}^{2d}, C)$$

where $p(x, \xi) \in C^\infty(\mathbb{R}^{2d})$ belongs to $I_{\mu, \nu, \theta}^{\infty}(\mathbb{R}^{2d}, C)$ if and only if

$$\sup_{\alpha, \beta \in \mathbb{N}_0^d} \sup_{x, \xi \in \mathbb{R}^{2d}} \frac{C_{|\alpha|+|\beta|}}{2^{\beta_1+\beta_2}} \langle \xi \rangle^{|\alpha|} \langle x \rangle^{|\beta|} e^{-\varepsilon|\xi|^s + |\xi|^s} |D_x^\alpha D_\xi^\beta p(x, \xi)| < \infty,$$

for every $\varepsilon > 0$. Obviously,

$$I_{\mu, \nu, \theta}^{\infty}(\mathbb{R}^{2d}) \subset I_{\lambda, \tau, \theta}^{\infty}(\mathbb{R}^{2d}) \subset S^{\lambda, \tau}_{L_1, L_2}(\mathbb{R}^{2d}).$$

Let us now show that for an anti-Wick symbol $\sigma \in \mathcal{S}(\mathbb{R}^{2d})$ the corresponding Weyl symbol $\sigma = (2^{2d}e^{-|\xi|^s})$ belongs to the introduced class of symbols. It is known [25] that $\sigma$ is of the form

$$\sigma(x, \xi) = \sum_{m,n \in \mathbb{N}_0^d} a_{m,n} D_x^m D_\xi^n F(x, \xi),$$

$$|a_{m,n}| \leq C \frac{h^{m|\alpha|+|\beta|}}{m!^{1/2\nu} n!^{1/2}}, \quad m, n \in \mathbb{N}_0^d$$

where $C, h > 0$ and $F$ is a continuous function on $\mathbb{R}^{2d}$ such that

$$|F(x, \xi)| \leq Ce^{C(x^s + |\xi|^s)}, \quad (x, \xi) \in \mathbb{R}^{2d},$$

for some $C, s > 0$.

**Proposition 2.** Let $\sigma(x, \xi) \in \mathcal{S}(\mathbb{R}^{2d})$. Then $(2^{2d}e^{-|\xi|^s}) \in S^{\lambda, \tau}_{L_1, L_2}$ for any $\lambda, \tau > s$, where $s$ is given in (13) and $0 \leq L_1, L_2 \leq 2^{-1/2}$.

**Proof.** We will use the estimate

$$|D_x^\alpha D_\xi^\beta| \leq Ce^{-\frac{1}{2}q^{1/2}} q^{1/2}, \quad q \in \mathbb{N}_0^{2d}$$

[22].
Let \((x, \xi) \in \mathbb{R}^{2d}\), \(\alpha, \beta, m, n \in \mathbb{N}_0^d\). The above estimate, (12) and (13) imply

\[ |D_x^\alpha D_\xi^\beta (2^{2d(\sigma - e^{-1/\gamma}})\langle x, \xi \rangle | \]

\[ \leq 2^{2d} \int_{\mathbb{R}^{2d}} \sum_{m, n \in \mathbb{N}_0^d} a_{m,n} D_x^\alpha F(y, \eta) D_\xi^\beta e^{-(\xi - \eta)^T (\xi - \eta)} dy d\eta \]

\[ \leq 2^{2d} \sum_{m, n \in \mathbb{N}_0^d} |a_{m,n}| \int_{\mathbb{R}^{2d}} |F(y, \eta)||D_x^\alpha D_\xi^\beta e^{-(\xi - \eta)^T (\xi - \eta)}| dy d\eta \]

\[ \leq C \sum_{m, n \in \mathbb{N}_0^d} |a_{m,n}| \int_{\mathbb{R}^{2d}} |F(y, \eta)| e^{-\frac{(\xi - \eta)^T (\xi - \eta)}{2}} \left( \alpha! \beta! \max(2^{|m| + |n|}, |\xi| + |\eta|) \right)^{1/2} dy d\eta \]

\[ \leq C(\alpha! \beta! \max(2^{|m| + |n|}, |\xi| + |\eta|))^{1/2} \sum_{m, n \in \mathbb{N}_0^d} \left( \frac{\alpha! \beta!}{\max(2^{|m| + |n|}, |\xi| + |\eta|)} \right)^{1/2} \int_{\mathbb{R}^{2d}} |F(x - y, \xi - \eta)| e^{-\frac{(\xi - \eta)^T (\xi - \eta)}{2}} dy d\eta \]

\[ \leq C(\alpha! \beta! \max(2^{|m| + |n|}, |\xi| + |\eta|))^{1/2} \int_{\mathbb{R}^{2d}} e^{-\frac{(\xi - \eta)^T (\xi - \eta)}{2} + \lambda |\xi|^2 - \frac{|\xi|^2}{2}} dy d\eta \]

\[ \leq C(\alpha! \beta! \max(2^{|m| + |n|}, |\xi| + |\eta|))^{1/2} e^{-\frac{|\xi|^2 + |\eta|^2}{2}}. \]

Therefore, since \(1/\gamma > 1/2\),

\[ \frac{2^{-\frac{1}{2}}(\alpha! + \beta!)}{\alpha! \beta!} \left| D_x^\alpha D_\xi^\beta (2^{2d(\sigma - e^{-1/\gamma}})\langle x, \xi \rangle \right| \leq C e^{\frac{2^{|m| + |n|}}{2 - \gamma}}. \]

it follows \(2^{2d(\sigma - e^{-1/\gamma}}) \in S_{L_1, L_2, \lambda}^{\alpha, \beta} \) for every \(\lambda, \tau \geq s\) and \(0 \leq L_1, L_2 \leq 2^{1/2}. \)

4.1. Main result

We first give Lemma 1 which will be used in the proof of Theorem 2.

**Lemma 1.** Let there be given \(L_1, L_2 \geq 0\), a Wilson basis of exponential decay \(\{\psi_{k,n}\}_{k \in \mathbb{N}_0^d, n \in \mathbb{Z}^d}\), and let

\[ A > 2^{\gamma L_1}, B > 2^{\gamma L_2}, \text{ where } \gamma = \left( \frac{4}{1 - \gamma^2} \right)^{1/2}. \]
Let \( \sigma(x, \xi) \in S_{A,B}^{\alpha, \gamma} \) and \( L_1 \geq \frac{1}{2 \gamma + \tau} \) if \( \tau < 0 \). There exists a positive constant \( C \) such that

\[
\left| \langle \psi_{\kappa', \eta}, \sigma(x, D)\psi_{\kappa, \eta} \rangle \right| \leq \frac{C \| \sigma \|_{S_{A,B}^{\alpha, \gamma}} e^{\frac{\| \kappa' \|_{\infty} + \| \eta' \|_{\infty} + \frac{1}{2} \kappa \kappa'}{\epsilon^2 (M(L_1|l| + \gamma) + M(L_2 + \gamma - \kappa'))}}}{e^{\frac{\| \kappa \|_{\infty} + \| \eta \|_{\infty} + \frac{1}{2} \kappa \kappa'}}}
\]  

(14)

where \( k = 2 \kappa + \kappa', \ k' = 2 \kappa' + \kappa', \ l, l' \in \mathbb{N}^d, \ k, k' \in \{0, 1\}^d \), \( l_i = k_i = 0 \) for \( k_i = 1 \), \( i \in \{1, \ldots, d\} \), \( l'_j = k'_j = 0 \) for \( k'_j = 1 \), \( j \in \{1, \ldots, d\} \) (In (14), \( M(\rho) \) is the associated function for the sequence \( (p^{1/V})_{p \in \mathbb{N}_0} \)).

The proof will be given in the separate subsection.

Remark 2. Under the same assumptions as in Lemma 1, using the inequality \( |a + b|^\gamma \geq |a|^\gamma - |b|^\gamma \), we obtain

\[
\left| \langle \psi_{\kappa', \eta}, \sigma(x, D)\psi_{\kappa, \eta} \rangle \right| \leq \frac{C \| \sigma \|_{S_{A,B}^{\alpha, \gamma}} e^{\frac{\| \kappa' \|_{\infty} + \| \eta' \|_{\infty} + \frac{1}{2} \kappa \kappa'}}}{e^{\frac{\| \kappa \|_{\infty} + \| \eta \|_{\infty} + \frac{1}{2} \kappa \kappa'}}}
\]  

(15)

Theorem 2. Let there be given an exp-type weight \( \omega_{r, \gamma} \), a Weyl symbol \( \sigma(x, \xi) \in S_{A,B}^{\alpha, \gamma} \), \( \lambda, \tau \in \mathbb{R} \), with \( A, B \) as in Lemma 1,

\[
L_1 > \left( 2(\kappa + |\tau|) + \frac{\tau}{2}\right) \gamma, \quad L_2 > \left( \lambda + \frac{\kappa + |\tau|}{2}\right) \gamma,
\]

and \( L_1 \geq \max \left\{ \frac{1}{2 \gamma + \tau}, \left( 2(\kappa + |\tau|) + \frac{\tau}{2}\right)^{\gamma} \right\} \) if \( \tau < 0 \).

