On the Stability of Wavelet and Gabor Frames (Riesz Bases)

Zhang Jing

Communicated by Akram Aldroubi

ABSTRACT. If the sequence of functions \( \{\phi_{j,k}\} \) is a wavelet frame (Riesz basis) or Gabor frame (Riesz basis), we obtain its perturbation system \( \{\psi_{j,k}\} \) which is still a frame (Riesz basis) under very mild conditions. For example, we do not need to know that the support of \( \phi \) or \( \psi \) (or \( \phi \) or \( \psi \)) is compact as in [14]. We also discuss the stability of irregular sampling problems. In order to arrive at some of our results, we set up a general multivariate version of Littlewood–Paley type inequality which was originally considered by Lemarie and Meyer [17], then by Chui and Shi [9], and Long [16].

1. Introduction

Frames were introduced by Duffin and Schaeffer [13] to study an irregular sampling problem, i.e., expansions of functions in \( L^2[0, 1] \) in complex exponents \( \exp(i\lambda_n x) \), where \( \lambda_n \) is not equivalent \( 2\pi n \) (see [20]). Frame analysis is an important topic in both wavelet theory and time-frequency analysis. Let \( H \) be a Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \|\cdot\| = \langle \cdot, \cdot \rangle^{\frac{1}{2}} \). A sequence \( \{f_n\} \subset H \) is called a frame if there are constants \( A \) and such that for every \( f \in H \)

\[
A\|f\|^2 \leq \sum_{n \in N} |\langle f, f_n \rangle|^2 \leq \|f\|^2,
\]

(1.1)

where \( N \) is the set of positive integers. The constants \( A \) and \( B \) are called bounds of the frame. If \( A = B \), we say it is a tight frame. If only the right-hand inequality is satisfied for all \( f \in H \), then \( \{f_n\} \) is called a Bessel sequence with bound \( B \). For a positive constant \( M \), we know that \( \{f_n\} \) is a Bessel sequence with bound \( M \) if and only if, for every finite sequence of scalars \( \{c_n\} \),

\[
\left\| \sum_n c_n f_n \right\|^2 \leq M \sum_n |c_n|^2.
\]

(1.2)

(see [20]) or if and only if (1.2) is satisfied for every sequence \( \{c_n\} \) in \( l^2 \) [8, 16].

Math Subject Classifications. 26B05, 42B10, 42C99.
Keywords and Phrases. frame, Gabor system, Riesz basis, stability, wavelet.
Acknowledgements and Notes. Dedicated to the memory of Professor Long Rui Lin.
It is widely known that every element of a separable Hilbert space can be represented as an infinite series by orthogonal bases. Although a frame needs not be a basis, it can be used to write an element of the space as a series. For any frame \( \{ f_n \} \), there exists a dual frame \( \{ \tilde{f}_n \} \) such that

\[
f = \sum_n < f, \tilde{f}_n > f_n = \sum_n < f, f_n > \tilde{f}_n, \quad \forall f \in H. \tag{1.3}
\]

If \( \{ f_n \}_{n \neq n_0} \) is not a frame for any \( n_0 \in \mathbb{N} \), we say it is exact. An exact frame is a basis. The expansion \( f = \sum_n c_n f_n \) is not unique when \( \{ f_n \} \) is an inexact frame. Among all choices of \( \{ c_n \} \), \( < f, \tilde{f}_n > \) is best because it minimizes the quantity \( \sum_n |c_n|^2 \) [10]. It is an advantage in applications that the elements in a frame may not be independent. We use Daubechies's example to explain the reason. Let \( u_1 = (1, 0), u_2 = (0, 1), e_1 = u_2, e_2 = -\sqrt{3}u_1 - \frac{1}{2}u_2, e_3 = \sqrt{3}u_1 - \frac{1}{2}u_2, (u_1, u_2) \) constitutes an orthonormal basis for \( \mathbb{R}^2 \), and \( (e_1, e_2, e_3) \) is a tight frame with bounds \( \frac{3}{2} \). For all elements \( f \) in \( \mathbb{R}^2 \), we have

\[
f = \sum_{j=1}^2 < f, u_j > u_j \quad \text{or} \quad f = \frac{2}{3} \sum_{j=1}^3 < f, e_j > e_j.
\]

If we add \( \alpha_j \epsilon \) to the coefficients \( < f, u_j > \), where \( \alpha_j \) are independent random variables with mean zero and variance 1, \( \epsilon \in \mathbb{R} \), then the expected error on the reconstruction will be

\[
E \left( \left\| f - \sum_{j=1}^2 (< f, u_j > + \alpha_j \epsilon) u_j \right\|^2 \right) = 2\epsilon^2.
\]

If we add \( \alpha_j \epsilon \) to the frame coefficients \( < f, e_j > \), then we have

\[
E \left( \left\| f - \sum_{j=1}^3 (< f, e_j > + \alpha_j \epsilon) e_j \right\|^2 \right) = \frac{4}{3}\epsilon^2.
\]

This is why we can store the coefficients \( < f, f_n > \) with low precision and still reconstruct \( f \) with comparatively much higher precision [10]. In signal and image processing, the redundancy of frames leads to the signal being less affected by the presence of noise, and reconstruction from sampling done at relatively low precision.

Riesz basis is a special kind of frame. A sequence \( \{ f_n \} \) of Hilbert space \( H \) is a Riesz basis if and only if it is the image of an orthonormal basis under a bounded invertible linear operator \( U : H \to H \) (see [20]). Riesz basis is a bounded unconditional basis. Another equivalent definition of Riesz basis in \( H \) is that \( \text{span}\{\sum_{i=1}^n c_i f_i\} = H \) and

\[
A \left( \sum_n |c_n|^2 \right) \leq \left\| \sum_n c_n f_n \right\|^2 \leq B \left( \sum_n |c_n|^2 \right), \quad \forall \{c_n\} \in l^2. \tag{1.4}
\]

A and B are called Riesz bounds for \( \{ f_n \} \). If we say that \( \{ f_n \} \) is a Riesz basis with bounds \( A \) and \( B \), we mean that \( A \) and \( B \) are its frame bounds since its Riesz bounds and frame bounds coincide (see [6]). The terms exact frame, bounded unconditional basis, and Riesz basis are equivalent (see [20]).

In applications, we can reconstruct uniquely and stably any element \( f \in H \) from the sequence of coefficients \( < f, f_n > \) and the frame bounds \( A \) and \( B \) [10]. In some cases, however, it could happen that \( \{ f_n \} \) are not known exactly. For example, for the case of wavelet frames in \( \mathbb{R} \), i.e., sequences \( \{ \varphi_{j,k} \} \), \( \varphi \) is replaced by another function \( \psi \) because of problems of numerical computation. We
whenever $\sum |c_n|^2 \leq 1$.

Let $\delta \in R$, by Lemma 1, for all $x \in I_r$,

$$1 - e^{i\delta x} = \left(1 - \frac{\sin \pi \delta}{\pi \delta} e^{i2\pi \delta}\right) + \sum_{m=1}^{\infty} \frac{(-1)^m 2\delta \sin \delta \pi e^{i2\pi m \delta}}{m^2 - \delta^2} \cos mx$$

$$+ \sum_{m=1}^{\infty} \frac{(-1)^{m+r+1} 2\delta \cos \pi \delta e^{i2\pi m \delta}}{\pi \left(\left(m - \frac{1}{2}\right)^2 - \delta^2\right)} \sin \left(m - \frac{1}{2}\right)x.$$

Let $\{c_n\}$ be an arbitrary finite sequence of scalars such that $\sum |c_n|^2 \leq 1$. Since it is permitted to interchange the order of summation, by triangle inequality, we see

$$\left|\sum_n c_n(e^{inx} - e^{i\lambda_n x})\right| = \left|\sum_n c_n \left(1 - e^{i\lambda_n x}\right)\right| \leq S_1 + S_2 + S_3,$$

where $\delta_n = \lambda_n - n$, and

$$S_1 = \left|\sum_n \left(1 - \frac{\sin \pi \delta_n e^{i2\pi \delta_n}}{\pi \delta_n} \right) c_n e^{inx}\right| \leq 1 - \frac{\sin \pi L}{\pi L},$$

$$S_2 = \left|\sum_n c_n e^{inx} \sum_{m=1}^{\infty} \frac{(-1)^m 2\delta_n \sin \pi \delta_n e^{i2\pi m \delta_n}}{\pi \left(m^2 - \delta_n^2\right)} \cos mx\right| \leq \sum_{m=1}^{\infty} \frac{2L \sin \pi L}{\pi \left(m^2 - L^2\right)},$$

$$S_3 = \left|\sum_n c_n e^{inx} \sum_{m=1}^{\infty} \frac{(-1)^m 2\delta_n \cos \pi \delta_n e^{i2\pi m \delta_n}}{\pi \left(\left(m - \frac{1}{2}\right)^2 - \delta_n^2\right)} \sin \left(m - \frac{1}{2}\right)x\right|$$

$$\leq \sum_{m=1}^{\infty} \frac{2L \cos \pi L}{\pi \left(\left(m - \frac{1}{2}\right)^2 - L^2\right)}.$$ 

Because the series

$$\sum_{m=1}^{\infty} \frac{2L}{\pi \left(m^2 - L^2\right)} \quad \text{and} \quad \sum_{m=1}^{\infty} \frac{2L}{\pi \left((m - \frac{1}{2})^2 - L^2\right)}$$

are partial fraction expansions of the functions $\frac{1}{\pi L} - \cot \pi L$ and $\tan \pi L$, respectively [18, p. 62–64], we have

$$\left|\sum_n c_n \left(e^{inx} - e^{i\lambda_n x}\right)\right| \leq B_1(L) = 1 - \cos \pi L + \sin \pi L.$$ 

When $L < \frac{1}{4}$, $B_1(L) = 1 - \cos \pi L + \sin \pi L < 1$. The proof is over. \hfill \square

The following multivariate version of Kadec's $\frac{1}{4}$-Theorem is due to Favier and Zalik [14].

**Theorem 1.**

Assume that $|n_m - \lambda_{nm}| \leq L < \frac{1}{4}$, $m = 1, \ldots, d$, then $\{e^{inx} - e^{i\lambda_n x}\}_{n \in \mathbb{Z}^d}$ is a Bessel sequence in $L^2(I_r)$ with bound

$$B_1(L) = 1 - \cos \pi L + \sin \pi L, \quad B_d(L) = \left(B_{d-1}^{\frac{1}{2}} + B_1^{\frac{1}{2}} \left(1 + B_{d-1}^{\frac{1}{2}}\right)\right)^2 \quad (2.1)$$
want to know under what conditions if \( \{ \varphi_{j,k} \} \) is a wavelet frame or Riesz basis (Gabor frame or Riesz basis), then \( \{ \psi_{j,k} \} \) is also a frame or Riesz basis. In frame theory, the stability is also a very interesting problem [20].

