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Abstract

We present extensions of the classical Poisson summation formula in which the sequence of sampling knots, normally a lattice, can be taken from a relatively wide class of sequences.

Keywords: Poisson summation formula, complete interpolating sequence, amalgam space.

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1 Introduction

1.1 Background

Suppose \( f \) and \( F \) are a Fourier transform pair, specifically

\[
F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) e^{ix\xi} d\xi.
\]

Then, if both \( f(\xi) \) and \( F(x) \) are continuous and decay sufficiently rapidly, the classical Poisson summation formula reads

\[
\sum_{m=-\infty}^{\infty} f(2\pi m) = \sum_{n=-\infty}^{\infty} F(n)
\]

where both sums converge absolutely. More precise statements can be found in various texts on Fourier analysis including [12, 22, 31, 33] and examples of more recent work on the subject can be found in [1, 2, 3, 6, 7, 9, 11, 16, 19, 21].

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As is evident from the above citations, the Poisson summation formula is well known and has several variations and interpretations. Most of these exploit a lattice structure or related group theoretic phenomena. Indeed, it is known that formulas of the type
\[
\sum_{m=\infty}^{\infty} f(\xi_m) = c \sum_{n=\infty}^{\infty} F(x_n)
\]
are valid only when the sequences \(\{\xi_m\}\) and \(\{x_n\}\) are appropriate dual lattices, see [9].

On the other hand replacing \(f(\eta)\) by \(f(\xi + \eta)\) in (1.1.2) results in
\[
(1.1.3) \quad \sum_{m=-\infty}^{\infty} f(\xi + 2\pi m) = \sum_{n=-\infty}^{\infty} F(n) e^{-in\xi},
\]
which may be interpreted in a wider sense than (1.1.2). For instance, as a function of the variable \(\xi\), convergence of the sums may be taken in some mean sense and equality may be valid only almost surely.

### 1.2 Contents

It is extensions of identity (1.1.3) with which we will primarily be concerned. Indeed, the objective of this note is to present such an extension where the sequence of sampling knots on the right hand side, the integer lattice \(\mathbb{Z} = \{n\}\), is replaced with a more general sequence \(X = \{x_n\}\). This, of course, also involves appropriate adjustments to the left hand side as well. See (3.4.1)

The class of allowable sequences \(X\) in our extensions are the so-called complete interpolating sequences for the classical Paley-Wiener space. This includes a very wide range of irregular sampling sequences. For example, every sufficiently small perturbation of the integer lattice is a complete interpolating sequence. See Section 4 for more examples and citations.

The classical Poisson summation formula has many applications. For example it provides the justification, in terms of error bounds, for the use of the discrete Fourier transform in lieu of the continuous Fourier transform in certain numerical calculations. This and other examples are detailed in various texts and articles on applied Fourier analysis such as [5, 29]. It is expected that our formulas will also find meaningful applications.
1.3 Remark

For the sake of completeness we mention that if, in addition to translation, dilations and modulations are also used then variants of (1.1.2) which are significantly more complicated than (1.1.3) can be obtained, for instance see [4]. However we will not go there.

2 Setup

The setting for our extension of (1.1.3) are certain so-called amalgam spaces. Various amalgam type spaces are well known and have been studied in many contexts, for example see [13, 14, 15] and the recent survey article [17] which contains a brief history and a bibliography on the subject. We begin by defining two specific examples of these spaces and viewing (1.1.3) as both an extension and summation procedure.

2.1 The amalgam spaces $l^1(L^2)$ and $l^\infty(L^2)$

Let $Q = [\pi, \pi] = \{\xi : -\pi \leq \xi \leq \pi\}$ and for a measurable function $f$ defined on $Q + 2\pi m = \{\xi : (2m - 1)\pi \leq \xi \leq (2m + 1)\pi\}$ let

$$
\|f\|_{L^2(Q+2\pi m)} = \left\{ \int_{Q+2\pi m} |f(\xi)|^2 d\xi \right\}^{1/2}.
$$

The amalgam space $l^1(L^2)$ is the class of those measurable functions $f$ on $\mathbb{R} = (-\infty, \infty)$ such that

$$
\|f\|_{l^1(L^2)} = \sum_{m=-\infty}^{\infty} \|f\|_{L^2(Q+2\pi m)}
$$

is finite.