The corresponding operator \( \sigma(x, D) : M_{p,q}^{\omega_{r, \gamma}} \to M_{p,q}^{\omega_{r, \gamma}} \) is a bounded linear operator, where \( 1 \leq p, q < \infty \) and \( \omega_{r, \gamma}(x, \xi) = \omega_{r, \gamma}(x, \xi)e^{r \xi^2 + i\gamma \xi^2} \), i.e. there exists \( C > 0 \) such that

\[
\| \sigma(x, D) f \|_{M_{p,q}^{\omega_{r, \gamma}}} \leq C \| \sigma \|_{S_{A,B}^{\alpha, \gamma}} \| f \|_{M_{p,q}^{\omega_{r, \gamma}}}, \quad f \in M_{p,q}^{\omega_{r, \gamma}}
\]

Corollary 1. Observe a pseudodifferential equation \( \sigma(x, D)u = f \), where the Weyl symbol of \( \sigma(x, D) \) belongs to the class \( S_{A,B}^{\alpha, \gamma} \), with \( A, B \) as in Lemma 1. If \( u \in \mathcal{S}^{(\gamma)} \), then \( f \in \mathcal{S}^{(\gamma)} \) as well. Moreover, the mapping \( \sigma(x, D) : \mathcal{S}^{(\gamma)} \to \mathcal{S}^{(\gamma)} \) is continuous.

Proof of Corollary 1. Since \( \mathcal{S}^{(\gamma)} = \text{proj} \lim_{\gamma \to \infty} M_{2,2}^{\omega_{r, \gamma}} \) (see also 2). It suffices to prove that \( f \in M_{2,2}^{\omega_{r, \gamma}} \), where \( \psi_{r, \gamma}(x, \xi) = e^{r \xi^2 + i\gamma \xi^2} \), for every \( \rho > 0 \). Put \( \psi_{r, \gamma}(x, \xi) = e^{2r \xi^2 + i\gamma \xi^2} \psi_{r, \gamma}(x, \xi) \) and

\[
w_{r, \gamma} = \psi_{r, \gamma}(x, \xi)e^{2r \xi^2 + i\gamma \xi^2} = \psi_{r, \gamma}(x, \xi)e^{2r \xi^2 + i\gamma \xi^2}, \quad \lambda, \tau \geq 0, x, \xi \in \mathbb{R}^d.
\]
We have $\|u\|_{L_p^\varepsilon} \leq \|u\|_{L_q^\varepsilon}$, $1 \leq p, q < \infty$. Now, since $u \in \mathcal{S}^{(q)}$, Theorem 2 implies

\[ \|f\|_{L_2^\varepsilon} \leq C\|\sigma\|_{A,B} \|u\|_{L_2^\varepsilon} \leq C\|\sigma\|_{A,B} \|u\|_{L_2^\varepsilon} < \infty. \]

Another consequence is the boundedness of the operator $\sum_{|a| \leq n} a_a D^a$, on $\mathcal{S}^{(q)}$. It follows from the fact that its Weyl symbol belongs to the class $S^{A,B}_{A,B}$ for any choice of $A, B > 0$.

We now prove Theorem 2.

**Proof.** We prove the special case $p = q = 2$. This is only a technical restriction. By Theorem 1(d) it is enough to show

\[
\sum_{k' \in \mathbb{N}^d} \sum_{n' \in \mathbb{Z}^d} \left| \langle \psi_{k',n'}, \sigma(x, D)f \rangle \right|^2 \nu_{n'}^2(n'/2, k') \leq C(||\sigma||_{A,B}^2) \sum_{k \in \mathbb{N}^d} \sum_{n \in \mathbb{Z}^d} \left| \langle \psi_{k,n}, f \rangle \right|^2 \nu_{n}^2(n/2, k).
\]

(16)

Put $c_{k,n} = \langle \psi_{k,n}, f \rangle$, $k = 2l + \kappa$, $l \in \mathbb{N}^d$, $\kappa \in \{0, 1\}^d$, $(l_i = \kappa_i = 0$ if $k_i = 1$, $i \in \{1, \ldots, d\})$. By Theorem 1(d), (15) and Hölder's inequality we have

\[
\left| \langle \psi_{k',n'}, \sigma(x, D)f \rangle \right|^2 = \left| \sum_{k \in \mathbb{N}^d} \sum_{n \in \mathbb{Z}^d} \langle \psi_{k',n'}, \sigma(x, D)\psi_{k,n} \rangle c_{k,n} \right|^2 \leq C(||\sigma||_{A,B}^2) \sum_{k \in \mathbb{N}^d} \sum_{n \in \mathbb{Z}^d} \frac{e^{2(M(2|\kappa| + |l|/2) + |l|/2)}}{\nu_{n}^2(n/2, k)} |c_{k,n}|^2.
\]

From $\lambda \left( \left| \frac{n}{2} \right|^2 - \left| \frac{k}{2} \right|^2 \right) \leq \left| \frac{n}{2} \right|^2$, the definition of $\tilde{w}_y$ and the fact that it is an exp-type weight, we have

\[
\tilde{w}_y(n'/2, k') \leq w_y(n/2, k) \frac{e^{2(s+2|l|)(|d-n|/2)^s + (s+|l|)(|k'-k|^{s-2})}}{e^{2\lambda(|d|/2 + |k|^{s-2})}}.
\]

Therefore,

\[
\sum_{n' \in \mathbb{Z}^d} \left| \langle \psi_{k',n'}, \sigma(x, D)f \rangle \right|^2 \nu_{n'}^2(n'/2, k') \leq C(||\sigma||_{A,B}^2) \sum_{n' \in \mathbb{Z}^d} \sum_{k \in \mathbb{N}^d} \sum_{n \in \mathbb{Z}^d} \frac{e^{2(s+2|l|)(|d-n|/2)^s + (s+|l|)(|k'-k|^{s-2})}}{e^{2\lambda(|d|/2 + |k|^{s-2})}}.
\]
The series
\[
\sum_{n' \in \mathbb{Z}_d} \frac{e^{2\lambda (\frac{L^2}{2}|n'|^2 - \frac{1}{2}|n|)}}{e^{2\pi n'|\frac{L^2}{2}|(L_2 - (s+\lambda)|n'||)}} \cdot \sqrt{n,n'}
\]

\begin{equation}
\tag{17}
\end{equation}

can be estimated by a constant which does not depend on \( n \) for
\[
L_2 > \left( |\lambda| + \frac{s + \lambda}{2\pi} \right)^{1/2}
\]
since, putting \( n'' = n - n' \), we conclude that (17) is less than or equal to
\[
\sum_{n'' \in \mathbb{Z}_d} \frac{e^{2\lambda (\frac{L^2}{2}|n''|^2 - \frac{1}{2}|n''|)}}{e^{2\pi n''|\frac{L^2}{2}|(L_2 - (s+\lambda)|n''||)}}
\]

Since \( M(L|n - n'|) \sim (L_2|n - n'|)^{1/2}, |n - n'| \to \infty \), we have
\[
\sum_{k' \in \mathbb{N}_0} \sum_{n' \in \mathbb{Z}_d} |\langle \psi_{k',n'}, \sigma(x,D)f \rangle|^2 \sqrt{n''/2}, k')
\]
\[
\leq C(||\sigma||_{A,B}^2) \sum_{k' \in \mathbb{N}_0} \sum_{n' \in \mathbb{Z}_d} \frac{e^{2\epsilon ((s+\lambda)/2)^{1/2} - 2\epsilon |n'}}{e^{2\pi n''|\frac{L^2}{2}|(L_2 - (s+\lambda)|n''||)}}
\]
\[
\times \sum_{n'' \in \mathbb{Z}_d} |c_{k,n}|^2 \sqrt{n''/2}, k'),
\]

Further on, for \( L_1 > (2\pi(s + \frac{1}{2})^{1/2} \) the series
\[
\sum_{k' \in \mathbb{N}_0} \sum_{n' \in \mathbb{Z}_d} \frac{e^{2\epsilon ((s+\lambda)/2)^{1/2} - 2\epsilon |n'|}}{e^{2\pi n''|\frac{L^2}{2}|(L_2 - (s+\lambda)|n''||)}}
\]
is convergent uniformly in \( l \in \mathbb{N}_0 \). Hence
\[
\sum_{k' \in \mathbb{N}_0} \sum_{n' \in \mathbb{Z}_d} |\langle \psi_{k',n'}, \sigma(x,D)f \rangle|^2 \sqrt{n''/2}, k')
\]
\[
\leq C||\sigma||_{A,B}^2 \sum_{k' \in \mathbb{N}_0} \sum_{n' \in \mathbb{Z}_d} |c_{k,n}|^2 \sqrt{n''/2}, k'),
\]
where \( M(\rho) \sim \rho^{1/2}, \rho \to \infty \), is used again. Finally, by Theorem 1(d) we obtain
\[
||\sigma(x,D)f||_{M_2^{p ''}} \leq C||\sigma||_{A,B}^2 ||f||_{M_{1,2}^{p '}}.
\]
4.2. Proof of Lemma 1

Proof. Let \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \in \{0, 1\}^d \), \( \varepsilon' = (\varepsilon'_1, \ldots, \varepsilon'_d) \in \{0, 1\}^d \) and \( l_\varepsilon = (-1)^\varepsilon l \), \( l'_{\varepsilon'} = (-1)^{\varepsilon'} l' \), for \( l, l' \in \mathbb{N}^d \). We introduce

\[
\tilde{\xi}_{\varepsilon, \varepsilon'} = -\xi - \frac{l_\varepsilon + l'_{\varepsilon'}}{2} \quad \text{and} \quad \tilde{x} = \tilde{x} = -\frac{n + n'}{2} - \frac{\kappa + \kappa'}{4}, \quad x, \varepsilon \in \mathbb{R}^d,
\]

for \( l_\varepsilon, l'_{\varepsilon'} \in \mathbb{N}^d \), \( \kappa, \kappa' \in \{0, 1\}^d \), \( n, n' \in \mathbb{Z}^d \). From (5) and (6), after an easy but cumbersome calculation, we obtain

\[
W(\psi_{2L+x, \eta}, \psi_{2L'+x', \eta'})(x, \xi)
= e^{-2\pi i n(n' + (\kappa - \kappa')/2)} \sum_{\varepsilon, \varepsilon' \in \{0, 1\}^d} G_{\varepsilon, \varepsilon'} e^{-2\pi i (l_\varepsilon - l'_{\varepsilon'}) x} W(\varphi, \varphi)(\tilde{\xi}_{\varepsilon, \varepsilon'}, \tilde{x}),
\]

