ON THE UPPER AND LOWER MAJORANT PROPERTIES IN $L^p(G)$

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1. Introduction

Let $G$ be a compact abelian group with dual group $\Gamma$. We denote the circle group by $T$ and its dual group, the integers, by $Z$. For $f \in L^1(G)$ the Fourier transform, $\hat{f}$, is defined by

$$\hat{f}(\gamma) = \int_G f(x)\gamma(x)\,dx, \quad \gamma \in \Gamma,$$

where 'dx' denotes normalized Haar measure.

For $f, g \in L^1(G)$ we say as in (7) that $g$ is a majorant of $f$ if $|\hat{f}| \leq \hat{g}$. For $1 \leq p < \infty$ we say as in (3) that $L^p(G)$ has the upper majorant property (with constant $A(p)$) if

$$(1) \quad \|f\|_p \leq A(p)\|g\|_p \text{ whenever } f, g \in L^p(G) \text{ and } g \text{ is a majorant of } f.$$

As shown by Hardy and Littlewood in (7), $L^p(T)$ has the upper majorant property with unit constant when $p$ is an even integer. Their argument goes through, mutatis mutandis, for general $G$. The upper majorant property is equivalent to

$$(1') \quad \text{If } \phi \text{ is a complex-valued function on } \Gamma \text{ and } |\phi| \leq \hat{g} \text{ for some } g \in L^p(G) \text{ then } \phi = \hat{f} \text{ for some } f \in L^p(G) \text{ and } \|f\|_p \leq A(p)\|g\|_p$$

(see e.g. Theorem 3).

We say as in (3) that $L^p(G)$ has the lower majorant property (with constant $B(p)$) if

$$(2) \quad \text{Given } f \in L^p(G) \text{ then } f \text{ has a majorant } g \in L^p(G) \text{ such that } \|g\|_p \leq B(p)\|f\|_p.$$ 

We show in Theorem 3 that the lower majorant property is equivalent to

$$(2') \quad \text{Every function in } L^p(G) \text{ has a majorant in } L^p(G).$$

Assume now that $1 < p < \infty$ and $1/p + 1/q = 1$. Hardy and Littlewood proved that if $L^p(T)$ has the upper majorant property then $L^q(T)$ has the lower majorant property [(7) Theorem 1]. In this paper we improve and extend this theorem by proving:

**Theorem 1.** If $L^p(G)$ has the upper majorant property then $L^q(G)$ has the lower majorant property with the same constant (i.e. $B(q)$ may be taken to be $A(p)$).

The converse of Theorem 1 has been proved by Boas [see (3) Theorem 2].

The proof of Theorem 1 follows along the lines of Hardy and Littlewood's proof. We include most details for the sake of completeness. The main difference is our circumvention of a theorem of M. Riesz [(3) Lemma 4, (5) 100], which has no analogue for arbitrary groups. This is what allows us to extend their theorem to any compact abelian group and also to conclude that the constants are the same. (In (3) it is stated that Hardy and Littlewood proved the constants to be the same in [(7) Theorem 1]. However, this is not the case, since in the proof of [(7) Lemma 5] the constants change although the symbols do not.)

We denote by $N^p(\Gamma)$ the set of complex-valued functions $\phi$ on $\Gamma$ such that $|\phi| \leq \hat{f}$, for some $f \in L^p(\Gamma)$, with norm

$$
\|\phi\|_{N^p} = \inf\{\|f\|_p : |\phi| \leq \hat{f}\}.
$$

The set of functions on $\Gamma$ with finite support is denoted by $c_{00}(\Gamma)$. We prove:

**Theorem 2.** (i) $N^p(\Gamma)$ is a Banach space in which $c_{00}(\Gamma)$ is dense.

(ii) $N^p(\Gamma)$ is identified isometrically as the dual space of $N^q(\Gamma)$ under the pairing

$$
\langle \phi, \psi \rangle = \sum_{\gamma} \overline{\phi(\gamma)} \psi(\gamma), \quad \phi \in N^p(\Gamma), \ \psi \in N^q(\Gamma)
$$

where $\sum_{\gamma} |\phi(\gamma)\psi(\gamma)| < \infty$. In particular, $N^p(\Gamma)$ is reflexive.