We shall introduce the content of this paper. First, we supplement the general multivariate version of Kadec’s \( \frac{1}{4} \)-Theorem which was proved by Favier and Zalik [14]. Next, we discuss the stability of wavelet frames or Riesz bases for perturbation problems of mother wavelet and sampling. Finally, we study the stability problems of Gabor frames or Riesz bases.

In order to get some of our stability results, we deduce a general Littlewood–Paley type inequality in \( \mathbb{R}^d \) which was originally considered by Lemarié and Meyer [17], then by Chui and Shi [9], and Long [16].

We explain our notations throughout this paper.

If we do not note the domain of integration, we mean it is \( \mathbb{R}^d, d \in \mathbb{N} \).

Let the functions \( f \) and \( g \) be in \( L^2(\mathbb{R}^d) \), we define the Fourier transform, inner product, and norm as

\[
\hat{f}(\xi) = \int f(x) e^{-ix \cdot \xi} \, dx, \quad <f, g> = (2\pi)^{-d} \int f(x) \overline{g(x)} \, dx, \quad \|f\| = <f, f>^{\frac{1}{2}}.
\]

\( x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d, \quad \xi = (\xi_1, \xi_2, \ldots, \xi_d) \in \mathbb{R}^d, \quad x \cdot \xi = x_1 \xi_1 + x_2 \xi_2 + \cdots + x_d \xi_d, \quad dx = dx_1 dx_2 \cdots dx_d.

\( dx \) and \( d\xi \) are Lebesgue measure. By (1.5), we have

\[
\int f(x) \overline{g(x)} \, dx = (2\pi)^{-d} \int \hat{f}(\xi) \overline{\hat{g}(\xi)} \, d\xi, \quad \|f\|^2 = (2\pi)^{-2d} \int |\hat{f}(\xi)|^2 \, d\xi.
\]

If \( I \subset \mathbb{R}^d \) is a Lebesgue measurable set, we denote its measure as \( |I| \). If \( a = (a_1, a_2, \ldots, a_d) \in \mathbb{R}^d \), then \( |a| = (a_1^2 + a_2^2 + \cdots + a_d^2)^{\frac{1}{2}} \).

\( H \) is always used to denote Hilbert space.

\[
\text{supp } f = \{ x \in \mathbb{R}^d, f(x) \neq 0 \}.
\]

\( \chi_S(x) \) is the characteristic function of the set \( S \).

## 2. Kadec’s \( \frac{1}{4} \)-Theorem

In this section, \( \| \cdot \|_I = \| \cdot \|_{L^2(I)} \), \( I \subset \mathbb{R}^d, t = (t_1, t_2, \ldots, t_d), \lambda_k = (\lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kd}) \), where \( t, \lambda_k \in \mathbb{R}^d \).

For \( d = 1 \), we have the following two lemmas. Lemma 1 is an obvious corollary of Young [20, p. 14 Problem 9]. Lemma 2 is due to Kadec [20, p. 42–44]. He proved it when \( I = [-\pi, \pi] \). If \( I_r = I + 2r\pi, \) the proof is similar.

**Lemma 1.**

\( \{1, \cos nx, \sin(n - \frac{1}{2})x\}_{n=1}^{\infty} \) is an orthonormal basis in \( L^2(I_r) \), where \( I_r = [-\pi, \pi] + 2r\pi, \) \( r \in \mathbb{Z} \), and \( \mathbb{Z} \) is the set of integers.

**Lemma 2.**

If \( \{ \lambda_n \} \) is a sequence of real numbers for which \( |\lambda_n - n| \leq \frac{1}{4}, \forall n \in \mathbb{Z} \), then \( \{e^{in\pi} - e^{i\lambda_n\pi}\} \) is a Bessel sequence in \( L^2(I_r) \) with bound \( B_1(L) = 1 - \cos \pi L + \sin \pi L, \) where \( I_r \) is the same as Lemma 1.

**Proof.** It is sufficient to show that

\[
\left\| \sum c_n \left( e^{in\pi} - e^{i\lambda_n\pi} \right) \right\| \leq B_1(L) = 1 - \cos \pi L + \sin \pi L < 1,
\]
where \( n = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d, \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{nd}) \in \mathbb{R}^d, I_r = [-\pi, \pi]^d + 2r\pi, r \in \mathbb{Z}^d. \)

**Proof.** The assertion will be proved by induction on \( d. \) For \( d = 1, \) it is true by Lemma 2.

Let \( \{c_n\} \) be an arbitrary finite sequence, \( H_1 \subseteq \mathbb{R}^{d-1}, \) such that \( I_r = [-\pi, \pi]^d + 2r\pi = H_1 \times [(2r_d - 1)\pi, (2r_d + 1)\pi], \tilde{x} = (x_1, x_2, \ldots, x_{d-1}) \in H_1, \tilde{K} = (k_1, k_2, \ldots, k_{d-1}) \in \mathbb{Z}^{d-1}, \tilde{k}_d \) will be used instead of \( K. \) Since

\[
\left\| \sum_{n \in \mathbb{Z}^d} c_n \left( e^{in \cdot x} - e^{i\lambda_n \cdot x} \right) \right\|_r \leq \left\| \sum_n c_n e^{in \tilde{x} \cdot \tilde{x}} \left( e^{i\tilde{\lambda}_n \cdot \tilde{x}} - e^{i\lambda_n \cdot \tilde{x}} \right) \right\|_r + \left\| \sum_n c_n e^{i\lambda_n \cdot \tilde{x}} \left( e^{in \tilde{x} \cdot \tilde{x}} - e^{i\lambda_n \cdot \tilde{x}} \right) \right\|_r = I_1 + I_2,
\]

using inductive hypothesis and Parseval identity, we have

\[
I_1^2 = (2\pi)^{-d} \int_{\pi(2r_d - 1)}^{\pi(2r_d + 1)} \left( \sum_{n \in \mathbb{Z}^d} c_{\tilde{\lambda}_n} e^{i\tilde{\lambda}_n \cdot \tilde{x}} \right) \left( e^{i\tilde{\lambda}_n \cdot \tilde{x}} - e^{i\lambda_n \cdot \tilde{x}} \right) d\tilde{x} \leq B_{d-1}(L) \sum_{n \in \mathbb{Z}^d} |c_n|^2.
\]

\[
I_2^2 = (2\pi)^{-d} \int_{H_1} \int_{\pi(2r_d - 1)}^{\pi(2r_d + 1)} \left( \sum_{\tilde{\lambda}_d \in \mathbb{Z}} c_{\tilde{\lambda}_n} e^{i\tilde{\lambda}_n \cdot \tilde{x}} \right) \left( e^{i\tilde{\lambda}_n \cdot \tilde{x}} - e^{i\lambda_n \cdot \tilde{x}} \right) d\tilde{x} d\tilde{x} \leq (2\pi)^{-d} B_1(L) \sum_{n \in \mathbb{Z}^d} \left( \sum_{\tilde{\lambda}_d \in \mathbb{Z}} |c_{\tilde{\lambda}_n}|^2 \right)^{1/2} \left( \sum_{n \in \mathbb{Z}^d} |c_n|^2 \right)^{1/2},
\]

By triangle inequality,

\[
\left( (2\pi)^{-d} \int_{H_1} \left( \sum_{\tilde{\lambda}_d \in \mathbb{Z}} c_{\tilde{\lambda}_n} e^{i\tilde{\lambda}_n \cdot \tilde{x}} \right)^2 d\tilde{x} \right)^{1/2} \leq \left( \sum_{\tilde{\lambda}_d \in \mathbb{Z}} |c_{\tilde{\lambda}_n}|^2 \right)^{1/2} + B_{d-1}(L) \left( \sum_{n \in \mathbb{Z}^d} |c_n|^2 \right)^{1/2},
\]

thus,

\[
I_2 \leq B_1^2(L) \left( 1 + B_{d-1}^2(L) \right) \left( \sum_{n \in \mathbb{Z}^d} |c_n|^2 \right)^{1/2}.
\]

It follows that

\[
\left\| \sum_{n \in \mathbb{Z}^d} c_n \left( e^{in \cdot x} - e^{i\lambda_n \cdot x} \right) \right\|_r^2 \leq (I_1 + I_2)^2 \leq \left( B_{d-1}^2(L) + B_1^2(L) \left( 1 + B_{d-1}^2(L) \right) \right)^2 \sum_{n \in \mathbb{Z}^d} |c_n|^2.
\]

That is

\[
B_d(L) = \left( B_{d-1}^2(L) + B_1^2(L) \left( 1 + B_{d-1}^2(L) \right) \right)^2.
\]

This completes the proof. \( \square \)
Remark 1.

1. Favier and Zalik consider that for \( d = 1 \), Theorem 1 follows from Young [20] by a change of variable of the form \( x = y + (r + 1)\pi \). If \( x \in [r\pi, (r + 2)\pi] \), then \( y \in [-\pi, \pi] \), but

\[
\int_{r\pi}^{(r+2)\pi} \left| \sum_{n \in \mathbb{Z}} c_n \left( e^{in\pi} - e^{i\lambda_n x} \right) \right|^2 \, dx = \int_{-\pi}^{\pi} \left| \sum_{n \in \mathbb{Z}} c_n \left( e^{in(r+1)\pi} e^{in\pi y} - e^{i\lambda_n (y+(r+1)\pi)} \right) \right|^2 \, dy
\]

So the fact that Theorem 1 is true in \([-\pi, \pi]\) does not imply that it is still true in \( I_r \) directly by the change of variable. We have proven the result step by step following Lemma 1 and Lemma 2.

2. Parseval identity is true if \( \{e^{inx}\} \) is an orthonormal basis. Define

\[
\langle f, g \rangle = (2\pi)^{-d} \int_{I_r} f(x) \overline{g(x)} \, dx.
\]

The definition of this inner product is necessary to ensure that \( \{e^{inx}\} \) is an orthonormal basis in \( I_r \). If we define \( \langle f, g \rangle = \int_{I_r} f(x) g(x) \, dx \) as usual, then we have \( B_1(L) = 2\pi (1 - \cos \pi L + \sin \pi L) \). Thus, \( B_d(L) \) is different from (2.1). It will be more complicated. In the proof of Favier and Zalik, they neglect \( 2\pi \).