(18)

where

\[
G_{\varepsilon, \varepsilon'} = c_{\varepsilon'} (-1)^{\varepsilon_1 (\kappa_1 + \kappa_1')} e^{2\pi i (l_\varepsilon - l'_{\varepsilon'})/2}, \quad c_{\varepsilon'} = c_{\varepsilon'} = (1/ \sqrt{2})^d,
\]

\[
W(\varphi, \varphi)(\tilde{\xi}_{\varepsilon, \varepsilon'}, \tilde{x}) = W(\varphi, \varphi)(\tilde{\xi}_{\varepsilon_1, \varepsilon_1'}, \tilde{x}_1) \otimes \cdots \otimes W(\varphi, \varphi)(\tilde{\xi}_{\varepsilon_d, \varepsilon_d'}, \tilde{x}_d).
\]

The special cases \( W(\psi_{k, \eta}, \psi_{k', \eta'}) \), where \( k_1 = 1 \) (or \( k'_1 = 1 \), \( i \in \{1, \ldots, d\} \)), are also included in (18). For example, if \( d = 1 \), the case \( k = 1, k' \neq 1 \) is obtained by putting \( \varepsilon = l_\varepsilon = \kappa = 0, c_1 = c_1 = 1 \) in (18).

We now have

\[
\langle \psi_{2L+x, \eta}, \sigma(x, D) \psi_{2L+x, \eta} \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sigma(x, \xi) W(\psi_{2L+x, \eta}, \psi_{2L+x', \eta'})(x, \xi) dx d\xi
= \sum_{\varepsilon, \varepsilon' \in \{0, 1\}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sigma(x, \xi) e^{-2\pi i n(n' + (\kappa - \kappa')/2)} e^{-2\pi i (l_\varepsilon - l'_{\varepsilon'}) x} \times G_{\varepsilon, \varepsilon'} W(\varphi, \varphi)(\tilde{\xi}_{\varepsilon, \varepsilon'}, \tilde{x}) dx d\xi.
\]

Integrating by parts, we obtain

\[
\langle \psi_{2L+x, \eta}, \sigma(x, D) \psi_{2L+x, \eta} \rangle
= \sum_{\varepsilon, \varepsilon' \in \{0, 1\}^d} G_{\varepsilon, \varepsilon'} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-2\pi i n(n' + (\kappa - \kappa')/2)} e^{-2\pi i (l_\varepsilon - l'_{\varepsilon'}) x} \int_{\mathbb{R}^d} P_{L_1}(l - l'_{\varepsilon'}) P_{L'1}(n - n' + (\kappa - \kappa')/2) dx d\xi,
\]

(19)
where

\[ I_{x,x'} = P_{L_1}(D_x) P_{L_2}(D_{x'}) [\sigma(x, x') W(\varphi, \varphi)(\xi_{x,x'}, x)] \]

\[ P_{L_1}(D_x) = \prod_{i=1}^{d} \sum_{\alpha \in \mathbb{N}_0^d} \frac{L_{\xi}^2 D_{i,x}^2}{\alpha!} \sum_{\alpha \in \mathbb{N}_0^d} a_{\alpha} D_{i,x}^\alpha, \]

\[ P_{L_2}(D_{x'}) = \prod_{i=1}^{d} \sum_{\beta \in \mathbb{N}_0^d} \frac{L_{\xi}^2 D_{i,x'}^2}{\beta!} \sum_{\beta \in \mathbb{N}_0^d} b_{\beta} D_{i,x'}^\beta. \] (20)

We have

\[ I_{x,x'} = \sum_{|\alpha| \geq 0} a_{\alpha} D_x^\alpha \sum_{|\beta| \geq 0} b_{\beta} D_{x'}^\beta [\sigma(x, x') W(\varphi, \varphi)(\xi_{x,x'}, x)] \]

= \sum_{|\alpha|, |\beta| \geq 0} a_{\alpha} b_{\beta} D_x^\alpha \sum_{r \leq \alpha} \left( \begin{array}{c} \beta \\ r \end{array} \right) D_{x'}^{\beta-r} \sigma(x, x') D_{x'}^r W(\varphi, \varphi)(\xi_{x,x'}, x)

= \sum_{|\alpha|, |\beta| \geq 0} a_{\alpha} b_{\beta} \sum_{r \leq \alpha} \left( \begin{array}{c} \beta \\ r \end{array} \right) \sum_{k \leq \alpha} \left( \begin{array}{c} \alpha \\ k \end{array} \right) D_x^{\alpha-k} D_{x'}^{\beta-r} \sigma(x, x')

\times D_x^k D_{x'}^r W(\varphi, \varphi)(\xi_{x,x'}, x).

By (4), there exist positive constants \( C_1 \) and \( C_2 \) such that

\[ |a_{\alpha}| \leq C_1 \left( \frac{\|L_1\|_{L_1}}{\alpha! r^{1/\nu}} \right), \quad |b_{\beta}| \leq C_2 \left( \frac{\|L_2\|_{L_1}}{\beta! r^{1/\nu}} \right), \quad \alpha, \beta \in \mathbb{N}_0^d. \]

Since \( \alpha! |r| \leq (\alpha + |r|)!, \) for \( 2 < E \leq \frac{\beta}{\gamma}, 2 < F \leq \frac{\beta}{\gamma}, \) we have

\[ |I_{x,x'}| \leq C \sum_{|\alpha|, |\beta| \geq 0} \frac{1}{E^{\|L_1\|_{L_1}}} \frac{1}{F^{\|L_2\|_{L_1}}} \sum_{r \leq \alpha} \sum_{k \leq \alpha} \left( \begin{array}{c} \beta \\ r \end{array} \right) \left( \begin{array}{c} \alpha \\ k \end{array} \right)

\times \left( \frac{\|L_1\|_{L_1}}{E^{\|L_1\|_{L_1}}} \right)^{|\alpha-k|} \left( \frac{\|L_2\|_{L_1}}{F^{\|L_2\|_{L_1}}} \right)^{|r-\beta|} D_x^{\alpha-k} D_{x'}^{\beta-r} \sigma(x, x')

\times D_x^k D_{x'}^r W(\varphi, \varphi)(\xi_{x,x'}, x)

\leq C \sum_{|\alpha|, |\beta| \geq 0} \frac{1}{E^{\|L_1\|_{L_1}}} \frac{1}{F^{\|L_2\|_{L_1}}} \sum_{r \leq \alpha} \sum_{k \leq \alpha} \left( \begin{array}{c} \beta \\ r \end{array} \right) \left( \begin{array}{c} \alpha \\ k \end{array} \right)

\times \left( \frac{\|L_1\|_{L_1}}{E^{\|L_1\|_{L_1}}} \right)^{|k|} \left( \frac{\|L_2\|_{L_1}}{F^{\|L_2\|_{L_1}}} \right)^{|r|} D_x^{\alpha-k} D_{x'}^{\beta-r} \sigma(x, x')

\times D_x^k D_{x'}^r W(\varphi, \varphi)(\xi_{x,x'}, x). \] (21)
Therefore, by (19) and (21), we have

\[ |\langle \psi_{2L+n', \sigma(x, D)} \psi_{2L+n, n} \rangle| \leq \sum_{e, e' \in \{0,1\}^d} C||\sigma||_{A,B}^{1/2} e^{(\beta/2)^{1/2} + \gamma/2} \left( \frac{L_1 E^{[1]} + \gamma}{k^1/2} \right)^{1/2} \]

\[ \times \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{(\beta+1)\frac{r}{2}} \left( \frac{L_2 D_{\xi} W(\phi, \phi)(\xi, x)}{r^{1/2}} \right) dx d\xi. \]

By (7) and (3), i.e.

\[ |P_{L_1}(L_1 - L_1)| \geq e^{2M(L_1|L_1 - L_1)}, \]

\[ |P_{L_2}(L_2(n - n' + (\kappa - \kappa')/2))| \geq e^{2M(L_2)|n - n' + (\kappa - \kappa')/2)}, \]

it follows

\[ |\langle \psi_{2L+n', \sigma(x, D)} \psi_{2L+n, n} \rangle| \leq \sum_{e, e' \in \{0,1\}^d} C||\sigma||_{A,B}^{1/2} e^{(\beta/2)^{1/2} + \gamma/2} \left( \frac{L_1 E^{[1]} + \gamma}{k^1/2} \right)^{1/2} \]

To obtain (14) we use the fact that for every \( \tau \geq 0 \) and every \( e, e' \in \{0,1\}^d \) the inequality \( |a + b| \geq |a| - |b| \) implies

\[ \frac{e^{\tau|L_1 - L_1|/2^\tau}}{e^{\tau|L_1 - L_1|/2^\tau}} \leq e^{\tau|L_1 - L_1|/2^\tau}. \]

For \( \tau < 0 \), the same is true when \( L_1 \geq \frac{|L_1 - L_1|}{2^\tau} \). This completes the proof. \( \square \)

5. A class of elliptic pseudodifferential operators

We consider a class of symbols \( \sigma \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d) \) satisfying:

(S1) \( \sigma(z) \geq 1, z = (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d \).