The proof of Theorem 2 is similar to that of Theorem 1 and its converse. In fact, Theorem 1 and its converse can be deduced from Theorem 2 by a short argument (see Remark 1).

Theorems 1 and 2 are proved in § 3. In § 2 we discuss for which $p$ and $G$ the upper majorant property holds. We include an argument, communicated by Y. Katznelson, which expands on an example of Boas to show that $L^p(T)$ does not have the upper majorant property when $p$ is greater than 2 and not an even integer.

In § 4 we consider the relationship between $N^p(\Gamma)$ and $S^p(G)$, the set of functions in $L^p(G)$ with an unconditionally converging Fourier series in the $L^p$ norm. We show in Theorem 3 that $L^p(G)$ has the upper majorant property if and only if $S^p(G)^* = N^p(\Gamma)$. Thus when $L^p(G)$ has the upper majorant property, the dual space of $S^p(G)$ is identified with $N^q(\Gamma)$ under the pairing

$$
\langle f, \psi \rangle = \sum_{\gamma} \hat{f}(\gamma) \overline{\psi(\gamma)}, \quad f \in S^p(G), \ \psi \in N^q(\Gamma).
$$

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2. The upper majorant property

As mentioned above, the argument of Hardy and Littlewood shows that $L^p(G)$ has the upper majorant property with unit constant when $p$ is an even integer. Also, they showed that $L^3(T)$ does not have the upper majorant property with unit constant [see (7)]. Boas showed that $L^p(T)$ does not have the upper majorant property with unit constant when $p$ is greater than 2 and not an even integer [see (3)].

The following argument, which has been used by several authors in different contexts, shows that $L^p(T)$ does not have the upper majorant property when $p$ is greater than 2 and not an even integer. The argument was shown to this author by Y. Katznelson.

One first observes that if $P$ is a trigonometric polynomial, and if

$$P_m(e^{i\theta}) = P(e^{im\theta}) \ (m = 1, 2, \ldots),$$

then

$$\lim_{m \to \infty} \|PP_m\|_p = \|P\|_p^3.$$

Secondly, if $Q$ is a majorant of $P$, then $Q_m$ is a majorant of $P_m$, and

$$|(PP_m)^* (n)| = |\hat{P} \ast \hat{P}_m(n)| \leq |\hat{P}| \ast |\hat{P}_m| (n) \leq Q \ast \hat{Q}_m(n) = (QQ_m)^* (n),$$

($n \in \mathbb{Z}$)

(where $\ast$ denotes convolution), so that $QQ_m$ is a majorant of $PP_m$, $m = 1, 2, \ldots$.

Now the example of Boas gives trigonometric polynomials $P$ and $Q$, with $Q$ a majorant of $P$, and a constant $A > 1$ such that

$$\frac{\|P\|_p^p}{\|Q\|_p} > A.$$

Thus, choosing $m$ sufficiently large, one obtains trigonometric polynomials $Q' = QQ_m$ and $P' = PP_m$ with $Q'$ a majorant of $P'$ and

$$\frac{\|P'\|_p^p}{\|Q'\|_p} > A^2.$$

Since this process can be iterated, no constant $A(p)$ can exist. Thus $L^p(T)$ does not have the upper majorant property when $p$ is greater than 2 and not an even integer. Clearly the same is true of $L^p(G)$ whenever $T$ is a quotient group of $G$, i.e. whenever $\Gamma$ contains an element of infinite order.

As observed by Boas in (3), $L^p(T)$ does not have the upper majorant
property when $1 \leq p < 2$. This is the case whenever $G$ is infinite, since otherwise one would have

$(a)$ if $f \in L^p(G)$ and $\hat{f} \not\geq 0$, then $\phi \hat{f} \in L^p(G)$ for all $\phi$ with $|\phi(\gamma)| \leq 1$.