3. Wavelet

Given a function \( \varphi : \mathbb{R}^d \to \mathbb{C} \), for \( j \in \mathbb{Z}, k \in \mathbb{Z}^d \), define

\[
\psi_{j, k}(x) = a_j^d \varphi \left( a_j^d x - kb \right), \quad \psi_{j, k}^{(p)}(x) = a_j^d \varphi \left( a_j^d x - \lambda_j k b \right) .
\]

In this section, we shall consider two problems:

1. If \( \{\varphi_{j, k}\} \) is a wavelet frame (Riesz basis) for \( L^2(\mathbb{R}^d) \), find conditions to ensure that \( \{\psi_{j, k}\} \) is also a frame (Riesz basis) when \( \{\psi_{j, k}\} \) is in some sense “close” to the \( \{\varphi_{j, k}\} \).

2. If \( \{\varphi_{j, k}\} \) is a wavelet frame (Riesz basis) for \( L^2(\mathbb{R}^d) \), then under what conditions is \( \{\psi_{j, k}^{(p)}\} \) also a frame (Riesz basis).

We need three lemmas. Lemma 3 is due to Christensen and Heil [7], Lemma 4 and Lemma 5 are due to Favier and Zalik [14].

Lemma 3.

Let \( \{f_n\} \) be a frame (Riesz basis) in Hilbert space \( H \) with bounds \( A \) and \( B \). Assume \( \{g_n\} \subseteq H \) is such that \( \{f_n - g_n\} \) is a Bessel sequence with bound \( M < A \). Then \( \{g_n\} \) is a frame (Riesz basis) with bounds \( A \left[ 1 - (M/A)^{\frac{1}{2}} \right]^2 \) and \( B \left[ 1 + (M/B)^{\frac{1}{2}} \right]^2 \).

Lemma 4.

Let \( \{f_n\} \) be a frame (Riesz basis) in \( H \) with bounds \( A \) and \( B \), assume that

\[
M = \sum \| f_n - g_n \|^2 < A,
\]

then \( \{g_n\} \) is a frame (Riesz basis) in \( H \) with bound \( A \left[ 1 - (M/A)^{\frac{1}{2}} \right]^2 \) and \( B \left[ 1 + (M/B)^{\frac{1}{2}} \right]^2 \).

Lemma 5.

Let \( 0 < \alpha < 1, 0 < L < \frac{1}{2} \). If \( 0 < L < \pi^{-1} \cos^{-1}(\frac{1-\alpha \cdot 2^{1-d}}{\sqrt{2}}) - \frac{1}{4} \), then \( B_d(L) < \alpha \).
We know that if $\varphi \in L^2(R^d)$, $\{\varphi_{j,k}\}$ is a wavelet frame with bounds $A$ and $B$ for $L^2(R^d)$, then $\varphi$ has some implied properties. For example, there exist constants $C_1$ and $C_2$, such that

$$0 < C_1 \leq \int \frac{|\hat{\varphi}(\xi)|^2}{|\xi|^j} d\xi \leq C_2,$$

where $C_1, C_2$ are related to $a, b$ and $A, B$ (see [10]). Another property of $\varphi$ is that $\hat{\varphi}$ is essentially bounded since we can obtain a Littlewood–Paley type inequality as Theorem 2 shows. It was first considered by Meyer and Lemarié when $\hat{\varphi}$ has enough decay conditions, Chui and Shi when the decay conditions are canceled off in $L^2(R)$, and Long when $\varphi$ belongs to $L^2(R^d)$ and $a = 2, b = 1$ (see [9, 16, 17]).

**Theorem 2.**

If $\{\varphi_{j,k}\}$ is a wavelet frame with bounds $A$ and $B$, then

$$(2\pi b)^d A \leq \sum_{j \in \mathbb{Z}} \left|\hat{\varphi}(a^j \xi)\right|^2 \leq (2\pi b)^d B, \quad \text{a.e. } \xi \in R^d. \quad (3.3)$$

Especially, $\hat{\varphi}$ is essentially bounded.

**Proof.** By the Plancherel theorem and Parseval identity, we have

$$\sum_{k \in \mathbb{Z}^d} \left|<f, \varphi_{j,k}>\right|^2 = (2\pi)^{-2d} \sum_k \left|\int f(x) \overline{\varphi_{j,k}(x)} dx\right|^2 = (2\pi)^{-2d} \sum_k a_{j,k}^d \left|\int \hat{f}(a^j \xi) \overline{\hat{\varphi}(\xi)} e^{i b \cdot \xi} d\xi\right|^2$$

$$= (2\pi)^{-2d} \sum_k a_{j,k}^d \left|\sum_{n \in \mathbb{Z}^d} \int_{[-\pi/b, \pi/b]^d + 2\pi/n} \hat{\varphi}(a^j \xi) \overline{\hat{\varphi}(\xi + 2\pi/n)} e^{i b \cdot \xi} d\xi\right|^2$$

$$= (2\pi)^{-2d} \sum_k a_{j,k}^d \left|\int_{[-\pi/b, \pi/b]^d} \sum_n \hat{\varphi}(a^j (\xi + 2\pi/n)) \overline{\hat{\varphi}(\xi + 2\pi/n)} e^{i b \cdot \xi} d\xi\right|^2$$

$$= (2\pi)^{-2d} (2\pi/b)^d a_{j,d} \int_{[-\pi/b, \pi/b]^d} \sum_n \hat{\varphi}(a^j (\xi + 2\pi/n)) \overline{\hat{\varphi}(\xi + 2\pi/n)} d\xi$$

The last step is true because for arbitrary fixed $n$, the projection $m \rightarrow m - n$ is a bijection, $m$ runs over $\mathbb{Z}^d$ if and only if $l = m - n$ runs over $\mathbb{Z}^d$. The functions $f$ and $\varphi$ being in $L^2$, interchanging the order among summations and integrations is admitted.

Assume $E_j$ is the set of Lebesgue points for $|\hat{\varphi}(a^j \xi)|^2$, then $|E_j^c| = 0$. Thus, $|\bigcup_j E_j^c| = 0$. That is, almost every point of $R^d$ is a Lebesgue point for all $|\hat{\varphi}(a^j \xi)|^2$, $j \in \mathbb{Z}$. 

```latex
\text{...}
```
Let \( \xi_0 \) be an arbitrary Lebesgue point for all \( |\hat{\phi}(a^j \xi)|^2, j \in \mathbb{Z} \). For every fixed positive integer \( M \), set
\[
\hat{f}(\xi) = (2\pi)^d \chi_{Q_0}(\xi) \delta^{-d/2}.
\]
where \( Q_0 = Q(\xi_0, \delta) \) is a cube centered at \( \xi_0 \) with side \( \delta \). Choose \( \delta < a^{-M}b^{-1} \pi \), obviously, \( \|f\| = 1 \).

Now we have
\[
\sum_{|j| \leq M} \sum_{k} |<f, \varphi_{j,k}||^2 = (2\pi)^{-d} b^{-d} \sum_{|j| \leq M} \int_{Q(\xi_0, \delta)} (2\pi)^{2d} \delta^{-d} |\hat{\phi}(a^{-j} \xi)|^2 d\xi
\]
\[
= (2\pi b)^{-d} \sum_{|j| \leq M} \int_{Q(\xi_0, \delta)} |\hat{\phi}(a^{-j} \xi)|^2 d\xi \leq B.
\]

Let \( \delta \to 0 \) and \( M \to \infty \) consecutively, by Lebesgue Differential Theorem, we prove the right inequality.

Let \( \xi_0, \delta, M \) be stated as above, and
\[
\hat{f}(\xi) = (2\pi)^d \delta^{-d/2} \chi_{Q_0}(\xi),
\]
where \( Q_0 \) is a cube with side \( \delta, \xi_0 \in Q_0 \).

Consider
\[
I = \sum_{j > -M} \sum_{k} |<f, \varphi_{j,k}||^2
\]
\[
= (2\pi)^{-3d} b^{-d} \sum_{j > -M} a^{jd} \int \sum_{l} \hat{f}(a^j \xi) \overline{\hat{f}(a^l (\xi + 2l\pi b^{-1}))} \overline{\hat{\phi}(\xi)} \hat{\phi}(\xi + 2l\pi b^{-1}) d\xi
\]
\[
= (2\pi)^{-3d} b^{-d} \sum_{j > -M} a^{jd} \int |\hat{f}(a^j \xi)|^2 |\hat{\phi}(\xi)|^2 d\xi,
\]

since two points \( a^j \xi \) and \( a^l (\xi + 2l\pi b^{-1}) \) cannot be contained in \( Q_0 \) simultaneously if \( l \neq 0 \). By the definition of the frame,
\[
I = (2\pi b)^{-d} \sum_{j > -M} a^{jd} \delta^{-d} \int \chi_{Q_0} (a^j \xi) |\hat{\phi}(\xi)|^2 d\xi \geq A - \sum_{j > -M} \sum_{k} |<f, \varphi_{j,k}||^2 = A - I^*,
\]
where \( I^* = \sum_{j \leq -M} \sum_{k} |<f, \varphi_{j,k}||^2 \). If we can prove \( I^* = o(1) \), then by letting \( \delta \to 0 \) and \( M \to \infty \) consecutively, the left inequality is proved. Consider
\[
I^* = \sum_{j \leq -M} \sum_{k} |<f, \varphi_{j,k}||^2
\]
\[
= (2\pi)^{-3d} b^{-d} \sum_{j \leq -M} a^{jd} \int \sum_{l} \hat{f}(a^j \xi) \overline{\hat{f}(a^l (\xi + 2l\pi b^{-1}))} \overline{\hat{\phi}(\xi)} \hat{\phi}(\xi + 2l\pi b^{-1}) d\xi
\]
\[
\leq (2\pi)^{-3d} b^{-d} \sum_{j \leq -M} \sum_{l} \left( \int |\hat{\phi}(\xi)|^2 \right)^{\frac{1}{2}} \left( \int |\hat{\phi}(\xi + 2a^l \pi b^{-1})|^2 d\xi \right)^{\frac{1}{2}}.
\]

Observe that for fixed \( j < -M \), the number of summation index \( l \) is bounded by \( (\delta/2a^l \pi b^{-1})^d = (\delta b/2\pi)^d a^{-jd} \), so that
\[
I^* \leq (2\pi)^{-3d} b^{-d} \sum_{j \leq -M} (\delta b/2\pi)^d a^{-jd} \int |\hat{\phi}(\xi + 2a^l \pi b^{-1})|^2 d\xi
\]
On the Stability of Wavelet and Gabor Frames (Riesz Bases)

\[ \sum_{j \leq -M} a^{-jd} \int (2\pi)^{2d} \delta^{-d} \chi_{Q_d}(\xi) \left| \hat{\phi}(a^{-j} \xi) \right|^2 d\xi = (2\pi)^{-2d} \sum_{j \leq M} \int_{Q_{a^{-j}}} \left| \hat{\phi}(\xi) \right|^2 d\xi , \]

where \( Q_{a^{-j}} \) is a cube with side \( a^{-j} \delta \), and \( a^{-j} \xi_0 \in Q_{a^{-j}} \).