(S2) \( \exists C > 0, \exists \eta > 0 \) such that

\[ \sigma(z + w) \leq C e^{\eta|w|} \sigma(w), \quad z, w \in \mathbb{R}^{2d}. \]
(S3) \((\forall h \geq 0) \ (\exists C > 0) \ (\exists \delta > 0)\) such that

\[
\sup_{z \in \mathbb{N}^d, |z| \geq 1} \left| \frac{h^{2^{d-1}} D^2 \sigma(z)}{|z|^{1/2}} \right| \leq C \frac{\sigma(z)}{(1 + |z|)^{\delta}}, \quad z \in \mathbb{R}^d.
\]

(S4) \(\sigma(x, \xi) \leq \sigma(x, \xi')\) for all \(\xi, \xi' \in \mathbb{R}^d\) such that \(|\xi| \leq |\xi'|\).

Without losing generality, instead of (S1), we can assume \(\sigma(z) \geq C_0, z \in \mathbb{R}^d \times \mathbb{R}^d\), for some \(C_0 > 0\). The above form of (S1) simplifies the notation. Note, by (S2), (S3) and \(|z| \leq |x| + |\xi'|\), we have \(\sigma(x, \xi) \in S^p_{2,1}, 0 < p, q \leq 1\), for every \(L_1, L_2 \geq 0\). Also, condition (S2) implies that \(\sigma\) is an exp-type weight and \(\sigma(z) \geq \frac{C_0}{|z|^\delta}\) for all \(z \in \mathbb{R}^d\).

For example, \(\sigma(x, \xi) = e^{(1 + |x|^2 + |\xi'|^2)/2}, 0 < \gamma < 1, x, \xi \in \mathbb{R}^d\) satisfies (S1)-(S4).

Another example is the symbol of the Schrödinger operator \(-\Delta + V\), with an increasing potential \(V\). More precisely, if \(V\) is a real valued function which satisfies the conditions:

\[
(V1) \quad V \in C^\infty(\mathbb{R}^d), \quad V \geq 1, \quad \forall x \in \mathbb{R}^d, \quad x \to \infty \Rightarrow V(x) \to \infty.
\]

\[
(V2) \quad \exists C > 0, \exists \eta > 0 \text{ such that } \forall x, y \in \mathbb{R}^d, \quad V(x + y) \leq C e^{\eta |x|^\gamma} V(y),
\]

\[
(V3) \quad \forall h > 0, \exists C > 0 \text{ such that } \forall x \in \mathbb{R}^d,
\]

\[
\sup_{z \in \mathbb{N}^d, |z| \geq 1} \left| \frac{h^{2^{d-1}} D^2 V(x)}{|z|^{1/2}} \right| \leq CV(x),
\]

then \(-\Delta + V\) satisfies (S1)-(S4).

To shorten the notation, we put \(\sigma_{l, n} = \sigma(n + \xi, l),\)

\[
\theta = \sigma \left( n + \kappa + \kappa' \frac{l + l'}{2} \right) - \frac{n + n'}{2} + \kappa + \kappa' \frac{l + l'}{2}, \quad N = \frac{n + n'}{2} + \kappa + \kappa' \frac{l + l'}{2}, \quad L = \frac{l + l'}{2},
\]

\[
l, l' \in \mathbb{N}^d, \quad \kappa, \kappa' \in \{0, 1\}^d, \quad n, n' \in \mathbb{Z}^d.
\]

The following simple observation will be used in the proof of Theorem 3. By (S2) we have

\[
\theta = \sigma(N, L)
\]

\[
= \sigma \left( n + \kappa + \kappa' \frac{l + l'}{2} \right) - \frac{n + n'}{2} + \kappa + \kappa' \frac{l + l'}{2}
\]

\[
\leq \sigma \left( n + \kappa \frac{l}{2} \right) e^{\left( \frac{n + n'}{4} + \frac{\kappa + \kappa'}{4} \right)}.
\]
and similarly
\[ \theta \leq \sigma_{\rho,\nu} e^\left(\frac{\frac{n+n-k+\frac{x}{2}}{2}}{4} + \frac{x}{2}\right), \]
where
\[ \theta \leq (\sigma_{l,n} \sigma_{\rho,\nu})^{1/2} e^\left(\frac{\frac{n+n-k+\frac{x}{2}}{2}}{4} + \frac{x}{2}\right), \]
\[ \theta \leq \sigma_{n,\nu} e^\left(\frac{\frac{n+n-k+\frac{x}{2}}{2}}{4} + \frac{x}{2}\right). \]

**Theorem 3.** Assume that \( \{\psi_{k,n}, k \in \mathbb{N}^d, n \in \mathbb{Z}^d\} \) is a Wilson basis of exponential decay and that \( \sigma(x, \xi) \) satisfies (S1)–(S4). Then

(a) For every \( a, b > 0 \) it holds
\[ |\langle \psi_{2+k+n, \rho} \sigma(x, D) \psi_{2+k+n, n} \rangle | \leq C_{\sigma_{l,n}}^1 \left( \frac{1}{(1 + |L|)^2} + \frac{1}{(1 + |N|)^2} + \frac{1}{(1 + |N| + |L|)^2} \right), \]
where \( N \) and \( L \) are given in (22), and \( s \) is the constant from (S3).

(b) For
\[ \sigma(x, D) \psi_{2+k+n, n}(x) = \sigma_{l,n} \psi_{2+k+n, n}(x) + r_{l,n}(x) \]
there exists a positive constant \( C \) such that
\[ \|r_{l,n}(x)\| \leq C \sigma_{l,n}. \]

(c) Let \( 1 \leq p, q < \infty \) and \( s \geq 0 \). For every \( f \in M_{p, q}^{\mu, \nu} \) there exist positive constants \( C_1, C_2 \) and \( C_3 \) such that
\[ C_1 \|f\|_{M_{p, q}^{\mu, \nu}} \leq \|\sigma(x, D)f\|_{M_{1, q}^{\mu, \nu}} + C_2 \|f\|_{M_{1, q}^{\mu, \nu}} \leq C_3 \|f\|_{M_{p, q}^{\mu, \nu}}. \]  \hspace{1cm} (23)

**Corollary 2.** If, additionally,
\[ \sigma(x) \geq C e^{\frac{|x|}{K}} \quad \text{for} \quad |x| \geq K, \]
for some positive constants \( C, \mu \) and \( K \), and if \( \sigma(x, D)f \in M_{2, 2}^{1, 1, \mu} \) for \( f \in L^2(\mathbb{R}) \), then \( f \) belongs to \( M_{2, 2}^{1, 1, \mu+\nu} \).
Remark 3. An immediate consequence of Theorem 3 is the continuity of the mapping \( \sigma : M^*_{p,q,d} \rightarrow M^*_{p,q,d} \). Moreover, \( \sigma(M^*_{p,q,d}) \), the image of \( M^*_{p,q,d} \) under \( \sigma \), is a Banach subspace of \( M^*_{p,q,d} \).

5.1. Proof of Theorem 3

As in the previous section put
\[
\tilde{z}_{x,\tilde{\varepsilon}} = -\xi - \frac{l_x + \ell_x'}{2} \quad \text{and} \quad \tilde{x} = x - \frac{n + n'}{2} - \frac{\kappa + \kappa'}{4}, \quad x, \xi \in \mathbb{R}^d,
\]
\( l_x = (-1)^x l, \quad l_x' = (-1)^x l' \in \mathbb{N}^d, \quad \kappa, \kappa' \in \{0,1\}^d, \quad n, n' \in \mathbb{Z}^d. \)

Proof. We give only the proof of (a) and (c) since (b) follows immediately from (a). Using the above notation and integration by parts, we obtain
\[
\begin{align*}
\langle \psi_{2F + n', p', n}, \psi_{2F + n, n} \rangle &= \langle \psi_{2F + n', p, n}, \psi_{2F + n, n} \rangle - \theta \langle \psi_{2F + n', p, n}, \psi_{2F + n, n} \rangle \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \langle (\sigma(x, \xi) - \theta) e^{-2\pi i (n - n' + (\kappa - \kappa')/2)} e^{-2\pi i (l_x - l_x')}, (x, \xi) \rangle \, dx \, d\xi \\
&= \sum_{e, e' \in \{0,1\}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \langle (\sigma(x, \xi) - \theta) e^{-2\pi i (n - n' + (\kappa - \kappa')/2)} e^{-2\pi i (l_x - l_x')}, (x, \xi) \rangle \\
&\quad \times G_{x,e} W(\varphi, \varphi) (\tilde{z}_{x,e'}, \tilde{x}) \, dx \, d\xi \\
&= \sum_{e, e' \in \{0,1\}^d} G_{x,e} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} P_L (l_x - l_x') P_L (n - n' + (\kappa - \kappa')/2) \, dx \, d\xi,
\end{align*}
\]
where \( I_{e,e'} \)
\[
= P_L (D_x) P_L (D_{x'}) \left[ \left( \sigma(x, \xi) - \sigma \left( \frac{n + n'}{2} + \frac{\kappa + \kappa'}{4}, \frac{l_x + l_x'}{2} \right) \right) W(\varphi, \varphi) (\tilde{z}_{x,e'}, \tilde{x}) \right],
\]
and operators \( P_L (D_x), P_L (D_{x'}) \) are given by (20). Further on
\[
|I_{e,e'}| \leq C \left( \left| \left( \sigma(x, \xi) - \sigma \left( \frac{n + n'}{2} + \frac{\kappa + \kappa'}{4}, \frac{l_x + l_x'}{2} \right) \right) W(\varphi, \varphi) (\tilde{z}_{x,e'}, \tilde{x}) \right| + |J|,
\]
where

\[
J = \sum_{|\alpha + \beta| > 1} \frac{1}{E^{|\alpha|}} \frac{1}{F^{|\beta|}} \sum_{r<\beta} \sum_{k<\alpha} \binom{\beta}{r} \binom{\alpha}{k} \left( \frac{\beta}{r} \right) \left( \frac{\alpha}{k} \right) \\
\times \left( \frac{\mathcal{G}_1 E^{\alpha-k}}{(\alpha-k)!^{1/2}} \frac{(\mathcal{G}_2 F)^{|\beta-r|}}{(\beta-r)!^{1/2}} \right) D_x^\alpha D_\xi^\beta \sigma(x, \xi)
\]

and \( E > 2, F > 2 \). We first estimate the case \( \varepsilon = \varepsilon' = 0 \), and write \( \tilde{\xi}_{0,0} = \tilde{\xi} \) for short. By (S2), (S3) and (S4) we have

\[
|\sigma(x, \xi) - 0| = |\sigma(x', \tilde{\xi})||\tilde{x}|| + |\sigma_\varepsilon(x', \tilde{\xi})||\tilde{\xi}|
\]

\[
\leq C \frac{|\sigma(x', \tilde{\xi'})|}{(1 + |x'| + |\tilde{\xi}'|)^j} (|\tilde{x}| + |\tilde{\xi}|)
\]

\[
\leq C \frac{|\sigma(x', \tilde{\xi}')|}{(1 + |x'| + |\tilde{\xi}'|)^j} (|\tilde{x}| + |\tilde{\xi}|)
\]

where

\[
\begin{align*}
\varepsilon' &= \frac{n + n' + \kappa + \kappa'}{4} + \tau \left( \varepsilon - \frac{n + n' + \kappa + \kappa'}{4} \right) = N - \tau \xi, \\
\xi' &= \frac{l + l'}{2} + \tau \left( -\xi - \frac{l + l'}{2} \right) = L - \tau \xi.
\end{align*}
\]