It would then follow that

$(b)$ if $f \in L^p(G)$ and $\hat{f} \not\geq 0$, then $f \in L^2(G)$

[see e.g. (5)]. Now $(b)$ fails to hold when $G = T$ [see (10) 128] and hence also if $T$ is a quotient of $G$. There remains the case when $\Gamma$ is an infinite torsion group. The following argument shows that $(b)$ fails to hold in this case as well.

Choose a sequence $\{\Gamma_n\} \subset \Gamma$ such that $\Gamma_n \cap \Gamma_m = \emptyset$, $n \neq m$, and each $\Gamma_n$ is the coset of a subgroup of order $r_n$, where $r_n \geq 2^n$. If $\hat{P}_n$ is the characteristic function of $\Gamma_n$, then

$$||P_n||_p = ||\hat{P}_n||_q = r_n^{1/q}$$

[see (8) Theorem 43.5]. Let $Q_n = (r_n^{-1}) \hat{P}_n$. Then for $p < 2$,

$$||Q_n||_p = r_n^{-1+1/q},$$

so

$$\sum_n ||Q_n||_p \leq \sum_n 2^{n(-1+1/q)} < \infty.$$  

Thus $\sum Q_n$ converges in $L^p(G)$ to (say) $f$ and $\hat{f} \not\geq 0$. Since $||Q_n||_2 = 1$ and $Q_n Q_m = 0$, $n \neq m$, $f \notin L^2(G)$.

3. The main results

We assume throughout that $1 < p < \infty$ and $1/p + 1/q = 1$. We first deal with some technicalities.

**Lemma 1.** If $f \in L^p(G)$, $g \in L^q(G)$ and $\hat{f} \geq 0$, $\hat{g} \geq 0$, then

$$\int_G f(x)\overline{g(x)} \, dx = \sum_{\gamma} \hat{f}(\gamma)\hat{g}(\gamma).$$

**Proof** [cf. (3), proof of Theorem 2]. There exists a sequence of trigonometric polynomials, $\{P_n\}$, such that $P_n * f \rightarrow \hat{f}$ in $L^p(G)$ and $0 \leq \tilde{P}_n \leq 1$. Thus

$$\sum_{\gamma} \tilde{P}_n(\gamma)\hat{f}(\gamma)\hat{g}(\gamma) = \int_G (P_n * f)(x)\overline{g(x)} \, dx \rightarrow \int_G f(x)\overline{g(x)} \, dx \text{ as } n \rightarrow \infty,$$

so $\sum_{\gamma} \hat{f}(\gamma)\hat{g}(\gamma)$ converges to $\int_G f(x)\overline{g(x)} \, dx$.

Recall that $N^p(\Gamma)$ is the set of complex-valued functions $\phi$ on $\Gamma$ such that $|\phi| \leq \hat{f}$ for some $f \in L^p(G)$, with norm

$$||\phi||_{N^p} = \inf\{||f||_p : |\phi| \leq \hat{f}\}.$$  

Since $L^p(G)$ is reflexive, the infimum above is assumed, and since $L^p(G)$ is uniformly convex, it is assumed uniquely.
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For $\phi \in N^p(\Gamma)$ we denote by $m_p(\phi)$ (or $m_p(\phi)$ if $\phi = \hat{\phi}$) the unique element $f \in L^p(G)$ such that $\|f\|_p = \|\phi\|_{N^p}$ and $\hat{f} \geq |\phi|$. Note that $m_p(|\phi|) = m_p(\phi)$.

If $f \in L^p(G)$, let
\[
\lambda_q(f) = \frac{|f|^{p-1}}{\|f\|_{p-1}^p} \text{sgn} f, \quad \text{where} \quad \text{sgn} z = \begin{cases} \frac{z}{|z|}, & z \neq 0 \\ 0, & z = 0. \end{cases}
\]

Then $\lambda_q(f)$ is the unique element $h \in L^q(G)$ such that $\|h\|_q = 1$ and $\int_G f(x) \overline{h(x)} \, dx = \|f\|_p$.