For every \( \varepsilon > 0 \), choose \( M \) to satisfy the following two inequalities:

\[ \max_i |\xi_0^i| > a^{-M}(a+1) \pi \quad \text{and} \quad \int_{E_M} |\hat{\phi}(\xi)|^2 d\xi < \varepsilon (2\pi)^{2d} , \]

where

\[ \xi_0 = (\xi_0^1, \xi_0^2, \ldots, \xi_0^d) \in Q_{a^{-j} \delta}, E_M = \left\{ \xi = (\xi_1, \xi_2, \ldots, \xi_d), \text{ and } \max_i |\xi_i| \geq \max_i |\xi_0^i| a^{M-1} \right\} . \]

Then for \( j \leq -M \), \( Q_{a^{-j} \delta} \subset E_M \) since \( \forall \xi \in Q_{a^{-j} \delta} \).

\[ \max_i |\xi_i| \geq \max_i |a^{-j} \xi_0^i| - a^{-j} \delta \geq a^{-j} \left( \max_i |\xi_0^i| - \delta \right) \geq a^{-j} \max_i |\xi_0^i| \left( 1 - \frac{a-1}{a+1} \right) = a^{-j} \frac{2}{a+1} \max_i |\xi_0^i| |\xi_0| = a^{M-1} \max_i |\xi_0^i| . \]

Assume \( j_1 < j_2 \leq -M \), we have \( Q_{a^{-j_1} \delta} \cap Q_{a^{-j_2} \delta} = \emptyset \). If there exists \( \xi \in Q_{a^{-j_1} \delta} \cap Q_{a^{-j_2} \delta} \), then for \( i = 1, 2, \ldots, d \)

\[ |\xi_i - a^{-j_1} \xi_0^i| \leq a^{-j_1} \delta, \quad |\xi_i - a^{-j_2} \xi_0^i| \leq a^{-j_2} \delta, \]

since \( a^{-j_1} \xi_0 \in Q_{a^{-j_1} \delta}, a^{-j_2} \xi_0 \in Q_{a^{-j_2} \delta} \). By triangle inequality,

\[ |a^{-j_1} - a^{-j_2}| |\xi_0^i| \leq \left( a^{-j_1} + a^{-j_2} \right) \delta, \quad |\xi_0^i| \leq \frac{a^{-j_1} + a^{-j_2}}{a^{-j_1} - a^{-j_2}} \delta = a^{-j_1 + j_2} - a^{-j_1} - j_2 \delta \leq a + 1 \delta . \]

This contradicts the choice of \( M \). Thus, \( I^* \leq (2\pi)^{-2d} \int_{E_M} |\hat{\phi}(\xi)|^2 d\xi < \varepsilon \). We arrive at our conclusion. \( \square \)

Remark 2. If \( \{\varphi_{j,k}\} \) is an orthonormal basis for \( L^2(R^d) \), then \( \sum_{j,k} |<f, \varphi_{j,k}>|^2 = \|f\|^2 \), \( A = B = 1 \), \( \sum_j |\hat{\phi}(a^j \xi)|^2 = (2\pi b)^d \).

Next we shall consider the stability of frame and Riesz basis when the mother wavelet \( \varphi \) has perturbation.

Theorem 3.

Let \( \varphi, \psi \in L^2(R^d), \{\varphi_{j,k}\} \) be a wavelet frame (Riesz basis) with bounds \( A \) and \( B \) for \( L^2(R^d) \). If

\[ \text{esssup} \sum_{j \in \mathbb{Z}} \left| \hat{\phi} \left( a^{-j} \xi + 2l \pi b^{-1} \right) - \hat{\psi} \left( a^{-j} \xi + 2l \pi b^{-1} \right) \right|^2 < (2\pi b)^d A , \]

then \( \{\psi_{j,k}\} \) is a wavelet frame (Riesz basis) with bounds \( A[1 - (M/A)\frac{1}{d}]^2 \) and \( B[1 + (M/A)\frac{1}{d}]^2 \), where

\[ M = (2\pi b)^{-d} \text{esssup} \sum_{j \in \mathbb{Z}} \left| \hat{\phi} \left( a^{-j} \xi + 2l \pi b^{-1} \right) - \hat{\psi} \left( a^{-j} \xi + 2l \pi b^{-1} \right) \right|^2 . \]
Proof. By (3.4) and Hölder inequality, for every $f \in L^2(R^d)$, we have

$$
\sum_{j \in \mathbb{Z}, k \in \mathbb{Z}^d} |\langle f, \varphi_{j,k} \rangle|^2 \\
= (2\pi)^{-3d} b^{-d} \sum_{j} a^{id} \int \sum_{l \in \mathbb{Z}^d} \hat{f}(a^j \xi) \hat{\varphi}(\xi) \hat{f}(a^j \xi + 2l \pi b^{-1}) \hat{\varphi}(\xi + 2l \pi b^{-1}) d\xi \\
\leq (2\pi)^{-3d} b^{-d} \sum_{j} \sum_{l} \left( a^{id} \int \left| \hat{f}(a^j \xi) \hat{\varphi}(\xi + 2l \pi b^{-1}) \right|^2 d\xi \right)^{\frac{1}{2}} \\
\leq (2\pi)^{-3d} b^{-d} \sum_{j} \left( \sum_{l} \int \left| \hat{f}(\xi) \hat{\varphi}(a^{-j} \xi + 2l \pi b^{-1}) \right|^2 d\xi \right)^{\frac{1}{2}} \\
\leq (2\pi)^{-3d} b^{-d} \sum_{j} \left( \sum_{l} \int \left| \hat{f}(\xi) \hat{\varphi}(a^{-j} \xi + 2l \pi b^{-1}) \right|^2 d\xi \right)^{\frac{1}{2}} \\
= (2\pi)^{-3d} b^{-d} \sum_{j} \int \left| \hat{f}(\xi) \hat{\varphi}(a^{-j} \xi + 2l \pi b^{-1}) \right|^2 d\xi.
$$

Substitute $\varphi = \tilde{\psi}$ for $\varphi$, we have

$$
\sum_{j,k} |\langle f, \varphi_{j,k} - \psi_{j,k} \rangle|^2 \\
\leq (2\pi)^{-3d} b^{-d} \sum_{j,l} \int \left| \hat{f}(\xi) \left( \hat{\varphi}(a^{-j} \xi + 2l \pi b^{-1}) - \hat{\psi}(a^{-j} \xi + 2l \pi b^{-1}) \right) \right|^2 d\xi \\
\leq (2\pi b)^{-d} \left( \operatorname{esssup} \sum_{j,l} \left| \hat{\varphi}(a^{-j} \xi + 2l \pi b^{-1}) - \hat{\psi}(a^{-j} \xi + 2l \pi b^{-1}) \right|^2 \right) \|f\|^2.
$$

The conclusion follows from Lemma 3. \qed

We shall consider an important special case that $\operatorname{supp} \varphi$ and $\operatorname{supp} \tilde{\psi}$ are compact.

**Theorem 4.**

Let $\varphi, \psi \in L^2(R^d), \{\varphi_{j,k}\}$ be a wavelet frame (Riesz basis) with bounds $A$ and $B$ for $L^2(R^d)$. If there exists $k_0 \in \mathbb{Z}^d$ such that

$$
\operatorname{supp} \varphi, \operatorname{supp} \tilde{\psi} \subset I_{k_0,b} = [-\pi/b, \pi/b]^d + 2k_0 \pi/b,
$$

and

$$
|\hat{\varphi}(\xi) - \hat{\psi}(\xi)| \leq \lambda |\hat{\varphi}(\xi)|, \quad \text{a.e. } \xi \in R^d,
$$

then $(\varphi_{j,k})$ is an unconditional basis of $L^2(R^d)$.
where \( \lambda < (A/B)^{1/2} \), then \( \{\psi_{j,k}\} \) is a wavelet frame (Riesz basis) with bounds \( A[1 - \lambda(A/B)^{1/2}]^2 \) and \( B[1 + \lambda(A/B)^{1/2}]^2 \).

**Proof.** Using the same trick as in the proof of Theorem 2, we have

\[
\left| \sum_{j \in \mathbb{Z}, k \in \mathbb{Z}^d} \langle f, \varphi_{j,k} \rangle \right|^2 = (2\pi)^{-2d} \sum_{j,k} \left| \int f(x) \overline{\varphi_{j,k}(x)} \, dx \right|^2
\]

\[
= (2\pi)^{-4d} \sum_{j,k} \left| \int \hat{f}(\xi) a^{-jd/2} \overline{\hat{\varphi}(a^{-j} \xi)} e^{ia^{-j} jk \xi} \, d\xi \right|^2
\]

\[
= (2\pi)^{-4d} \sum_{j,k} \left| \int_{I_{k_0,b}} \hat{f}(\xi) a^{jd/2} \overline{\hat{\varphi}(\xi)} e^{ibk \xi} \, d\xi \right|^2
\]

\[
= (2\pi)^{-4d} (2\pi b^{-1})^{-d} \sum_{j} a^{jd} \int \left| \hat{f}(a^{j} \xi) \overline{\hat{\varphi}(\xi)} \right|^2 \, d\xi
\]

\[
\leq (2\pi)^{-3d} b^{-d} \lambda^2 \sum_{j} a^{jd} \int \left| \hat{f}(a^{j} \xi) \overline{\hat{\varphi}(\xi)} \right|^2 \, d\xi = \lambda^2 \sum_{j,k} |\langle f, \varphi_{j,k} \rangle|^2 \leq \lambda^2 B \|f\|^2.
\]

If \( \lambda < (A/B)^{1/2} \), then \( M = \lambda^2 B < A \), we finish our proof. \( \square \)

**Remark 3.**

1. One may ask whether Theorem 4 is valid if \( \text{supp} \hat{\varphi} \) and \( \text{supp} \hat{\psi} \) are compact but not contained in \( I_{k_0,b} \). We say if \( \text{supp} \hat{\varphi} \) and \( \text{supp} \hat{\psi} \) are compact and for some \( \alpha > 0 \), \( \gamma > \alpha + d \)

\[
|\hat{\varphi}(\xi)| \leq C|\xi|^{\alpha}(1 + |\xi|)^{-\gamma},
\]

in Theorem 3 can be applied, because there always exists \( b_1 \) such that \( \text{supp} \hat{\varphi}, \text{supp} \hat{\psi} \subset [-\pi b_1^{-1}, \pi b_1^{-1}]^d \).