Using the fact that there exists \( \lambda > 0 \) such that \( W(\varphi, \phi)(\tilde{x}) = \mathcal{O}(e^{-\lambda|x|}), \tilde{x} = (x, \tilde{\xi}) \in \mathbb{R}^{2d} \) and putting

\[
A = e^{n|\tilde{\xi}|} (|\tilde{x}| + |\tilde{\xi}|)|W(\varphi, \phi)(\tilde{x})|
\]

we obtain

\[
\begin{align*}
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} A & \frac{A}{(1 + |x'| + |\tilde{\xi}'|)^j} d\tilde{x} |d\tilde{\xi}| \\
\leq & \int_{\mathbb{R}^d} A |d\tilde{\xi}| \left( \int_{|\tilde{x}| < N} A (1 + |x'| + |\tilde{\xi}'|)^j d\tilde{x} + \int_{|\tilde{x}| > N} A (1 + |x'| + |\tilde{\xi}'|)^j d\tilde{x} \right) \\
\leq & \int_{\mathbb{R}^d} A |d\tilde{\xi}| \left( \int_{|\tilde{x}| < N} A (1 + |x'| + |\tilde{\xi}'|)^j d\tilde{x} + \int_{|\tilde{x}| > N} A (1 + |N| + |\tilde{\xi}'|)^j d\tilde{x} \right) \\
\leq & \int_{\mathbb{R}^d} A |d\tilde{\xi}| \left( \int_{|\tilde{x}| < N} A (1 + |\tilde{\xi}'|)^j d\tilde{x} \right)
\end{align*}
\]
It follows

\[ \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\sigma(x; \xi) - \theta) W(\varphi, \psi)(\xi) \, dx \, d\xi \right| \leq C \theta \left( \frac{1}{(1 + |\xi|)^2} + \frac{1}{(1 + |\sigma|)^2} + \frac{1}{(1 + |\sigma| + |\xi|)^2} \right). \]  

(24)

We now observe \( J \). Let \( \alpha, \beta \in \mathbb{N}^d, k \leq \alpha, r \leq \beta \) and \( h > 0 \). Then, by (S2) and (S3),

\[ \frac{h^{k-r}}{(x-k)^{|\alpha-k|}} \frac{h^{l-r}}{(y-l)^{|\beta-r|}} |D_x^{k-r}D_y^{l-r}\sigma(x, \xi)| \leq C \frac{\sigma(x)}{(1 + |x|)^2} \]

\[ \leq C \frac{\theta}{(1 + |x|)^2} \epsilon^{d|r|}. \]

The similar estimates as in the proof of (24) imply

\[ \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J \, dx \, d\xi \right| \leq C \theta \left( \frac{1}{(1 + |\xi|)^2} + \frac{1}{(1 + |\sigma|)^2} + \frac{1}{(1 + |\sigma| + |\xi|)^2} \right). \]

Since \( \sigma(x, \xi) = \sigma(x, |\xi|) \), the case \( \epsilon = \epsilon' = 1 \) is analogous. Let \( \epsilon_i \neq \epsilon_i' \) for some \( i \in \{1, \ldots, d\} \), and let

\[ \bar{\theta} = \sigma(N, (l - l')/2) = \sigma(N, (l' - l)/2). \]

We have

\[ |\bar{\theta} - \theta| \leq |\sigma(N, (l - l')/2) - \sigma(N, (l + l')/2)| \]

\[ \leq |\sigma(N, (l - l')/2 + \tau((l + l')/2) - (l - l')/2)| \]

\[ \leq |\sigma(N, (l - l')/2 + \tau((l + l')/2) - (l - l')/2)|. \]
Since 

\[ (\sigma(x, \xi) - \bar{\sigma}) W(\varphi, \varphi)(\xi_{e', e}, \bar{x}) = (\sigma(x, \xi) - \theta) W(\varphi, \varphi)(\xi_{e', e}, \bar{x}) + (\bar{\sigma} - \theta) W(\varphi, \varphi)(\xi_{e', e}, \bar{x}), \]

and \( |l' - l| \leq |l + l'| \) it follows

\[
\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J_{l'e'} dx d\xi \right| \leq C \theta \left( \frac{1}{(1 + |L|)^3} + \frac{1}{(1 + |N|)^3} \frac{1}{(1 + |N| + |L|)^3} \right)
\]

and also

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|J_{l'e'}|}{P_{L_1}(l_e - l'_e) P_{L_2}(n - n' + (\kappa - \kappa')/2)} dx d\xi \leq \frac{P_{L_1}(l_e - l'_e) P_{L_2}(n - n' + (\kappa - \kappa')/2)}{C \theta} \left( \frac{1}{(1 + |L|)^3} + \frac{1}{(1 + |N|)^3} \frac{1}{(1 + |N| + |L|)^3} \right)
\]

\[ + \frac{P_{L_1}(l_e - l'_e) P_{L_2}(n - n' + (\kappa - \kappa')/2)}{C \theta \max\{l, l'\}} \frac{1}{(1 + |N| + |L|)^3}. \]

The last term is, by (3), less than or equal to

\[
\frac{C \theta}{e^{2(L_1 - 1)|l - l'|^2} e^{2(L_2|n - n' + (\kappa - \kappa')/2|)} (1 + |N| + |L|)^3}.
\]

Therefore,

\[
\left| \langle \psi_{2l' + \kappa', e}, \sigma(x, D)\psi_{2l + \kappa, n} \rangle - \sigma_{l,n} \langle \psi_{2l' + \kappa', e}, \psi_{2l + \kappa, n} \rangle \right| \]

\[
\leq C \frac{P_{L_1}(l - l')} P_{L_2}(n - n' + (\kappa - \kappa')/2) \left( \frac{1}{(1 + |L|)^3} + \frac{1}{(1 + |N|)^3} \frac{1}{(1 + |N| + |L|)^3} \right)
\]

\[ + \frac{e^{2(L_1 - 1)|l - l'|^2} e^{2(L_2|n - n' + (\kappa - \kappa')/2|)} (1 + |N| + |L|)^3}{C \theta}. \]
For given \(a,b>0\), choosing

\[
(a_1, b_1) \in \mathbb{P}(1+|N|+|L|)^2
\]

we finally obtain

\[
1 < \left( a_1, b_1 \right)^{1/2} \left( \frac{1}{1+|N|+|L|} \right)^{1/2} \left( \frac{1}{1+|N|+|L|} \right)^{1/2}
\]

\[
= C \frac{(\sigma_{1,n} \sigma_{1,n'})^{1/2}}{e^{2(L^2_1-L^2_1)|x-y|}} \frac{1}{(1+|N|+|L|)^{1/2}} \left( \frac{1}{1+|N|+|L|} \right)^{1/2} \left( \frac{1}{1+|N|+|L|} \right)^{1/2}
\]

(c) Operator \(\sigma(x, D)\) whose Weyl symbol \(\sigma(x, \xi)\) satisfies (S1)-(S4) is continuous on \(\mathcal{S}(\mathbb{R})\). It follows from Corollary 1, since \(\sigma(x, \xi) \in S_{L_1, L_2}^0\) for every \(L_1, L_2 \geq 0\). We define \(\sigma(x, D) : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R})\) by \(g, \sigma(x, D)f = \langle \sigma(x, D)g, f \rangle\) for all \(g \in \mathcal{S}(\mathbb{R})\) and \(f \in \mathcal{S}(\mathbb{R})\). Then \(\sigma(x, D)\) is a continuous operator on \(\mathcal{S}(\mathbb{R})\).