The following lemma is the essential ingredient in the proofs of Theorems 1 and 2. It characterizes those functions $g \in L^q(G)$ with non-negative transform for which $\lambda_q(g)$ also has non-negative transform.

**Lemma 2.** Let $g \in L^p(G)$, $g \geq 0$, and let $f = m_p(g)$. Then

(i) $[\text{cf. (7) Lemma 3}] \lambda_q(f)^\wedge \geq 0$ and $\lambda_q(f)^\wedge(\gamma) = 0$ if $\hat{f}(\gamma) > \hat{g}(\gamma)$,

(ii) $\lambda_q(g)^\wedge \geq 0$ if and only if $g = f$,

(iii) $m_q(\lambda_q(f)) = \lambda_q(f)$.

**Proof.** (i) Let $h = \lambda_q(f)$. Since $\hat{f} \geq 0, f(-x) = \overline{f(x)}$. Thus
\[
h(-x) = \overline{h(x)},
\]
so $h$ is real-valued.

For $\gamma \in \Gamma$, define
\[
F_\gamma(t) = \|f + t\gamma\|_p, \quad t \text{ real}.
\]
Then $F_\gamma'(0)$ exists [see (9) 230-1], and
\[
F_\gamma'(0) = \text{Re} \, \hat{h}(\gamma) \quad [\text{cf. (7) Lemma 2, (9) Lemma 15.10}]
\]
\[
= \hat{h}(\gamma).
\]
If $t \geq 0$, then $(f + t\gamma)^\wedge \geq \hat{\gamma}$, so
\[
F_\gamma'(t) = \|f + t\gamma\|_p \geq \|f\|_p = F_\gamma'(0).
\]
Thus
\[
\hat{h}(\gamma) = F_\gamma'(0) \geq 0.
\]
If $\hat{f}(\gamma) > \hat{g}(\gamma)$, then $(f + t\gamma)^\wedge \geq \hat{\gamma}$ for small negative $t$ as well as for $t \geq 0$, so $\hat{h}(\gamma) = F_\gamma'(0) = 0$.

(ii) Let $k = \lambda_q(g)$. If $\hat{k} \geq 0$, then
\[
\|g\|_p = \int_G g(x) \overline{k(x)} \, dx = \sum_\gamma \hat{g}(\gamma) \hat{k}(\gamma) \leq \sum_\gamma \hat{f}(\gamma) \hat{k}(\gamma) \leq \|f\|_p,
\]
so $g = f$. Statement (iii) follows in a similar manner, since
\[
\lambda_p(\lambda_q(f)) = f||f||_p.
\]

We may now give

**Proof of Theorem 1.** Suppose that $L^p(G)$ has the upper majorant property with constant $A(p)$, and that $f \in L^q(G)$. We will show that $f$ has a majorant $g \in L^q(G)$ such that $\|g\|_q \leq A(p)\|f\|_q$.

Choose a sequence of trigonometric polynomials, $\{P_n\}$, such that
$P_n \to f$ in $L^q(G)$. For each $n$, let $Q_n = m_q(P_n)$ and let $R_n = \lambda_p(Q_n)$.

Then by Lemma 2,

$$\hat{R}_n \geq 0 \quad \text{and} \quad \hat{R}_n(\gamma) = 0 \quad \text{if} \quad \hat{Q}_n(\gamma) \geq |\hat{P}_n(\gamma)|.$$ 

Thus

$$\|Q_n\|_q = \int_G Q_n(x)\overline{R_n(x)} \, dx = \sum_{\gamma} \hat{Q}_n(\gamma)\hat{R}_n(\gamma) = \sum_{\gamma} |\hat{P}_n(\gamma)|\hat{R}_n(\gamma) = \sum_{\gamma} \hat{P}_n(\gamma)\overline{S_n(\gamma)},$$

where

$$S_n = \sum_{\gamma} (\text{sgn} \hat{P}_n(\gamma))\overline{R_n(\gamma)}.$$ 

Now $|\hat{S}_n| \leq \hat{R}_n$, so $\|S_n\|_p \leq A(p)\|R_n\|_p \leq A(p)$.