Suppose \( \{a^{jd/2} \varphi(a^{j} x - kb_2)\} \) is a wavelet frame which implies

\[
(2\pi b)^d A \leq \sum_{j \in \mathbb{Z}} |\hat{\varphi}(a^{j} \xi)|^2 \leq (2\pi b)^d B, \quad \text{a.e. } \xi \in \mathbb{R}.
\]

By Theorem 2, we take arbitrary \( b_0 \leq \min(b_1, b_2) \), then \( \{a^{jd/2} \varphi(a^{j} x - kb_0)\} \) is also a wavelet frame and \( \text{supp} \hat{\varphi}, \text{supp} \hat{\psi} \subset [-\pi b_0^{-1}, \pi b_0^{-1}]^d \) (see [10]).

2. When the mother wavelet \( \varphi \) has perturbation, by Theorem 3 and Theorem 4 we see that if \( \varphi \) and \( \psi \) are not smooth enough, then they must be "close" enough and if \( \varphi \) and \( \psi \) are infinitely differentiable, then \( \hat{\varphi} \) and \( \hat{\psi} \) need not be so "close." The Paley–Wiener Theorem says that the infinitely differentiable property of \( \varphi \) is a necessary condition for \( \text{supp} \hat{\varphi} \) being compact.

We shall study the sampling perturbation of wavelet frame (Riesz basis) in the following two theorems.

**Theorem 5.**

Let \( \varphi(x), |\xi||\hat{\varphi}(\xi)| \in L^2(\mathbb{R}^d), \{\varphi_{j,k}\} \) be a wavelet frame (Riesz basis) with bounds \( A \) and \( B \). If

\[
\sum_{j,k} |k - \lambda_{j,k}|^2 < (2\pi)^{2d} b^{-2} \left( \int |\xi|^2 |\hat{\varphi}(\xi)|^2 \, d\xi \right)^{-1} A,
\]
then \( \{\varphi_{j,k}^{(p)}\} \) is a wavelet frame (Riesz basis) with bounds \( A[1 - (M/A)^{\frac{1}{2}}] \) and \( B[1 + (M/A)^{\frac{1}{2}}] \), where

\[
M = (2\pi)^{-2d} b^2 \int |\xi|^2 |\hat{\varphi}(\xi)|^2 d\xi \sum_{j,k} |k - \lambda_{j,k}|^2.
\]

**Proof.** By the Plancherel theorem, we have

\[
\|\varphi_{j,k} - \varphi_{j,k}^{(p)}\|^2 = (2\pi)^{-d} \int |\varphi_{j,k}(x) - \varphi_{j,k}^{(p)}(x)|^2 dx
\]

\[
= (2\pi)^{-2d} a^{-jd} \int |\hat{\varphi}(a^{-j} \xi)|^2 |e^{-ia^{-j} b k \xi} - e^{-ia^{-j} b \lambda_{j,k} \xi}|^2 d\xi
\]

\[
= (2\pi)^{-2d} \int |\hat{\varphi}(\xi)|^2 |e^{-ibk \xi} - e^{-ib\lambda_{j,k} \xi}|^2 d\xi
\]

\[
= (2\pi)^{-2d} b^2 \sum_{j,k} |k - \lambda_{j,k}|^2 \int |\xi|^2 |\hat{\varphi}(\xi)|^2 d\xi.
\]

So that

\[
\sum_{j,k \in \mathbb{Z}^d, k \in \mathbb{Z}^d} \left\| \varphi_{j,k} - \varphi_{j,k}^{(p)} \right\|^2 \leq (2\pi)^{-2d} b^2 \sum_{j,k} |k - \lambda_{j,k}|^2 \int |\xi|^2 |\hat{\varphi}(\xi)|^2 d\xi = M.
\]

The conclusion follows from Lemma 4.

**Remark 4.** The condition \( |\xi| \hat{\varphi}(\xi) \in L^2(\mathbb{R}^d) \) is easily satisfied, e.g., if \( \varphi \) satisfies

\[
|\hat{\varphi}(\xi)| \leq C|\xi|^{-1}(1 + |\xi|)^{-\alpha}, \quad \xi \neq 0,
\]

for \( \alpha > d/2 \), then \( |\xi| \hat{\varphi}(\xi) \in L^2(\mathbb{R}^d) \).

**Theorem 6.**

Let \( \varphi(x) \in L^2(\mathbb{R}^d), \{\varphi_{j,k}\} \) be a wavelet frame (Riesz basis) with bounds \( A \) and \( B \), supp \( \hat{\varphi} \subset I_{k_0,b} = [-\pi/b, \pi/b]^d + 2k_0 \pi/b \). If

\[
|k - \lambda_{j,k}| \leq L < \min \left( \frac{1}{4}, \pi^{-1} \cos^{-1} \left( \frac{1 - \alpha g^{1-d}}{\sqrt{2}} \right) \right),
\]

then \( \{\varphi_{j,k}^{(p)}\} \) is a wavelet frame (Riesz basis) with bounds \( A[1 - (M/A)^{\frac{1}{2}}] \) and \( B[1 + (M/A)^{\frac{1}{2}}] \), where \( n \in \mathbb{Z}^d, \alpha < AB^{-1}, M = B \cdot B_d(L) \).

**Proof.** By Lemma 5, we have \( B_d(L) < \alpha < AB^{-1} \leq 1 \). By Theorem 1 and Theorem 2,

\[
\sum_{j,k \in \mathbb{Z}^d, k \in \mathbb{Z}^d} \left| \int f(x) \varphi(x) \varphi_{j,k}^{(p)}(x) dx \right|^2
\]

\[
= (2\pi)^{-2d} \sum_{j,k} \left| \int f(x) a^{-jd/2} \left( \varphi(a^{-j} x - k b) - \varphi(a^{-j} x - \lambda_{j,k} b) \right) dx \right|^2
\]

\[
= (2\pi)^{-4d} \sum_{j,k} \left| \int \hat{f}(\xi) a^{-jd/2} \hat{\varphi}(a^{-j} \xi) \left( e^{ia^{-j} b k \xi} - e^{ia^{-j} b \lambda_{j,k} \xi} \right) d\xi \right|^2
\]

\[
= (2\pi)^{-4d} b^{-2d} \sum_{j,k} a^{-d} \left| \int_{I_{k_0,b}} \hat{f}(a^{-1} b^{-1} \xi) \hat{\varphi}(b^{-1} \xi) \left( e^{ik \xi} - e^{i\lambda_{j,k} \xi} \right) d\xi \right|^2
\]
On the Stability of Wavelet and Gabor Frames (Riesz Bases)

\begin{align*}
(2\pi)^{3d}b^{2d}B_d(L) & \sum_j a_j b^{1 - \alpha} \int_{[-b,a]} \left| \hat{f}(a^{-1}b^{-1} \xi) \overline{\hat{\phi}(b^{-1} \xi)} \right|^2 d\xi \\
& \leq (2\pi b)^{-d}B_d(L) \text{esssup}_j \left| \hat{\phi}(a^{-1}b^{-1} \xi) \right|^2 \| f \|^2 \leq B \cdot B_d(L) \| f \|^2 .
\end{align*}

That completes our proof. \qed

**Remark 5.**

1. Theorem 6 is valid on the condition that \( \text{supp} \hat{\phi} \) is compact. See Remark of Theorem 4.

2. By Favier and Zalik [14], if \( \| \varphi(x + h) - \varphi(x) \| \leq C |h|^\alpha, 0 < \alpha \leq 1 \), \( \{ \psi_{j,k} \} \) is a wavelet frame (Riesz basis) with bounds \( A \) and \( B \) for \( L^2(\mathbb{R}^d) \), and \( \delta = b^{2a}C^2 \sum_{j,k} |k - \lambda_{j,k}|^{2a} < A \), then \( \{ \psi_{j,k}^{(p)} \} \) is also a frame (Riesz basis) in \( L^2(\mathbb{R}^d) \). Their result and our Theorem 5 both say that if \( \varphi \) is not infinitely differentiable, then some of those \( \{ \lambda_{j,k} \} \) which satisfy \( \lim_{|k| \to \infty} |k - \lambda_{j,k}| = 0 \) for every \( j \) and \( \lim_{|j| \to \infty} |k - \lambda_{j,k}| = 0 \) for every \( k \) keep \( \{ \psi_{j,k}^{(p)} \} \) being a frame. So that for arbitrary fixed \( \varepsilon, |k - \lambda_{j,k}| < \varepsilon \) cannot guarantee the stability. Our Theorem 6 says that if \( \text{supp} \hat{\phi} \) is a bounded set, then we can relax the coercive condition on \( \{ \lambda_{j,k} \} \), that is, for any small enough \( L, |k - \lambda_{j,k}| < L \) keeps the stability. By the Paley–Wiener Theorem we conclude that the smoothness of the mother wavelet \( \varphi \) decides the stability of the wavelet frame \( \{ \psi_{j,k} \} \) in a sampling perturbation problem.

4. **Gabor Systems**

We first introduce the notations in this section.

\[
\begin{align*}
& a = (a_1, a_2, \ldots, a_d) \\
& b = (b_1, b_2, \ldots, b_d) \\
& [a^{-1}, b^{-1}] = [-a_1^{-1}, b_1^{-1}] \times \cdots \times [-a_d^{-1}, b_d^{-1}] \\
& j = (j_1, j_2, \ldots, j_d) \in \mathbb{Z}^d \\
& k = (k_1, k_2, \ldots, k_d) \in \mathbb{Z}^d \\
& n = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d \\
& \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \\
& \beta = (\beta_1, \beta_2, \ldots, \beta_d) \\
& \varphi_{j,k}(x) = e^{ib \cdot x} \varphi(x - k) \\
& \psi_{j,k}(x) = e^{ib \cdot x} \varphi(x - k) \\
& \text{where the function } \varphi : \mathbb{R}^d \to \mathbb{C} .
\end{align*}
\]

Let \( \{ \psi_{j,k} \} \) be a Gabor frame (Riesz basis). We shall discuss two questions as in Section 3.

1. Find conditions to guarantee that \( \{ \psi_{j,k} \} \) is still a frame (Riesz basis), where \( \psi \) is a perturbation of \( \varphi \). See Theorems 8, 9, and 10.