Now we show the second inequality in Theorem 3(c). For \(f \in \mathcal{S}(\mathbb{R})\), we have

\[
\|\sigma(x, D)f\|_{M_{q/p}^r} \leq C \left( \sum_{n \in \mathbb{N}^d} \left( \sum_{k \in \mathbb{Z}^d} \left| \langle \psi_{k,n}, \sigma(x, D)f \rangle \right|^q e^{-sp(|k|^2+|n/2|^2)} \right)^{1/q} \right)^{1/p}.
\]

where we used the fact that \(\sigma(x, D)f \in \mathcal{S}(\mathbb{R}) \subset M_{q/p}^r\) as well as Theorem 1(d). Put now \(\sigma_{k,n} = \sigma(n + \frac{k}{2}, f), k = 2l + \kappa \in \mathbb{N}^d\). Again by Theorem 1 and the continuity of \(\sigma(x, D)\) on \(\mathcal{S}(\mathbb{R})\) the last term does not exceed

\[
C \left( \sum_{k \in \mathbb{N}^d} \left( \sum_{n \in \mathbb{Z}^d} \langle \psi_{k,n}, f \rangle \sigma_{k,n} + \sum_{l \in \mathbb{N}^d} \left( \sum_{\kappa \in \mathbb{Z}^d} \langle \psi_{l,n}, f \rangle \langle \psi_{k,l}, \sigma(x, D) \psi_{l,n} \rangle \right)^{q/p} \right) \right)^{1/q} + II,
\]

\[
\leq I + II,
\]

(25)
where

\[ I = C \left( \sum_{k \in \mathbb{N}^d} \left( \sum_{n \in \mathbb{Z}^d} \left| \langle \psi_{k,n}, f \rangle \right|^p \sigma_{k,n}^p e^{\|k\|^2 + |n/2|^2} \right)^{q/p} \right)^{1/q}, \]

\[ II = C \left( \sum_{k \in \mathbb{N}^d} \left( \sum_{k' \in \mathbb{N}^d} \sum_{n \in \mathbb{Z}^d} \sum_{n' \in \mathbb{Z}^d} \left| \langle \psi_{k', n'}, f \rangle \right|^p \sigma_{k,n'}^p e^{\|k'\|^2 + |n'/2|^2} \right)^{q/p} \right)^{1/q}. \]

Since \(|k - k'| \leq 2|l - l'| + 1\), it follows

\[ \frac{1}{e^{2|l-l'|^2}} \leq \frac{C}{\|k - k'\|^q}. \]

By Theorem 3(a), II is less than or equal to

\[ C \left( \sum_{k \in \mathbb{N}^d} \left( \sum_{n \in \mathbb{Z}^d} \left( \sum_{k' \in \mathbb{N}^d} \sum_{n' \in \mathbb{Z}^d} \left| \langle \psi_{k', n'}, f \rangle \right|^p \sigma_{k', n'}^p e^{\|k'\|^2 + |n'/2|^2} \right)^{q/p} \right)^{1/q} \]

for arbitrary \(a, b > 0\). Hölder's inequality gives

\[ \left( \sum_{k' \in \mathbb{N}^d} \sum_{n' \in \mathbb{Z}^d} \left| \langle \psi_{k', n'}, f \rangle \right|^p \frac{\sigma_{k', n'} A e^{\|n/2|^2}}{e^{\|k'\|^2 + b |n' - k' - 2k'|}} \right)^{q/p} \]

\[ \leq \sum_{k' \in \mathbb{N}^d} \sum_{n' \in \mathbb{Z}^d} \left| \langle \psi_{k', n'}, f \rangle \right|^p \frac{\sigma_{k', n'} A e^{\|n/2|^2}}{e^{\|k'\|^2 + b |n' - k' - 2k'|}}. \]

Using

\[ \sum_{n \in \mathbb{Z}^d} e^{\|n/2|^2} \leq \sum_{n \in \mathbb{Z}^d} e^{\|n/2 + \frac{K_0}{2}|^2} \leq C e^{\|n/2|^2}, \]

\( N_0 \), and \( L \) are given by (22) and \( \ell \) is the constant from the condition (S3). Since \(|k - k'| \leq 2|l - l'| + 1\), it follows

\[ \frac{1}{e^{2|l-l'|^2}} \leq \frac{C}{\|k - k'\|^q}. \]
for $b > sp/2$, we obtain

$$
\left( \sum_{\alpha \in \mathbb{N}^d} \sum_{k' \in \mathbb{N}^d} \sum_{n' \in \mathbb{Z}^d} |\langle \Psi_{k',n'}, f \rangle|^p \frac{\alpha_{k',n'}^p A^p e^{-q/2}}{e^{ipk'} + e^{-ipk'}} \right)^{1/p} 
\leq C \left( \sum_{\alpha \in \mathbb{N}^d} \sum_{k' \in \mathbb{N}^d} \sum_{n' \in \mathbb{Z}^d} |\langle \Psi_{k',n'}, f \rangle|^p \sigma_{k',n'}^p e^{sp/2} \frac{e^{ipk'}}{e^{ipk'}} \right)^{1/p},
$$

Hence,

$$
\sum_{\alpha \in \mathbb{N}^d} \left( \sum_{k' \in \mathbb{N}^d} \left( \sum_{n' \in \mathbb{Z}^d} |\langle \Psi_{k',n'}, f \rangle|^p \sigma_{k',n'}^p e^{sp/2} \frac{e^{ipk'}}{e^{ipk'}} \right)^{q/p} \right)^{p/q} 
\leq C \sum_{\alpha \in \mathbb{N}^d} \left( \sum_{k' \in \mathbb{N}^d} \left( \sum_{n' \in \mathbb{Z}^d} |\langle \Psi_{k',n'}, f \rangle|^p \sigma_{k',n'}^p e^{sp/2} \frac{e^{ipk'}}{e^{ipk'}} \right)^{q/p} \right)^{p/q},
$$

where we have used

$$
\sum_{k' \in \mathbb{N}^d} e^{aqk'} \leq C e^{aqk'},
$$

for $a > spq$. Therefore,

$$
\Pi \leq C \left( \sum_{\alpha \in \mathbb{N}^d} \left( \sum_{k' \in \mathbb{N}^d} \left( \sum_{n' \in \mathbb{Z}^d} |\langle \Psi_{k',n'}, f \rangle|^p \sigma_{k',n'}^p e^{sp/2} \frac{e^{ipk'}}{e^{ipk'}} \right)^{q/p} \right)^{p/q} \right)^{1/q}.
$$

We conclude $||\sigma(x,D)f||_{\mathcal{M}_{L^2}} \leq C ||f||_{\mathcal{M}_{L^2}}$. This and $||f||_{\mathcal{M}_{L^2}} \leq ||f||_{\mathcal{M}_{L^2}}$, imply the second inequality in (23).

**Remark 4.** Since

$$
A \leq C \left( \frac{(1 + 1/2)^2}{(1 + 1/2)^2} + \frac{(1 + \|2 + \|2\|)^2}{(1 + \|2\|)^2} \right),
$$
we actually have
\[ II \leq C \left( \sum_{k \in \mathbb{N}^d} \left( \sum_{n \in \mathbb{Z}^d} |\langle \psi_{k,n}, f \rangle| \right)^p A_k^p |\varphi(k'|x'|/2^q)|^q \right)^{1/p}, \] (26)

where \( A_1 = \frac{1}{1+\|\psi_0\|^2} + \frac{1}{1+\|\psi_1\|^2}. \)

To prove the first inequality in Theorem 3 we set
\[ A_1 = \{(k,n) \in \mathbb{N}^d \times \mathbb{Z}^d : \sigma_{k,n} \leq K \}, \quad A_2 = \{(k,n) \in \mathbb{N}^d \times \mathbb{Z}^d : \sigma_{k,n} > K \}, \]
where \( K \) is a positive constant which will be determined later. If \( k \in \mathbb{N}^d \) and there exists \( n \in \mathbb{Z} \) such that \( (k,n) \in A_1 \) (\( A_2 \), respectively), we will write \( k \in A_1, k \) (\( A_2, k \) resp.). Similarly, \( n \in A_1, n \) (\( n \in A_2, n \) resp.) means that there exists \( k \in \mathbb{N}_0 \) such that \( (k,n) \in A_1 \) (\( A_2 \), resp.).

Let \( f = f_1 + f_2 \), where
\[ f_1 = \sum_{(k,n) \in A_1} \langle \psi_{k,n}, f \rangle \psi_{k,n} \quad \text{and} \quad f_2 = \sum_{(k,n) \in A_2} \langle \psi_{k,n}, f \rangle \psi_{k,n}. \]

Since \( ||f||_{M_{d,q}^\sigma} \leq ||f_1||_{M_{d,q}^\sigma} + ||f_2||_{M_{d,q}^\sigma} \) and, by Theorem 1(d), \( ||f_1||_{M_{d,q}^\sigma} \leq C ||f||_{M_{d,q}^\sigma} \), to prove the first inequality in (23), it remains to show
\[ ||f_2||_{M_{d,q}^\sigma} \leq C ||\sigma(x,D)f||_{M_{d,q}^\sigma} + C ||f||_{M_{d,q}^\sigma}. \] (27)

Let \( \sigma(x,D)f_2 = g_1 + g_2 \), where
\[ g_1 = \sum_{(k,n) \in A_1} \langle \psi_{k,n}, \sigma(x,D)f_2 \rangle \psi_{k,n} \quad \text{and} \quad g_2 = \sum_{(k,n) \in A_2} \langle \psi_{k,n}, \sigma(x,D)f_2 \rangle \psi_{k,n}. \]