Thus

$$\|Q_n\|_q = \sum_{\gamma} \hat{P}_n(\gamma)\overline{S_n(\gamma)} = \int_G P_n(x)S_n(-x) \, dx \leq \|P_n\|_q \|S_n\|_p \leq A(p)\|P_n\|_q.$$ 

Hence the sequence $\{Q_n\}$ is bounded in $L^q(G)$, so it has a subsequence converging weakly to (say) $g \in L^q(G)$. Since $\hat{P}_n(\gamma) \to \hat{f}(\gamma)$, $\gamma \in \Gamma$, and $\|P_n\|_q \to \|f\|_q$, we necessarily have that $|\hat{f}| \leq \hat{g}$ and $\|g\|_q \leq A(p)\|f\|_q$.

We may also give

**Proof of Theorem 2.**

(i) That $N^p(\Gamma)$ is a Banach space will follow from (ii). We now show that $c_{00}(\Gamma)$ is dense in $N^p(\Gamma)$.

If $\phi \in N^p(\Gamma)$, let $f = m_p(\phi)$. Choose a sequence of trigonometric polynomials $\{P_n\}$ such that $0 \leq P_n \leq 1$ and $P_n * f \to f$ in $L^p(G)$.

Let $\phi_n = \hat{P}_n \hat{f} \in c_{00}(\Gamma)$. Then

$$|\phi - \phi_n| = |(1 - \hat{P}_n(\gamma))\phi(\gamma)| \leq (1 - \hat{P}_n(\gamma))|\hat{f}(\gamma)| = (f - P_n * f)^\gamma, \quad \gamma \in \Gamma,$$

so $\|\phi - \phi_n\|_N \leq \|f - P_n * f\|_p \to 0$ as $n \to \infty$.

(ii) Let $N^q(\Gamma)^\ast$ denote the dual space of $N^q(\Gamma)$ and $\|\|_\ast$ the norm in $N^q(\Gamma)^\ast$.

First suppose that $\phi \in N^p(\Gamma)$. Let $f = m_p(\phi)$. If $\psi \in N^q(\Gamma)$, $g = m_q(\psi)$, then

$$\left| \sum_{\gamma} \phi(\gamma)\overline{\psi(\gamma)} \right| \leq \sum_{\gamma} |\phi(\gamma)||\psi(\gamma)| \leq \sum_{\gamma} |\hat{f}(\gamma)||\hat{g}(\gamma)| = \int_G f(x)g(x) \, dx \leq \|f\|_p \|g\|_q = \|\phi\|_N \|\psi\|_{N^q}.$$ 

Thus $\phi^\ast = \langle \phi, \cdot \rangle \in N^q(\Gamma)^\ast$ and $\|\phi^\ast\|_\ast \leq \|\phi\|_N$. 

Suppose now that $F \in \mathcal{N}^q(\Gamma)^*$. Define $\phi$ on $\Gamma$ by $\phi(\gamma) = F(\gamma)$. Since $c_0(\Gamma)$ is dense in $\mathcal{N}^q(\Gamma)$, to complete the proof it is enough to show that $\phi \in \mathcal{N}^p(\Gamma)$ and $\|\phi\|_{\mathcal{N}^q} \leq \|F\|_*$. Let $J$ denote the collection of finite subsets of $\Gamma$, directed by inclusion. For $J \in \mathcal{J}$, let $\chi_J$ denote the characteristic function of $J$ and let $\phi_J = \phi \chi_J$, $f_J = m_p(\phi_J)$. We claim that it is enough to show that $\|f_J\|_p \leq \|F\|_*$, since then the net $\{f_J\}_{J \in \mathcal{J}}$ has a subnet converging weakly to an element $f \in L^p(G)$. Now $f$ satisfies $\|f\|_p \leq \|F\|_*$ and $|\phi| \leq \hat{f}$, so that $\phi \in \mathcal{N}^p(\Gamma)$ and $\|\phi\|_{\mathcal{N}^p} \leq \|f\|_p \leq \|F\|_*$. For $J \in \mathcal{J}$, let $\hat{h}_J = \lambda_q(f_J)$. Lemma 2 implies that $\hat{h}_J \geq 0$ and $\hat{h}_J(\gamma) = 0$ if $f_J(\gamma) > |\phi_J(\gamma)|$. Thus, as in the proof of Theorem 1,