The amalgam space $l^\infty(L^2)$ is its dual. Namely, $l^\infty(L^2)$ is the class of those measurable functions $f$ on $\mathbb{R} = (-\infty, \infty)$ such that

$$
\|f\|_{l^\infty(L^2)} = \sup_{-\infty < m < \infty} \|f\|_{L^2(Q+2\pi m)}
$$

is finite.
2.2 The summation operator $S_Z$

If $f$ is in the amalgam space $l^1(L^2)$ then the left hand side of (1.1.3) is in $L^2(Q)$ and the right hand side is simply its Fourier series.

Indeed,

\begin{equation}
\| \sum_{m=-\infty}^{\infty} f(\xi + 2\pi m) \|_{L^2(Q)} \leq \|f\|_{l^1(L^2)}
\end{equation}

is just Minkowski’s inequality which implies that the left hand side of (1.1.3) may be viewed as a linear mapping of $l^1(L^2)$ onto $L^2(Q)$ of norm 1. We denote this mapping by $S_Z$, thus

$$S_Z f(\xi) = \sum_{m=-\infty}^{\infty} f(\xi + 2\pi m).$$

Also note that

\begin{equation}
\| f \|_{L^1(\mathbb{R})} \leq \sqrt{2\pi} \| f \|_{l^1(L^2)}
\end{equation}

follows from Hölder’s inequality and implies that

\begin{equation}
l^1(L^2) \subset L^1(\mathbb{R}).
\end{equation}

These observations can be summarized as follows:

2.3 Proposition

If $f$ is in $l^1(L^2)$ then

(i) $f$ is in $L^1(\mathbb{R})$ and $F(x)$ is continuous,

(ii) for $n = 0, \pm 1, \pm 2, \ldots$, 

$$F(n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) e^{in\xi} d\xi = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \sum_{m=-\infty}^{\infty} f(\xi + 2\pi m) \right\} e^{in\xi} d\xi$$

are the Fourier coefficients of

$$S_Z f(\xi) = \sum_{m=-\infty}^{\infty} f(\xi + 2\pi m),$$

and

(iii) both sums in (1.1.3) converge in the $L^2(Q)$ sense and (1.1.3) is valid for almost all $\xi$ in $Q$. 

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2.4 The prolongation operator $E_Z$

In view of 2.2 it is natural to consider the dual or adjoint of $S_Z$, the operator $S_Z^* = E_Z$, as an extension or prolongation operator from $L^2(Q)$ to $l^\infty(L^2)$.

If $f \in l^1(L^2)$ and $g \in L^2(Q)$ the routine calculation

$$\int_Q g(\xi) S_Z f(\xi) d\xi = \int_Q g(\xi) \sum_m f(\xi + 2\pi m) d\xi$$

$$= \sum_m \int_{Q+2\pi m} g(\xi - 2\pi m) \overline{f(\xi)} d\xi$$

$$= \sum_m \int_{Q+2\pi m} \tau_{2\pi m} g(\xi) \overline{f(\xi)} d\xi$$

$$= \int_{-\infty}^{\infty} \left\{ \sum_m P_m \tau_{2\pi m} g(\xi) \right\} \overline{f(\xi)} d\xi$$

shows that

$$E_Z g(\xi) = \sum_{m=-\infty}^{\infty} P_m \tau_{2\pi m} g(\xi)$$

where $\tau_\eta$ and $P_m$ are the translation and projection operators defined by $\tau_\eta g(\xi) = g(\xi - \eta)$ and $P_m f(\xi) = \chi_Q(\xi - 2\pi m) f(\xi)$ respectively. Here $\chi_Q$ is the indicator function of the interval $Q = [-\pi, \pi]$.

Note also that

$$g(\xi) = \sum_{n=-\infty}^{\infty} a_n e^{-i n \xi} \quad (2.4.1)$$

where

$$a_n = \frac{1}{2\pi} \int_Q g(\xi) e^{i n \xi} d\xi .$$

The right hand side of (2.4.1) makes sense for a. e. $\xi$ in $\mathbb{R}$ and is a continuation of $g$ from the interval $Q$ to all of $\mathbb{R}$. Indeed, we may write

$$\sum_{m=-\infty}^{\infty} P_m \tau_{2\pi m} g(\xi) = E_Z g(\xi) = \sum_{n=-\infty}^{\infty} a_n e^{-i n \xi} . \quad (2.4.2)$$

and view $E_Z g$ as the periodic continuation of $g$. 