2. Find conditions to ensure that \( \{ \psi_{j,k}^{(p)} \} \) (Theorems 12 and 13), \( \{ \psi_{j,k}^{(q)} \} \) (Theorems 14 and 15), and \( \{ \psi_{j,k}^{(p,q)} \} \) (Theorem 16) are still frames (Riesz bases).

In Theorem 7, we will show that if \( \{ \psi_{j,k} \} \) is a Gabor frame, then both \( \varphi \) and \( \hat{\phi} \) are essentially bounded.

**Theorem 7.**

Let \( \varphi(x) \in L^2(\mathbb{R}^d) \), \( \{ \psi_{j,k} \} \) be a Gabor frame (Riesz basis) with bounds \( A \) and \( B \), then we have Littlewood–Paley type inequalities

\[
(2\pi)^d A \leq \sum_{j \in \mathbb{Z}^d} |\hat{\phi}(\xi - jb)|^2 \leq (2\pi)^d B, \quad \text{a.e. } \xi \in \mathbb{R}^d . \tag{4.1}
\]
\[ \tilde{b} A \leq \sum_{k \in Z^d} |\varphi(x - ka)|^2 \leq \tilde{b} B, \quad \text{a.e. } x \in R^d. \] (4.2)

**Proof.** For every \( f \in L^2(R^d) \), let \( I = [-\pi a^{-1}, \pi a^{-1}] \), \( I_n = I + 2\pi na^{-1}, n \in Z^d \), then

\[
\begin{align*}
\sum_{k \in Z^d} |\langle f, \varphi_{j,k} \rangle|^2 &= (2\pi)^{-2d} \sum_k \left| \int f(x)e^{-i<x,jb>}\varphi(x-ka)dx \right|^2 \\
&= (2\pi)^{-4d} \sum_k \left| \int \hat{f}(\xi)\hat{\varphi}(\xi - jb)e^{i<\xi,ka>}d\xi \right|^2 \\
&= (2\pi)^{-4d} \sum_k \left| \int_{I_n} \hat{f}(\xi)\hat{\varphi}(\xi - jb)e^{i<\xi,ka>}d\xi \right|^2 \\
&= (2\pi)^{-4d} \sum_k \left| \int_{I_n} \hat{f}(\xi)\hat{\varphi}(\xi - 2n\pi a^{-1} - jb)e^{i<\xi,ka>}d\xi \right|^2 \\
&= (2\pi)^{-3d} \sum_k \int_{I_n} \hat{f}(\xi)\hat{\varphi}(\xi - 2n\pi a^{-1} - jb)e^{i<\xi,ka>}d\xi \\
&= (2\pi)^{-3d} \sum_k \int_{I_n} \hat{f}(\xi)\hat{\varphi}(\xi - 2n\pi a^{-1} - jb)e^{i<\xi,ka>}d\xi.
\end{align*}
\]

If \( \xi_0 \) is a Lebesgue point of \( \hat{\varphi}(\xi + 2\pi a^{-1} - jb) \) for every \( n \) and \( j \), let

\[
\hat{f}(\xi) = (2\pi)^d \chi_{Q(\xi_0, \delta)}(\xi)\delta^{-d/2},
\]

where \( 0 < \delta < \min_{1 \leq m \leq d}(\pi a_m^{-1}) \). \( Q(\xi_0, \delta) \) is a cube centered at \( \xi_0 \) with side \( \delta \).

Since \( ||f|| = 1 \), by the frame condition

\[
A \leq \sum_{j,k \in Z^d} |\langle f, \varphi_{j,k} \rangle|^2 = (2\pi)^{-d} \sum_k \delta^{-d} \int_{Q(\xi_0, \delta)} |\hat{\varphi}(\xi - jb)|^2 d\xi \leq B.
\]

Let \( \delta \to 0 \), by Lebesgue Differential Theorem, (4.1) is obtained. The proof of (4.2) is similar because

\[
\begin{align*}
\sum_{j \in Z^d} |\langle f, \varphi_{j,k} \rangle|^2 &= (2\pi)^{-2d} \sum_j \left| \int f(x)e^{-i<x,jb>}\varphi(x-ka)dx \right|^2 \\
&= (2\pi)^{-4d} \sum_j \left| \int_{I} f(x)f(x + 2\pi b^{-1})\varphi(x)\varphi(x + 2\pi b^{-1} - ka)dx \right|^2 \\
&= (2\pi)^{-4d} \sum_j \left| \int_{I} \varphi(x)\varphi(x + 2\pi b^{-1} - ka)dx \right|^2 \\
&= (2\pi)^{-3d} \int_{I} \varphi(x)\varphi(x + 2\pi b^{-1} - ka)dx.
\end{align*}
\]

Let \( x_0 \) be a Lebesgue point of \( |\varphi(x + 2\pi b^{-1} - ka)|^2 \) for every \( l, k \in Z^d \), and

\[
f(x) = \chi_{Q_{\delta}(x_0)}(x)\delta^{-d/2}(2\pi)^{d/2},
\]

where \( 0 < \delta < \min_{1 \leq m \leq d}(b_m^{-1}) \). \( Q_{\delta}(x_0) \) is a cube centered at \( x_0 \) with side \( \delta \), then

\[
A \leq \sum_{j,k \in Z^d} |\langle f, \varphi_{j,k} \rangle|^2 = \delta^{-1} \sum_{k} \delta^{-d} \int_{Q_{\delta}(x_0)} |\varphi(x - ka)|^2 dx \leq B.
\]
Let $\delta \to 0$, then (4.2) is proved.

**Remark 6.** If $\{\psi_{j,k}\}$ is an orthonormal basis of $L^2(R^d)$, then $A = B = 1$, $\sum_j |\hat{\psi}(\xi - jb)|^2 = (2\pi)^d a$, and $\sum_k |\varphi(x - ka)|^2 = b$.

**Theorem 8.**

Let $\varphi, \psi \in L^2(R^d)$, and $\{\varphi_{j,k}\}$ be a Gabor frame (Riesz basis) with bounds $A$ and $B$ for $L^2(R^d)$. If

$$M' = \text{esssup} \sum_{k,l \in Z^d} \left| \varphi \left( x + 2l\pi b^{-1} - ka \right) - \psi \left( x + 2l\pi b^{-1} - ka \right) \right|^2 \leq b A,$$

then $\{\psi_{j,k}\}$ is a Gabor frame (Riesz basis) with bounds $A[1 - (M/A)^{1/2}]^2$ and $B[1 + (M/A)^{1/2}]^2$, where $M = b^{-1}M'$.

**Proof.** By (3.4) and the Hölder inequality,

$$\sum_{j,k \in Z^d} \left| \langle f, \varphi_{j,k} - \psi_{j,k} \rangle \right|^2$$

$$= (2\pi)^{-2d} \sum_{j,k} \left| \int f(x) \left( \varphi(x - ka) - \psi(x - ka) \right) e^{-i \langle jb, x \rangle} \, dx \right|^2$$

$$= (2\pi)^{-d} b^{-1} \sum_k \int_{(-\pi b^{-1}, \pi b^{-1})} \left| \sum_{l,l} f \left( x + 2l\pi b^{-1} - ka \right) \left( \varphi \left( x + 2\pi nb^{-1} - ka \right) - \psi \left( x + 2\pi nb^{-1} - ka \right) \right) \right|^2 \, dx$$

$$= (2\pi)^{-d} b^{-1} \sum_{k,l} \left| \int f(x) \left( \varphi \left( x + 2\pi lb^{-1} - ka \right) - \psi \left( x + 2\pi lb^{-1} - ka \right) \right) \right|^2 \, dx$$

$$\leq (2\pi)^{-d} b^{-1} \sum_{k,l} \left( \int \left| f \left( x + 2\pi lb^{-1} - ka \right) - \psi \left( x + 2\pi lb^{-1} - ka \right) \right|^2 \, dx \right)^{1/2} \left( \int \left| f \left( x + 2\pi lb^{-1} - ka \right) \right|^2 \, dx \right)^{1/2}$$

$$\leq (2\pi)^{-d} b^{-1} \left( \sum_{k,l} \int \left| f \left( x + 2\pi lb^{-1} - ka \right) \right|^2 \, dx \right)^{1/2} \left( \sum_{k,l} \int \left( \varphi \left( x - ka \right) - \psi \left( x - ka \right) \right)^2 \, dx \right)^{1/2}$$

$$= (2\pi)^{-d} b^{-1} \sum_{k,l} \int \left| f \left( x + 2\pi lb^{-1} - ka \right) \right|^2 \, dx$$

$$\leq b^{-1} \text{esssup} \sum_{k,l} \left| \varphi \left( x + 2\pi lb^{-1} - ka \right) - \psi \left( x + 2\pi lb^{-1} - ka \right) \right|^2 \|f\|^2 = b^{-1}M'\|f\|^2.$$

The assertion follows from Lemma 3. □
Theorem 9.
Let \( \varphi, \psi \in L^2(\mathbb{R}^d), \{\varphi_{j,k}\} \) be a Gabor frame (Riesz basis) with bounds \( A \) and \( B \) for \( L^2(\mathbb{R}^d) \).
If
\[
M' = \text{esssup} \sum_{j,l} \left| \hat{\varphi} \left( \xi + 2\pi l a^{-1} + j b \right) - \hat{\psi} \left( \xi + 2\pi l a^{-1} + j b \right) \right|^2 < (2\pi)^d \bar{a} A ,
\]
then \( \{\psi_{j,k}\} \) is a Gabor frame (Riesz basis) with bounds \( A[1 - (M/A)^{\frac{1}{2}}]^2 \) and \( B[1 + (M/A)^{\frac{1}{2}}]^2 \), where \( M = (2\pi)^{-d} \bar{a}^{-1} M' \).