$$\|f_J\|_p = \sum_{\gamma} \phi_J(\gamma)\hat{S}_J(\gamma), \text{ where } |\hat{S}_J| \leq \hat{h}_J.$$ Therefore

$$\|f_J\|_p = F(\chi_J\hat{S}_J) \leq \|F\|_* \|\chi_J\hat{S}_J\|_{\mathcal{N}^q} \leq \|F\|_* \|\hat{h}_J\|_q = \|F\|_*.$$ 

Remark 1. If $L^p(G)$ has the upper majorant property with constant $A(p)$ then it follows that $\hat{f} \rightarrow f$ maps $\mathcal{N}^p(\Gamma)$ to $L^p(G)$ with norm not exceeding $A(p)$. Taking adjoints, one sees that $f \rightarrow \hat{f}$ maps $L^q(G)$ ($= L^p(G)^*$) into $\mathcal{N}^q(\Gamma)$ ($= \mathcal{N}^p(\Gamma)^*$) with norm not exceeding $A(p)$, so $L^q(G)$ has the lower majorant property with constant $A(p)$. In this way, Theorem 1 follows from Theorem 2. The converse of Theorem 1 follows in a similar manner.

Remark 2. For $J \in \mathcal{J}$, let $X^q(J)$ denote the set of all complex-valued functions $\phi$ on $J$, with norm

$$\|\phi\|_{X^q} = \inf\{\|f\|_p : \hat{f}(\gamma) \geq |\phi(\gamma)|, \gamma \in J\},$$
and let $Y^p(J)$ denote the same set, but with norm

$$\|\phi\|_{Y^p} = \inf\{\|g\|_p : \hat{g}(\gamma) \geq |\phi(\gamma)|, \gamma \in J, \hat{g}(\gamma) = 0, \gamma \notin J\}.$$

Then in a manner similar to the proof of Theorem 2, one shows that $Y^p(J)$ is isometric to the dual space of $X^q(J)$, under the pairing

$$\langle \phi, \psi \rangle = \sum_{\gamma} \phi(\gamma)\psi(\gamma), \quad \phi \in Y^p(J), \quad \psi \in X^q(J).$$

Suppose now that $G = T$, $J_n = \{0, \pm 1, \ldots, \pm n\} \subset \mathbb{Z}$. $\pi_n : X^q(J_n) \rightarrow L^q(T)$ is defined by

$$\pi_n(\phi)(e^{i\theta}) = \sum_{|k| \leq n} \psi(k)e^{ik\theta}, \quad \psi \in X^q(J_n),$$
and $\pi_n^* : L^p(T) \rightarrow Y^p(J_n)$ is defined by

$$\pi_n^*(g)(k) = \hat{g}(k), \quad |k| \leq n, \quad g \in L^p(T).$$
When $L^p(T)$ is identified with $L^q(T)^*$ in the usual way, and $Y^p(J_n)$ with $X^q(J_n)^*$ as above, then $\pi_n^*$ is (essentially) the adjoint of $\pi_n$. In particular, $\|\pi_n\| = \|\pi_n^*\|$. In the notation of (7), $\|\pi_n\| = \lambda_n(q)$ and $\|\pi_n^*\| = \mu_n(q)$, so that $\lambda_n(q) = \mu_n(q)$. This is precisely the statement of [(7) Lemma 3].