Theorem 1(d) implies
\[ ||g_2||_{M_{d,q}^\sigma} \geq C \left( \sum_{k \in A_2} \left( \sum_{n \in \mathbb{N}_0} |\langle \psi_{k,n}, g_2 \rangle| \right)^p e^{-2p(k'|x'|/2^q)} \right)^{1/p}, \]
\[ \geq C \left( \sum_{k \in A_2} \left( \sum_{n \in \mathbb{N}_0} |\langle \psi_{k,n}, f \rangle \sigma_{k,n} \right) \right)^{1/p}. \]
\[ \begin{align*}
&+ \sum_{(k',n') \in \Lambda_2} \langle \psi_{k,n}', \sigma(x, D) \psi_{k,n}' \rangle \\
&\quad - \sigma_{k,n} \langle \psi_{k,n}, \psi_{k,n}' \rangle \rho e^{sp(k^2 + n/2l)} \\left( q/p \right)^{1/q}
\end{align*} \]

\[ \geq I' - II', \]

where

\[ I' = C \left( \sum_{k \in \Lambda_1} \left( \sum_{n \in \Lambda_1} \left| \sum_{(k',n') \in \Lambda_2} \langle \psi_{k,n}, \sigma(x, D) \psi_{k,n}' \rangle \rho \sigma_{k,n}' e^{sp(k^2 + n/2l)} \right| \right)^{q/p} \right)^{1/q}, \]

\[ II' = C \left( \sum_{k \in \Lambda_1} \left( \sum_{n \in \Lambda_1} \left( \sum_{(k',n') \in \Lambda_2} \langle \psi_{k,n}, \sigma(x, D) \psi_{k,n}' \rangle - \sigma_{k,n} \langle \psi_{k,n}, \psi_{k,n}' \rangle \right) e^{sp(k^2 + n/2l)} \right)^{p/q} \right)^{1/q}. \]

Similar arguments as in the first part of the proof, together with (26) give

\[ \Pi' \leq C_1 \left( \sum_{k \in \Lambda_1} \left( \sum_{n \in \Lambda_1} \left| \sum_{(k',n') \in \Lambda_2} \langle \psi_{k,n}, \sigma(x, D) \psi_{k,n}' \rangle \rho \sigma_{k,n}' e^{sp(k^2 + n/2l)} \right| \right)^{q/p} \right)^{1/q}. \]

If we choose \( K \) such that \( C_1 A_1 \leq \frac{C}{2} \) holds for all \( (k, n) \in \Lambda_2 \), we obtain

\[ I' - II' \geq C \left( \sum_{k \in \Lambda_1} \left( \sum_{n \in \Lambda_1} \left| \sum_{(k',n') \in \Lambda_2} \langle \psi_{k,n}, \sigma(x, D) \psi_{k,n}' \rangle \rho \sigma_{k,n}' e^{sp(k^2 + n/2l)} \right| \right)^{q/p} \right)^{1/q}, \]

and \( \| f_2 \|_{M^p_{n,q}} \leq C \| g_2 \|_{M^p_{n,q}} \leq C \| \sigma(x, D) f_2 \|_{M^p_{n,q}} \), where Theorem 1 has been used again. By Minkowski's inequality we have

\[ \| \sigma(x, D) f_2 \|_{M^p_{n,q}} \leq C \| \sigma(x, D) f \|_{M^p_{n,q}} + C \| \sigma(x, D) f \|_{M^p_{n,q}} + C \| f \|_{M^p_{n,q}} \]

since \( \| \sigma(x, D) f \|_{M^p_{n,q}} \leq C \| f \|_{M^p_{n,q}} \leq C \| f \|_{M^p_{n,q}} \). We have proved (27). Thus Theorem 3 is proved. \( \square \)
6. Spectral asymptotics

We again use the notation $\sigma_{k,n} = \sigma(n + \frac{k}{\lambda}, l)$ for $k = 2l + \kappa \in \mathbb{N}^d$.

Lemma 2. Let there be given a Wilson basis of exponential decay $\{\psi_{k,n}, k \in \mathbb{N}^d, n \in \mathbb{Z}^d\}$. Let $f \in L^2$, symbol $\sigma(x, \xi)$ satisfy (S1)–(S4) and $\sigma(x, \xi) \to \infty$ as $|x| + |\xi| \to \infty$. Then:

(a) There exists $C > 0$ such that

\[
(1 - C) \sum_{k \in \mathbb{N}^d, \, n \in \mathbb{Z}^d} |\langle \psi_{k,n}, f \rangle|^2 \sigma_{k,n} \leq \langle f, \sigma(x, D)f \rangle \leq (1 + C) \sum_{k \in \mathbb{N}^d, \, n \in \mathbb{Z}^d} |\langle \psi_{k,n}, f \rangle|^2 \sigma_{k,n}.
\]

(b) There exist $C_1 \in \mathbb{R}, \, C_2 > 0, \, C_3 > 0$, such that

\[
C_1 \sum_{k \in \mathbb{N}^d, \, n \in \mathbb{Z}^d} |\langle \psi_{k,n}, f \rangle|^2 \sigma_{k,n} \leq \langle f, \sigma(x, D)f \rangle + C_2 \|f\|^2 \leq C_3 \sum_{k \in \mathbb{N}^d, \, n \in \mathbb{Z}^d} |\langle \psi_{k,n}, f \rangle|^2 \sigma_{k,n}.
\]

Proof. Part (a) can be proved using Theorem 3(a) and Hölder's inequality (see also [32, Lemma 5.1]). Part (b) then follows directly from (a), (S1) and

\[
\|f\|^2 = \sum_{k \in \mathbb{N}^d, \, n \in \mathbb{Z}^d} |\langle \psi_{k,n}, f \rangle|^2.
\]

It is sufficient to take $C_1 = 1 - C$ and $C_3 = 1 + C + C_2$, where $C$ is given in (a).

Theorem 4. Suppose that symbol $\sigma(x, \xi)$ satisfies the conditions (S1)–(S4) and

\[
\sigma(x, \xi) \sim e^{\alpha(\|x\|^2 + |\xi|^2)} \quad \text{when} \quad |x| + |\xi| \to \infty.
\]

(a) $\sigma(x, D)|_{\mathcal{D}(\sigma)}$ is essentially self-adjoint in $L^2$ and the domain of its unique self-adjoint extension $L$ is $\mathcal{M}_{\sigma}^{\mathcal{D}}$.

(b) The self-adjoint operator $L$ has only the discrete spectrum and its eigenfunctions belong to $\mathcal{F}(\sigma)$.

Proof. (a) We prove that $\sigma(x, D)|_{\mathcal{D}(\sigma)}$ is essentially self-adjoint by using [30, Theorem 26.1]. To that end we need to show the hypoellipticity. The difference to arguments
given in [32] is that we cannot use the hypoellipticity directly [30, Theorems 26.1 and 26.2]. Instead, due to Corollary 2, we have the $\mathcal{G}(\upsilon)$ hypoellipticity since $\mathcal{G}(\upsilon) = \cap_{b>0} M_{2,2}^{1/2}$.

(b) By the standard arguments, it is sufficient to prove that the set

$$\left\{ f \in L^2 : \sum_{k \in \mathbb{N}^r, n \in \mathbb{Z}^d} \left| \langle \psi_{k,n} f \rangle \right|^2 \sigma_{k,n} \leq 1 \right\}$$

is compact in $L^2$. This easily follows from (28) and $\sigma(x, \xi) \to \infty$ when $|x| + |\xi| \to \infty$.

The fact that eigenfunctions of $L$ belong to $\mathcal{G}(\upsilon)$ is derived by Corollary 2 and $\cap_{b>0} M_{2,2}^{1/2} = \mathcal{G}(\upsilon)$. □

Let $\lambda_1 \leq \lambda_2 \leq \ldots$ be eigenvalues of $L$, and $N(\lambda)$ the number of $\lambda_k$ less than or equal to $\lambda > 0$.

**Theorem 5.** Suppose that a symbol $\sigma(x, \xi)$ satisfies (S1)–(S4) and (29). We rearrange the set $\{\sigma_{k,n}\}_{k \in \mathbb{N}^r, n \in \mathbb{Z}^d}$ into a non-decreasing order and denote it by $\{\mu_i\}_{i \in \mathbb{N}}$.

(a) There exists a positive constant $C$ such that

$$(1 - C)\mu_i \leq \lambda_i \leq (1 + C)\mu_i, \quad i \in \mathbb{N}.$$

(b) Let $M(\lambda)$ be the number of elements of the set $\{i : \mu_i \leq \lambda\}$ for $\lambda > 0$. Then

$$M\left(\frac{\lambda}{1 + C}\right) \leq N(\lambda),$$

with the constant $C$ given by (a). Moreover, if $C \leq 1$, we have $N(\lambda) \leq M\left(\frac{\lambda}{1 + C}\right)$.

**Proof.** We will use the following max–min and min–max principles [37].

Let $L$ be a lower semi-bounded self-adjoint operator in $L^2$ with the domain $D(L)$ having only the discrete spectrum. Let $\lambda_1 \leq \lambda_2 \leq \ldots$ be eigenvalues of $L$. Then the $k$th eigenvalue of $L$ is characterized by:

$$\lambda_k = \sup_{M_{k-1}} \inf \{ \langle u, Lu \rangle : u \in D(L), \ |u| = 1, \ u \perp M_{k-1} \},$$

$$= \inf_{M_k \in D(L)} \sup \{ \langle u, Lu \rangle : |u| = 1, \ u \perp M_k \}.$$
where $M_k$ denotes a $k$-dimensional subspace of $L^2$ and inf and sup are taken over all $k-1$ and $k$-dimensional subspaces, respectively.

By Theorem 4(a) we have $D(L) = M_{2,2}^\sigma$, where $L$ is the self-adjoint extension of $\sigma(x, D)|_{\phi_{1,k}}$.

Since $\{\mu_l\}_{l \in \mathbb{N}}$ is the rearrangement of $\{\sigma_{k,n}\}_{k \in \mathbb{N}^d, n \in \mathbb{Z}^d}$, for every $l \in \mathbb{N}$, there exists $k \in \mathbb{N}^d$ and $n \in \mathbb{Z}^d$ such that $\mu_l = \sigma_{k,n}$. In this case we define $\phi_l = \psi_{k,n}$. Set

$$M_{l-1} = \text{linear span of } \{\phi_1, \ldots, \phi_{l-1}\}.$$ 

Let $u \in D(L)$, $\|u\| = 1$ and $u \perp M_{l-1}$. Then, by Lemma 2

$$(1 - C)\mu_l \leq \langle u, \sigma(x, D)u \rangle.$$

Using (30), we obtain $(1 - C)\mu_l \leq \lambda_l$. Similarly, for $u \in M_l$, $\|u\| = 1$, we have

$$(1 + C)\mu_l \geq \langle u, \sigma(x, D)u \rangle.$$

This and (31) imply $\lambda_l \leq (1 + C)\mu_l$.