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Finally we note that using the notation established above the summation operator may be re-expressed as

\[(2.4.3) \quad S_Zf(\xi) = \sum_{m=-\infty}^{\infty} \tau^{-2\pi m} P_m f(\xi)\]

At the moment this formalism may seem unnecessarily pompous but it gives a hint as to what to expect in what follows.

3 Poisson type summation formulas

Our extensions of (1.1.3) are based on appropriate generalizations of the summation and prolongation operators $S_Z$ and $E_Z$ defined in Section 2. Because $E_Z$ is intuitively easier to deal with we begin with its generalization.

3.1 The prolongation operator $E_X$

Recall the fact that $E_Z f$ is simply the periodic extension of a function $f$ defined on $Q$ to one defined on $\mathbb{R}$. If $f$ is in $L^2(Q)$ then this extension or prolongation can be described via its expansion in terms of the complete orthonormal system of functions $\{e^{-in\xi}\}_{n\in\mathbb{Z}}$ as in (2.4.2). Now suppose one wants to obtain a similar type prolongation of $L^2(Q)$ by using a sequence of frequencies, $X = \{x_n\}_{n\in\mathbb{Z}}$, other than the integers. It is clear that at the very least the corresponding sequence of exponentials $\{e^{-ix_n\xi}\}_{n\in\mathbb{Z}}$ should form some sort of nice basis for $L^2(Q)$. In what follows we consider the case when $\{e^{-ix_n\xi}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(Q)$.

Recall that a basis $\{f_n\}$ of a Hilbert space $\mathcal{H}$ is a Riesz basis if and only if there are positive constants $c_0$ and $c_1$ such that for every linear combination $f = \sum_n a_n f_n$ in $\mathcal{H}$ the bounds

\[(3.1.1) \quad c_0^2 \sum_n |a_n|^2 \leq \|f\|_\mathcal{H}^2 \leq c_1 \sum_n |a_n|^2\]

are valid. Such a basis may be alternatively defined as the image of a complete orthonormal system under an invertible bounded linear transformation, i.e. $f_n = Tu_n$ for all $n$ where $\{u_n\}$ is a complete orthonormal system and $T$ is an invertible bounded linear transformation. For other equivalent definitions and further information on Riesz bases see [8, 10, 23, 28, 32]. In the case
that $\mathcal{H}$ is $L^2(Q)$ and $f_n(\xi) = e^{-i\pi n \xi}$, $n = 0, \pm 1, \pm 2, \ldots$, the corresponding frequencies $\mathcal{X} = \{x_n\}_{n \in \mathbb{Z}}$ are referred to as a complete interpolation sequence for the classical Paley-Wiener space and, in what follows, abbreviated to PWCIS. For examples of PWCISs see Section 4 and the references mentioned there.

Let $\mathcal{X} = \{x_n\}_{n \in \mathbb{Z}}$ be a PWCIS so that $\{e^{-i\pi n \xi}\}_{n \in \mathbb{Z}}$ is a Riesz basis for $L^2(Q)$ and $\{\phi_n(\xi)\}_{n \in \mathbb{Z}}$ is the corresponding dual or biorthogonal Riesz basis. Namely, the functions $\{\phi_n(\xi)\}_{n \in \mathbb{Z}}$ satisfy

$$\frac{1}{2\pi} \int_Q \phi_m(\xi)e^{i\pi n \xi} d\xi = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{otherwise.} \end{cases}$$

(3.1.2)

Every function $f$ in $L^2(Q)$ may be uniquely expressed as

$$f(\xi) = \sum_{n \in \mathbb{Z}} a_n e^{-i\pi n \xi}$$

(3.1.3)

where

$$a_n = \frac{1}{2\pi} \int_Q f(\xi) \overline{\phi_n(\xi)} d\xi$$

and also as

$$f(\xi) = \sum_{n \in \mathbb{Z}} b_n \phi_n(\xi)$$

(3.1.4)

where

$$b_n = \frac{1}{2\pi} \int_Q f(\xi)e^{i\pi n \xi} d\xi.$$ 

For each $m$ in $\mathbb{Z}$ consider the linear transformation $A_m$ which maps $L^2(Q)$ onto itself and is defined by the formula