Proof. Using the same trick as in the proof of Theorem 8, we have
\[
\sum_{j,k\in\mathbb{Z}^d} |<f, \varphi_{j,k} - \psi_{j,k}>|^2
= (2\pi)^{-2d} \sum_{j,k} \left| \int f(x) \left( \varphi(x - ka) - \psi(x - ka) \right) e^{-i j b, x} \, dx \right|^2
= (2\pi)^{-4d} \sum_{j,k} \left| \int \hat{f}(\xi + j b) \left( \hat{\varphi}(\xi) - e^{i k a, \xi} \hat{\psi}(\xi) \right) \, d\xi \right|^2
\leq (2\pi)^{-3d} \bar{a}^{-1} \left( \sum_{j,l} \left| \int \hat{f}(\xi + j b) \left( \hat{\varphi}(\xi + 2\pi l a^{-1}) - \hat{\psi}(\xi + 2\pi l a^{-1}) \right) \, d\xi \right|^2 \right)^{\frac{1}{2}}
\leq (2\pi)^{-3d} \bar{a}^{-1} \sum_{j,l} \left| \int \hat{f}(\xi + j b) \left( \hat{\varphi}(\xi + 2\pi l a^{-1}) - \hat{\psi}(\xi + 2\pi l a^{-1}) \right) \, d\xi \right|^2
\leq (2\pi)^{-d} \bar{a}^{-1} M' \|f\|^2 .
\]
By Lemma 3, we arrive at our conclusion. \( \square \)

Remark 7. Theorem 8 and Theorem 9 are symmetrical since \( \{e^{i j b, x} \varphi(x - ka)\} \) is a Gabor frame with bounds \( A \) and \( B \) if and only if \( \{e^{i k a, \xi} \hat{\varphi}(\xi - j b)\} \) is a Gabor frame with the same bounds.

Theorem 10.
Let \( \varphi, \psi \in L^2(\mathbb{R}^d), \{\varphi_{j,k}\} \) be a Gabor frame (Riesz basis) with bounds \( A \) and \( B \) for \( L^2(\mathbb{R}^d) \), \( \text{supp} \varphi, \text{supp} \psi \subset I_{n_0, b} \). If
\[
M' = \text{esssup} \sum_k |\varphi(x - ka) - \psi(x - ka)|^2 < \bar{b} A ,
\]
then \( \{\psi_{j,k}\} \) is a Gabor frame (Riesz basis) with bounds \( A[1 - (M/A)^{\frac{1}{2}}]^2 \) and \( B[1 + (M/A)^{\frac{1}{2}}]^2 \), where \( I_{n_0, b} = [-\pi b^{-1}, \pi b^{-1}] + 2\pi n_0 b^{-1}, n_0 \in \mathbb{Z}^d, M = \bar{b}^{-1} M' \).

Proof. By the Parseval identity and (3.4)
\[
\sum_{j,k \in \mathbb{Z}^d} |<f, \varphi_{j,k} - \psi_{j,k}>|^2
\]
On the Stability of Wavelet and Gabor Frames (Riesz Bases)

\[ = (2\pi)^{-d} \delta^{-1} \sum_{k} \int_{[-\pi b^{-1}, \pi b^{-1}]} \left| \sum_{n} f \left( x + 2n\pi b^{-1} \right) \left( \varphi \left( x + 2\pi nb^{-1} - ka \right) - \psi \left( x + 2\pi nb^{-1} - ka \right) \right) \right|^2 dx \]

\[ = (2\pi)^{-d} \delta^{-1} \sum_{k} \sum_{l} \left( f(x) \left( \varphi \left( x + 2\pi lb^{-1} - ka \right) - \psi \left( x + 2\pi lb^{-1} - ka \right) \right) \right) \left( f \left( x + 2\pi lb^{-1} \right) \left( \varphi(x - ka) - \psi(x - ka) \right) \right) dx . \]

For arbitrary given \( k \), if \( (x - ka) \in I_{n_0, b} \), then \( (x - ka + 2\pi lb^{-1}) \) is not contained in \( I_{n_0, b} \) assuming \( l \neq 0 \), so that

\[ \sum_{j,k} \left| \langle f, \varphi_{j,k} - \psi_{j,k} \rangle \right|^2 = (2\pi)^{-d} \delta^{-1} \sum_{k} \int |f(x)|^2 |\varphi(x - ka) - \psi(x - ka)|^2 dx \leq \delta^{-1} M'\|f\|^2 . \]

The result is derived. \[ \square \]

Combined with Theorem 7, we have the following.

**Corollary 1.**

Let \( \varphi, \psi \) satisfy conditions in Theorem 10, if

\[ |\varphi(x) - \psi(x)| \leq \lambda |\varphi(x)|, \quad \text{a.e. } x \in \mathbb{R}^d , \]

where \( \lambda < (A/B)^{\frac{1}{2}} \), then \( \{\psi_{j,k}\} \) is a Gabor frame (Riesz basis) with bounds \( A[1 - \lambda(B/A)^{\frac{1}{2}}]^2 \) and \( B[1 + \lambda(B/A)^{\frac{1}{2}}]^2 \).

**Remark 8.** Theorem 10 can be applied to the case that \( \text{supp} \varphi \) and \( \text{supp} \psi \) are bounded. Because there exist \( b_1 \) and \( b_2 \) such that \( \text{supp} \varphi \) and \( \text{supp} \psi \subset [-\pi b_1^{-1}, \pi b_1^{-1}] \) and that \( \{e^{i\langle j, b_1 \rangle} \varphi(x - ka)\} \) is a Gabor frame implies (4.2) to be true. If \( |\varphi(x)| \leq C(1 + |x|)^{-r}, \quad r > d, \) then for every \( b_0 < \min(b_1, b_2) \), \( \{e^{i\langle j, b_0 \rangle} \varphi(x - ka)\} \) is a Gabor frame.

**Theorem 11.**

Let \( \varphi, \psi \in L^2(\mathbb{R}^d) \), \( \{\varphi_{j,k}\} \) be a Gabor frame (Riesz basis) with bounds \( A \) and \( B \) for \( L^2(\mathbb{R}^d) \), \( \text{supp} \varphi, \text{supp} \psi \subset I_{n_0, a} \). If

\[ M' = \text{esssup} \sum_{j} \left| \hat{\varphi}(\xi - jb) - \hat{\psi}(\xi - jb) \right|^2 < (2\pi)^d \bar{a} A , \]

then \( \{\psi_{j,k}\} \) is a Gabor frame (Riesz basis) with bounds \( A[1 - (M/A)^{\frac{1}{2}}]^2 \) and \( B[1 + (M/A)^{\frac{1}{2}}]^2 \), where \( I_{n_0, a} = [-\pi a^{-1}, \pi a^{-1}] + 2\pi n_0a^{-1}, n_0 \in \mathbb{Z}^d, \quad M = (2\pi)^{-d} \bar{a}^{-1} M' . \)

Especially, if

\[ \left| \hat{\varphi}(\xi) - \hat{\psi}(\xi) \right| \leq \lambda |\hat{\varphi}(\xi)|, \quad \text{a.e. } \xi \in \mathbb{R}^d , \]

where \( \lambda < (A/B)^{\frac{1}{2}} \), then \( \{\psi_{j,k}\} \) is a Gabor frame (Riesz basis) with bounds \( A[1 - \lambda(B/A)^{\frac{1}{2}}]^2 \) and \( B[1 + \lambda(B/A)^{\frac{1}{2}}]^2 \).

**Proof.** See Theorem 10 and its corollary. \[ \square \]

We shall study the effect of sampling perturbation in the following four theorems.
Theorem 12.
Let $\varphi \in L^2(\mathbb{R}^d)$, $\{\varphi_{j,k}\}$ be a Gabor frame (Riesz basis) with bounds $A$ and $B$ for $L^2(\mathbb{R}^d)$. If
\[ M' = \text{esssup} \sum_{k,l} |\varphi(x + 2\pi lb^{-1} - ka) - \varphi(x + 2\pi lb^{-1} - \lambda_ka)|^2 < \tilde{b}A, \]
then $\{\varphi_{j,k}^{(\mu)}\}$ is a Gabor frame (Riesz basis) with bounds $A[1 - (M/A)^{1/2}]$ and $B[1 + (M/A)^{1/2}]^2$, where $M = \tilde{b}^{-1}M'$.

Proof. Since
\[
\sum_{j,k \in \mathbb{Z}^d} |<f, \varphi_{j,k} - \varphi_{j,k}^{(\mu)}>|^2 = (2\pi)^{-2d} \sum_{j,k} \left| \int f(x) \left( \varphi(x - ka) - \varphi(x - \lambda_ka) \right) e^{-i<x,jb>} dx \right|^2
\]
\[
= (2\pi)^{-d} \tilde{b}^{-1} \sum_{k} \int_{[-\pi b^{-1}, \pi b^{-1}]} \left| \sum_{n} f \left( x + 2\pi nb^{-1} \right) \left( \varphi \left( x + 2\pi nb^{-1} - ka \right) - \varphi \left( x + 2\pi nb^{-1} - \lambda_ka \right) \right) \right|^2 dx
\]
\[
\leq (2\pi)^{-d} \tilde{b}^{-1} \sum_{k,l} \int f(x) \left( \varphi \left( x + 2\pi lb^{-1} - ka \right) - \varphi \left( x + 2\pi lb^{-1} - \lambda_ka \right) \right)^2 dx
\]
\[
\leq \tilde{b}^{-1} M'\|f\|^2,
\]
by Lemma 3, we finish our proof.  

Theorem 13.
Let $\varphi \in L^2(\mathbb{R}^d)$, $\{\varphi_{j,k}\}$ be a Gabor frame (Riesz basis) with bounds $A$ and $B$ for $L^2(\mathbb{R}^d)$, $\text{supp} \varphi \subset I_{n_0,b,c} = [-\pi b^{-1} + c, \pi b^{-1} - c] + 2\pi n_0 b^{-1}$, where $0 \leq c < \pi b^{-1}$. If $|k - \lambda_k| \leq c$ and $M' = \text{esssup} \sum_k |\varphi(x - ka) - \varphi(x - \lambda_ka)|^2 < \tilde{b}A$, then $\{\varphi_{j,k}^{(\mu)}\}$ is a Gabor frame (Riesz basis) with bounds $A[1 - (M/A)^{1/2}]$ and $B[1 + (M/A)^{1/2}]^2$, where $M = \tilde{b}^{-1}M'$.