(b) If $\mu_l \leq \frac{1}{1+C}$, then $\lambda_l \leq \lambda$ follows from (a). Therefore,

$$M\left(\frac{\lambda}{1+C}\right) \leq N(\lambda).$$

Similarly, if $C \in (0, 1)$, $\mu_l \geq \frac{1}{1+C}$ implies $\lambda_l \geq \lambda$. Hence, $N(\lambda) \leq M(\frac{\lambda}{1+C})$. □

7. Pseudodifferential operators and Gabor frames

As we already mentioned, in [7,28] a class of elliptic PDOs is approximately diagonalized by the means of local trigonometric bases, and Gabor frames, respectively. The same can be done with PDOs introduced in Section 5. We will shortly explain how to prove Theorems 3(c), 4 and 5 using Gabor frames instead of Wilson bases. Local trigonometric bases can be treated analogously, so we omit the details (see also [28]). We restrict to the one-dimensional case for the sake of simple notation only.

Gabor frames are indispensable tool in time frequency analysis, and its main properties are well known [5]. Here we give only definitions and properties necessary for our purpose and refer to [5,13,16] and references therein for more information on general frame theory. Recall, a set of functions $\{g_{k,n}, k \in \mathbb{Z}\}$ is called a Gabor system if

$$g_{k,n}(x) = e^{2\pi i k x} g(x - ak) = M_{ak}T_{ak}g, \quad k, n \in \mathbb{Z},$$
for a fixed function \( g \) and time-frequency shift parameters \( a, b > 0 \). A Gabor system is a Gabor frame in \( L^2(\mathbb{R}) \) if there exists \( 0 < A < B < \infty \) such that
\[
A \|f\|^2 \leq \sum_{k,n \in \mathbb{Z}} |\langle g_{k,n}, f \rangle|^2 \leq B \|f\|^2.
\]

If \( A = B \) the frame is tight.

Approximate diagonalization of \( \Psi \)DOs given in Theorem 3(a) can be reformulated for Gabor systems (not necessarily Gabor frames) as follows.

**Theorem 6.** Let \( \{g_{k,n} \mid k,n \in \mathbb{Z} \} \) be a Gabor system with \( g \in \mathcal{S}(\mathbb{R}) \), and symbol \( \sigma \) satisfy (S1)-(S4).

(a) For every \( \alpha, \beta > 0 \) we have
\[
|\langle g_{k',n'}, \sigma(x, D)g_{k,n} \rangle - \sigma_{k,n} \langle g_{k',n'}, g_{k,n} \rangle| \leq C_{\alpha, \beta} \frac{\left(\sigma_{k,n} \sigma_{k',n'}\right)^{1/2}}{\sigma_{k,n} \sigma_{k',n'}} \left(\frac{1}{1 + |a(k+k')|^2} + \frac{1}{1 + |b(n+n')|^2}\right)^{1/2} \left(1 + |a(k+k')|^2\right)^{1/2} \left(1 + |b(n+n')|^2\right)^{1/2},
\]
where \( \sigma_{k,n} = \sigma(ak, bn) \) and \( \delta \) is the constant from the condition (S3).

(b) Assume that \( \{g_{k,n} \mid k,n \in \mathbb{Z} \} \) is a tight Gabor frame with \( A = B = 1 \). For
\[
\sigma(x, D)g_{k,n}(x) = \sigma_{k,n}g_{k,n}(x) + r_{k,n}(x)
\]
there exists \( C > 0 \) such that \( \|r_{k,n}(x)\| \leq C \sigma_{k,n} \).

**Proof.** (a) For a Gabor system \( \{g_{k,n} \mid k,n \in \mathbb{Z} \} \) equality (18) becomes
\[
W(g_{k,n}, g_{k',n'})(x, \xi) = C_{k,n} e^{-2\pi i b(n+n')x-2\pi i a k x} \times W(g, g) \left( x - \frac{b(n+n')}{2}, \xi - \frac{a(k+k')}{2} \right),
\]
where \( |C_{k,n}| = 1 \). Since \( g \in \mathcal{S}(\mathbb{R}) \), we use (7) and repeat the proof of Theorem 3(a) step by step. Part (b) then follows immediately.

It is easy to see that Theorem 6 can be used to prove Theorem 3(c) in the same way as we used Theorem 3(a) in its proof. The only novelty is use of the reconstruction
formula for tight frames which gives
\[
\langle g_{k,n}, \sigma(x, D)f \rangle = \frac{1}{A} \left( \langle g_{k,n}, f \rangle \sigma_{k,n} + \sum_{k',n' \in \mathbb{Z}^d} \langle g_{k',n'}, f \rangle \right) \\
\left( \langle g_{k,n}, \sigma(x, D)g_{k',n'} \rangle - \sigma_{k,n} \langle g_{k,n}, g_{k',n'} \rangle \right)
\]
for every \( f \in \mathcal{G}^{d(i)} \).

To prove Theorem 4 with the use of Gabor frames we only need the equality
\[
\|f\|_2^2 = \sum_{k,n \in \mathbb{Z}^d} |\langle g_{k,n}, f \rangle|^2
\]
which is valid for tight frames with \( A = 1 \).

Finally, in order to estimate eigenvalues of the elliptic \( \Psi \)DOs introduced in Section 5 by the means of Gabor systems, we need the following modification of Lemma 2, which is easy to prove.

**Lemma 3.** Let \( f \in L^2 \) and symbol \( \sigma(x, \xi) \) satisfying (S1)–(S4), \( \sigma(x, \xi) \to \infty \) as \( |x| + |\xi| \to \infty \).

(a) For a tight Gabor frame \( \{g_{k,n} = M_{bn} T_{ak} \theta \}_{k,n \in \mathbb{Z}^d} \) with \( A = 1 \) there exists \( C_{a,b} > 0 \) such that for every \( f \in L^2(\mathbb{R}) \)
\[
(1 - C_{a,b}) \sum_{k,n \in \mathbb{Z}^d} |\langle g_{k,n}, f \rangle|^2 \sigma_{k,n} \\
\leq \langle f, \sigma(x, D)f \rangle \leq (1 + C_{a,b}) \sum_{k,n \in \mathbb{Z}^d} |\langle g_{k,n}, f \rangle|^2 \sigma_{k,n},
\]
where \( \sigma_{k,n} = \sigma(ak, bn) \).

(b) For an orthogonal Gabor system \( \{\tilde{g}_{k,n} = M_{dn} T_{ck} g \}_{k,n \in \mathbb{Z}^d} \) there exists \( C_{c,d} > 0 \) such that for every \( f \in \text{span}\{\tilde{g}_{k,n}\}_{k,n \in \mathbb{Z}^d} \)
\[
(1 - C_{c,d}) \sum_{k,n \in \mathbb{Z}^d} |\langle \tilde{g}_{k,n}, f \rangle|^2 \delta_{k,n} \\
\leq \langle f, \sigma(x, D)f \rangle \leq (1 + C_{c,d}) \sum_{k,n \in \mathbb{Z}^d} |\langle \tilde{g}_{k,n}, f \rangle|^2 \delta_{k,n},
\]
where \( \delta_{k,n} = \sigma(ck, dn) \).

Recall, if the elements of a Gabor system \( \{M_{dn} T_{ck} \theta \}_{k,n \in \mathbb{Z}^d} \) are mutually orthogonal then \( cd > 1 \). On the other hand if an atom \( g \) is a sufficiently "nice" function, e.g., the
Gaussian, and the family \( \{ M_{bn} T_{ak} \xi \}_{k,n \in \mathbb{Z}} \) is a tight Gabor frame, we have \( ab < 1 \) [13]. Actually Wilson bases were discovered in the search for Gabor system’s “critical density” \( ab = 1 \) [8]. We now give the statement analogous to Theorem 5.

**Theorem 7.** Suppose that a symbol \( \sigma(x, \xi) \) satisfies \( (S1)-(S4) \) and \( (29) \). We rearrange the families \( \{ \sigma_{k,n} \}_{k,n \in \mathbb{Z}^d} \) and \( \{ \tilde{\sigma}_{k,n} \}_{k,n \in \mathbb{Z}^d} \) into a non-decreasing order and denote them by \( \{ \mu_l \}_{l \in \mathbb{N}} \) and \( \{ \tilde{\mu}_l \}_{l \in \mathbb{N}} \).

(a) For \( a, b, c, d > 0 \), such that \( ab < 1 \) and \( cd > 1 \) there exists \( C_{ab}, C_{cd} > 0 \) such that

\[
(1 - C_{ab}) \mu_l \leq \lambda_l \leq (1 + C_{cd}) \tilde{\mu}_l, \quad l \in \mathbb{N}.
\]

(b) Let \( M(\lambda) \) and \( \tilde{M}(\lambda) \) be the number of elements of the family \( \{ l : \mu_l \leq \lambda \} \) and \( \{ l : \tilde{\mu}_l \leq \lambda \} \) respectively, for \( \lambda > 0 \). Then

\[
M\left( \frac{\lambda}{1 + C_{cd}} \right) \leq N(\lambda),
\]

where the constant \( C_{cd} \) is given in (a). Moreover, if the constant \( C_{ab} \) in (a) is less than 1, we also have \( N(\lambda) \leq M\left( \frac{\lambda}{1 - C_{ab}} \right) \).

**Proof.** The proof, based on Lemma 3 and the min-max principle, is a slight modification of the proof of Theorem 5. \( \square \)

**References**