4. Majorants and unconditionally converging series

We denote by $S^p(G)$ the set of functions in $L^p(G)$ with an unconditionally converging Fourier series in the $L^p$ norm. $S^p(G)$ is a Banach space with norm

$$\|f\|_{S^p} = \sup_{J \in \mathcal{J}} \left\| \sum_{\gamma \in J} \hat{f}(\gamma) \gamma \right\|_p, \quad f \in S^p(G),$$

where $\mathcal{J}$ as before denotes the collection of finite subsets of $\Gamma$. For unconditional convergence and related notions, the reader is referred to (4). The space $S^p(G)$ has been considered in (1) and (2). By a theorem of Grothendieck, $S^p(G) = L^2(G)$ when $1 \leq p < 2$.

For a subset $X$ of $L^p(G)$, $X^\wedge$ denotes the set $\{ \hat{f} : f \in X \}$. The following theorem shows the relationship between $S^p(G)$, $N^p(\Gamma)$, and the upper and lower majorant properties.

**Theorem 3.** Let $1 < p < \infty$, $1/p + 1/q = 1$. Then the following statements are equivalent.

1. $L^p(G)$ has the upper majorant property (with constant $A(p)$).
2. $N^p(\Gamma) \subset L^p(G)^\wedge$.
3. $N^p(\Gamma) = S^p(G)^\wedge$.
4. If $f \in L^p(G)$ and $|\hat{f}| \in L^p(G)^\wedge$, then $f \in S^p(G)$.
5. If $f \in L^p(G)$ and $|\hat{f}| \geq 0$, then $f \in S^p(G)$.
6. $L^q(G)$ has the lower majorant property (with constant $A(p)$).
7. $N^q(\Gamma) \supset L^q(G)$.
8. $S^p(G)^*$ is identified with $N^q(\Gamma)$ under the pairing

$$\langle f, \psi \rangle = \sum_{\gamma} \hat{f}(\gamma) \psi(\gamma), \quad f \in S^p(G), \psi \in N^q(\Gamma),$$

where $\sum_{\gamma} |\hat{f}(\gamma)\psi(\gamma)| < \infty$.

**Proof.** (1) $\Rightarrow$ (3). Since $f \in S^p(G)$ implies $|\hat{f}| \in S^p(G)$ [see e.g. (1) Lemma 1], one always has that $S^p(G)^\wedge \subset N^p(\Gamma)$. If $\phi \in N^p(\Gamma)$, $f = m_p(\phi)$, then

$$\left\| \sum_{\gamma} \phi(\gamma) \gamma \right\|_p \leq A(p)\|f\|_p, \quad J \in \mathcal{J},$$

because $|\phi| \leq \hat{f}$ and (1) holds. Since $L^p(G)$ is weakly complete, the series $\sum_{\gamma} \phi(\gamma) \gamma$ is weakly subseries convergent in $L^p(G)$, and hence

$$\langle f, \psi \rangle = \sum_{\gamma} \hat{f}(\gamma) \psi(\gamma), \quad f \in S^p(G), \psi \in N^q(\Gamma),$$

where $\sum_{\gamma} |\hat{f}(\gamma)\psi(\gamma)| < \infty$. 

This is precisely the statement of [(7) Lemma 3].
unconditionally convergent by the Orlicz–Pettis theorem [see (4) 60].

Thus \( \phi \in S^p(G)^\gamma \).

(3) \( \Rightarrow \) (4). If \( f \in L^p(G) \) and \( |\hat{f}| \in L^p(G)^\gamma \), then \( \hat{f} \in N^p(\Gamma) = S^p(G)^\gamma \), so \( f \in S^p(G) \).

(4) \( \Rightarrow \) (5) is immediate.