Proof. Because $\text{supp}(\varphi(x - ka) - \varphi(x - \lambda_ka)) \subset I_{n_0,b} = [-\pi b^{-1}, \pi b^{-1}] + 2\pi n_0 b^{-1}$, whenever $l \neq 0$, $\text{supp}(\varphi(x + 2\pi lb^{-1} - ka) - \varphi(x + 2\pi lb^{-1} - \lambda_ka))$ and $I_{n_0,b}$ do not overlap, we have
\[
\sum_{j,k \in \mathbb{Z}^d} |<f, \varphi_{j,k} - \varphi_{j,k}^{(\mu)}>|^2
\]
\[
= (2\pi)^{-2d} \sum_{j,k} \left| \int f(x) \left( \varphi(x - ka) - \varphi(x - \lambda_ka) \right) e^{-i<x,jb>} dx \right|^2
\]
\[
= (2\pi)^{-d} \tilde{b}^{-1} \sum_{k} \int_{[-\pi b^{-1}, \pi b^{-1}]} \left| \sum_{n} f \left( x + 2\pi nb^{-1} \right) \left( \varphi \left( x + 2\pi nb^{-1} - ka \right) - \varphi \left( x + 2\pi nb^{-1} - \lambda_ka \right) \right) \right|^2 dx
\]
\[
= (2\pi)^{-d} \tilde{b}^{-1} \sum_{k} \sum_{l} \int f(x) f \left( x + 2\pi lb^{-1} \right) \left( \varphi(x - ka) - \varphi(x - \lambda_ka) \right)^2 dx
\]
On the Stability of Wavelet and Gabor Frames (Riesz Bases)

\[ \left( \varphi \left( x + 2\pi b^{-1} - k a \right) - \varphi \left( x + 2\pi b^{-1} - \lambda a \right) \right) \ dx \leq \left( 2\pi \right)^{-d} \delta^{-1} \sum_{k} \int_{\mathbb{R}^{d}} \left| f(x) \left( \varphi(x - k a) - \varphi(x - \lambda a) \right) \right|^{2} \ dx \leq \delta^{-1} M' \| f \|^{2}. \]

Hence, we arrive at our assertion by Lemma 3.

**Theorem 14.**

Let \( \varphi \in L^{2}(\mathbb{R}^{d}), \{ \varphi_{j,k} \} \) be a Gabor frame (Riesz basis) with bounds \( A \) and \( B \) for \( L^{2}(\mathbb{R}^{d}) \). If

\[ M_{1} = \int |x|^{2} \sum_{k} |\varphi(x - k a)|^{2} \ dx < \infty, \quad M_{2} = \sum_{j} \left| (j - \beta_{j}) b \right|^{2} < (2\pi)^{d} M_{1}^{-1} A, \]

then \( \{ \varphi^{(q)}_{j,k} \} \) is a Gabor frame (Riesz basis) with bounds \( A[1 - (M/A)^{1/2}] \) and \( B[1 + (M/A)^{1/2}] \), where \( M = (2\pi)^{-d} M_{1} M_{2} \).

**Proof.** By Hölder inequality,

\[ \sum_{j,k \in \mathbb{Z}^{d}} \left| \langle f, \varphi_{j,k} - \varphi^{(q)}_{j,k} \rangle \right|^{2} \]

\[ = (2\pi)^{-2d} \sum_{j,k} \int \left| f(x) \varphi(x - k a) \left( e^{-i<x,jb>} - e^{-i<x,\beta_{j}b>} \right) \right|^{2} \ dx \]

\[ \leq (2\pi)^{-2d} \sum_{j,k} \int |f(x)|^{2} \ dx \int \left| \varphi(x - k a) \left( 1 - e^{-i<x,(j - \beta_{j})b>} \right) \right|^{2} \ dx \]

\[ \leq (2\pi)^{-d} \sum_{j,k} \int |\varphi(x - k a)|^{2} |x|^{2} \ dx \left| (j - \beta_{j}) b \right|^{2} \| f \|^{2} = (2\pi)^{-d} M_{1} M_{2} \| f \|^{2}. \]

Thus, we complete our proof by Lemma 3.

**Theorem 15.**

Let \( ab = 2\pi / m_{0}, m_{0} \in \mathbb{N}, \varphi \in L^{2}(\mathbb{R}^{d}), \{ \varphi_{j,k} \} \) be a Gabor frame (Riesz basis) with bounds \( A \) and \( B \) for \( L^{2}(\mathbb{R}^{d}) \), \( \text{supp} \varphi \subset I_{n,a} = [0, a] + n_{0}a, n_{0} \in \mathbb{N} \). If \( 0 < \alpha < A/B, |j - \beta_{j}| \leq L < \min (\frac{1}{4}, \pi^{-1} \cos^{-1} \left( \frac{(-a)^{d}}{2} \right) - \frac{1}{2}) \), then \( \{ \varphi^{(q)}_{j,k} \} \) is a Gabor frame (Riesz basis) with bounds \( A[1 - (M/A)^{1/2}] \) and \( B[1 + (M/A)^{1/2}] \), where \( M = B \cdot B_{d}(L) \).

**Proof.** By Lemma 5, \( B_{d}(L) < \alpha < A/B \). Theorem 1 is true if we substitute \( I_{n,a} = [0, 2\pi]^{d} + 2\pi n \), \( n \in \mathbb{Z}^{d} \). The proof is the same because both Lemma 1 and Lemma 2 are still true and \( B_{1}(L) \) remains the same.

Since \( a = \frac{1}{m_{0}} 2\pi b^{-1} \), that is, \( [0, a] \subset [0, 2\pi b^{-1}] \). For every \( k \in \mathbb{Z}^{d} \), let \( I_{k,a} = I_{n_{0},a} + ka \), then there exists \( k \in \mathbb{Z}^{d} \), such that \( I_{k,a} \subset I_{n,k,b} = [0, 2\pi b^{-1}] + 2\pi b^{-1} \). By Theorem 1 and Theorem 7, we have

\[ \sum_{j,k \in \mathbb{Z}^{d}} \left| \langle f, \varphi_{j,k} - \varphi^{(q)}_{j,k} \rangle \right|^{2} \]

\[ = (2\pi)^{-2d} \sum_{j,k \in \mathbb{Z}^{d}} \left| \int_{I_{k,a}} f(x) \varphi(x - k a) \left( e^{-i<x,jb>} - e^{-i<x,\beta_{j}b>} \right) \ dx \right|^{2} \]

\[ = (2\pi)^{-2d} \sum_{j,k \in \mathbb{Z}^{d}} \left| \int_{I_{n,k,b}} f(x) \varphi(x - k a) \left( e^{-i<x,jb>} - e^{-i<x,\beta_{j}b>} \right) \ dx \right|^{2} \]
\[
\leq (2\pi)^{-d} \tilde{b}^{-1} B_d(L) \sum_k \int |f(x)\overline{\varphi(x-ka)}|^2 \, dx \leq B \cdot B_d(L) \|f\|^2.
\]

The assertion follows from Lemma 3. \qed

**Theorem 16.**

Let \( \varphi \in L^2(\mathbb{R}^d) \), \( (\varphi_{j,k}) \) be a Gabor frame (Riesz basis) with bounds \( A \) and \( B \) for \( L^2(\mathbb{R}^d) \). If

\[
M = 2\tilde{b}^{-1} \operatorname{esssup}_{k,l} \sum_{j,k} \left| \varphi \left( x + 2\pi lb^{-1} - ka \right) - \varphi \left( x + 2\pi lb^{-1} - \lambda_k a \right) \right|^2
\]

\[
+ 2(2\pi)^d \sum_j |(j - \beta_j) b|^2 \int |x|^2 \sum_k |\varphi(x - \lambda_k a)|^2 \, dx < A,
\]

then \( (\varphi_{j,k}^{(q)}) \) is a Gabor frame (Riesz basis) with bounds \( A(1 - (M/A)^{\frac{1}{2}})^2 \) and \( B(1 + (M/A)^{\frac{1}{2}})^2 \).

**Proof.** Using the same trick as above, e.g., Theorem 8 and Theorem 14, we have

\[
\sum_{j,k \in \mathbb{Z}^d} \left| \langle f, \varphi_{j,k} - \varphi_{j,k}^{(q)} \rangle \right|^2
\]

\[
= (2\pi)^{-2d} \sum_{j,k} \left| \int f(x) \left( e^{-i<j,b,x> \varphi(x-ka)} - e^{-i<j,b,x> \varphi(x-\lambda_k a)} \right) \, dx \right|^2
\]

\[
\leq (2\pi)^{-2d} \sum_{j,k} \left( \left| \int f(x) e^{-i<j,b,x> \varphi(x-ka)} \left( \varphi(x-ka) - \varphi(x-\lambda_k a) \right) \, dx \right| + \left| \int f(x) \varphi(x-\lambda_k a) \left( e^{-i<j,b,x> - e^{-i<j,b,x>}} \right) \, dx \right|^2 \right)^2
\]

\[
\leq 2(2\pi)^{-2d} \left( \sum_{j,k} \left| \int f(x) e^{-i<j,b,x> \varphi(x-ka)} \left( \varphi(x-ka) - \varphi(x-\lambda_k a) \right) \, dx \right|^2 + \sum_{j,k} \left| \int f(x) \varphi(x-\lambda_k a) \left( e^{-i<j,b,x> - e^{-i<j,b,x>}} \right) \, dx \right|^2 \right)
\]

\[
= 2(2\pi)^{-2d} (I_1 + I_2).
\]

\[
I_1 = \left| \int f(x) e^{-i<j,b,x> \varphi(x-ka)} \left( \varphi(x-ka) - \varphi(x-\lambda_k a) \right) \, dx \right|^2
\]

\[
= (2\pi)^d \tilde{b}^{-1} \sum_k \int \left| \sum_n f \left( x + 2\pi nb^{-1} \right) \left( \varphi \left( x + 2\pi nb^{-1} - ka \right) - \varphi \left( x + 2\pi nb^{-1} - \lambda_k a \right) \right) \right|^2 \, dx
\]

\[
\leq (2\pi)^d \tilde{b}^{-1} \sum_{k,l} \left| \int f(x) \left( \varphi \left( x + 2\pi lb^{-1} - ka \right) - \varphi \left( x + 2\pi lb^{-1} - \lambda_k a \right) \right) \, dx \right|^2
\]

\[
\leq (2\pi)^{2d} \tilde{b}^{-1} \operatorname{esssup}_{k,l} \left| \varphi \left( x + 2\pi lb^{-1} - ka \right) - \varphi \left( x + 2\pi lb^{-1} - \lambda_k a \right) \right|^2 \|f\|^2.
\]

\[
I_2 = \sum_{j,k} \left| \int f(x) \varphi(x-\lambda_k a) \left( e^{-i<j,b,x> - e^{-i<j,b,x>}} \right) \, dx \right|^2
\]
\[
\leq (2\pi)^d \sum_j \left| (j - \beta_j) b \right|^2 \int |x|^2 \sum_k \left| \varphi (x - \lambda_k a) \right|^2 \, dx \parallel f \parallel^2.
\]

The proof is over. \(\square\)

Acknowledgments

I would like to thank the referee for correcting my English carefully and my advisers Professor Li Binren and Professor Peng Lizhong for their help.

References


Received May 27, 1997
Second revision received September 15, 1998
Institute of Mathematics, Academia sinica, Beijing, P.R. of China
e-mail: zhj@matho8.math.ac.cn