(5) \( \Rightarrow \) (3). If \( \phi \in N^p(\Gamma) \), \( f = m_\rho(\phi) \), then \( \hat{f} \geq 0 \) and \( f \in L^p(G) \), so \( f \in S^p(G) \). Since \( |\phi| \leq \hat{f} \), \( \sum \phi(\gamma)\gamma \) converges unconditionally in \( L^p(G) \), so \( \phi \in S^p(G)^\gamma \). Thus \( N^p(\Gamma) \subset S^p(G)^\gamma \), and hence \( N^p(\Gamma) = S^p(G)^\gamma \).

(3) \( \Rightarrow \) (2) is immediate.

(2) \( \Rightarrow \) (1). If \( \gamma \in \Gamma \), then the mapping \( \phi \rightarrow \phi(\gamma) \) is continuous on \( N^p(\Gamma) \). Since (2) holds, it follows from the Closed Graph Theorem that \( \hat{f} \rightarrow f \) is continuous when considered as a mapping from \( N^p(\Gamma) \) to \( L^p(G) \). Let the norm of this mapping be \( A(p) \). If \( f, g \in L^p(G) \) with \( g \) a majorant of \( f \), then

\[
\|f\|_p \leq A(p)\|\hat{f}\|_{N^p} = A(p)\|m_\rho(f)\|_p \leq A(p)\|g\|_p,
\]

so \( L^p(G) \) has the upper majorant property.

(3) \( \Rightarrow \) (8) follows directly from Theorem 2.

(8) \( \Rightarrow \) (7). Let \( g \in L^q(G) \). Then \( g \) gives rise to a continuous linear functional on \( S^p(G) \) via \( f \rightarrow \int_G f(x)g(x) \, dx \). Thus there exists \( \psi \in N^q(\Gamma) \) such that

\[
\int_G f(x)g(x) \, dx = \sum_{\gamma} \hat{f}(\gamma)\psi(\gamma), \quad f \in S^p(G).
\]

Now \( \hat{g}(\gamma) = \int_G \gamma^{-1}(x)g(x) \, dx = \psi(\gamma^{-1}), \gamma \in \Gamma \), so \( \hat{g} \in N^q(\Gamma) \).

(7) \( \Rightarrow \) (6) follows in a manner similar to the proof of (2) \( \Rightarrow \) (1).

(6) \( \Rightarrow \) (1) is [(3) Theorem 2].

Note that the proof of the equivalence of (1)–(5) is valid for \( p = 1 \).

We conclude with several comments about \( S^p(G)^\ast \).

Remark 3. Assume that \( L^p(G) \) has the upper majorant property. Then Theorems 2 and 3 imply that \( S^p(G) \) is reflexive. This is in fact the case for any \( p < \infty \) [see (2) Theorem 4.9]. Condition (8) of Theorem 3 says that, in the characterization of \( S^p(G)^\ast \) given by [(2) Theorem 4.9], it is sufficient to consider sums consisting of one term. The author conjectures that this is the case for any \( p < \infty \).

One always has that \( S^p(G) \) coincides with the 'Derived algebra' of Helgason [see (1) Theorem 2 or (2) Theorem 4.5]. The Derived algebra norm is given by

\[
\|f\|_0 = \sup\{\|f \ast g\|_p : g \in L^p(G), \|g\|_\infty \leq 1\}.
\]
One also has that
\[ \|f\|_0 = \sup \left\{ \left\| \sum_{\gamma} a(\gamma) \hat{f}(\gamma) \gamma \right\|_p : \|a\|_\infty \leq 1 \right\} \]
[see (2) Lemma 4.3, Theorem 4.6].

Assume now that \( L^p(G) \) has the upper majorant property with unit constant. If \( f \in S^p(G) \) and \( \hat{g} = |\hat{f}| \), then clearly
\[ \|f\|_0 = \|g\|_p = \|\hat{f}\|_{N^p}. \]
Thus the identification in (8) is isometric when \( S^p(G) \) is endowed with the derived algebra norm.

Added in proof: John Fournier has informed me that he has shown that \( L^p(G) \) does not have the upper majorant property when \( \Gamma \) is an infinite torsion group and \( p \) is greater than two and not an even integer.

REFERENCES


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