$$A_m f(\xi) = \sum_{n \in \mathbb{Z}} a_n e^{-i\pi n 2\pi m} e^{-i\pi n \xi}$$

where the coefficients $\{a_n\}$ are uniquely determined by $f$ via (3.1.3). Because $\{e^{-i\pi n \xi}\}_{n \in \mathbb{Z}}$ is a Riesz basis the linear transformations $A_m$ are well defined, bounded, and invertible. Indeed, see 3.6 below, it is not difficult to see that

$$\frac{c_0}{c_1} \|f\|_{L^2(Q)} \leq \|A_m f\|_{L^2(Q)} \leq \frac{c_1}{c_0} \|f\|_{L^2(Q)}$$

(3.1.5)
where $c_0$ and $c_1$ are the Riesz basis constant as in (3.1.1).

Note that $A_m = A_0^n$ for all integers $m$; in particular $A_0 = I$, the identity operator. Also note that $A_m = I$ for all $m$ in the case $X = \mathbb{Z}$.

Next observe that

$$\|A_m f(\xi)\|_{L^2(\xi \in Q)} = \left\| \sum_{n \in \mathbb{Z}} a_n e^{-ix_n \xi} \right\|_{L^2(\xi \in \mathbb{Q} + 2\pi m)}$$

so that in view of (3.1.5) we have

$$\frac{c_0}{c_1} \|f\|_{L^2(\xi \in Q)} \leq \left\| \sum_{n \in \mathbb{Z}} a_n e^{-ix_n \xi} \right\|_{L^2(\xi \in \mathbb{Q} + 2\pi m)} \leq \frac{c_1}{c_0} \|f\|_{L^2(\xi \in Q)}$$

where, as in (3.1.5), $c_0$ and $c_1$ are constants from relation (3.1.1) associated with the Riesz basis $\{e^{-ix_n \xi}\}_{n \in \mathbb{Z}}$.

In view of (3.1.6) the right hand side of (3.1.3), initially defined for $\xi$ in the interval $Q$, is well defined for almost all real $\xi$ and is a function in the amalgam space $l^\infty(L^2)$. We define the right hand side of (3.1.3) as the prolongation of $f$ and denote it by $E_X f$. This prolongation can also be defined in terms of the operators $A_m$. Thus

$$\sum_{m=-\infty}^{\infty} P_m \tau_{2\pi m} A_m f(\xi) = E_X f(\xi) = \sum_{n=-\infty}^{\infty} a_n e^{-ix_n \xi}$$

where $\tau_\eta$ and $P_m$ are, respectively, the translation and projection operators defined in Section 2.

We summarize these observations as follows:

3.2 Proposition

Suppose $X = \{x_n\}_{n \in \mathbb{Z}}$ is a PWCIS. If $f$ is in $L^2(\xi \in \mathbb{Q})$ and enjoys the representation (3.1.3) then the mapping of $f$ into $E_X f$ defined via (3.1.7) is a bounded linear transformation from $L^2(\xi \in \mathbb{Q})$ into $l^\infty(L^2)$ whose norm depends only on the constants associated with the Riesz basis $\{e^{-ix_n \xi}\}_{n \in \mathbb{Z}}$.

3.3 The summation operator $S_X$

Suppose $X = \{x_n\}$ is a PWCIS. In view of the case $X = \mathbb{Z}$ the summation operator $S_X$ should simply be the dual or adjoint of the prolongation operator...
\( \mathbf{E}_X \) and map \( l^1(L^2) \) into \( L^2(Q) \). To obtain an explicit expression take \( f \) and \( g \) in \( l^1(L^2) \) and \( L^2(Q) \) respectively and write

\[
\int_Q S_X f(\xi) \overline{g(\xi)} d\xi = \int_{-\infty}^{\infty} f(\xi) \overline{E_X g(\xi)} d\xi = \int_{-\infty}^{\infty} f(\xi) \sum_{m=-\infty}^{\infty} \overline{P_m^{\tau_2\pi m} A_m g(\xi)} d\xi \\
= \sum_{m=-\infty}^{\infty} \int_{Q+2\pi m} f(\xi) \tau_2\pi m A_m g(\xi) d\xi = \int_Q \left\{ \sum_{m=-\infty}^{\infty} A_m^* \tau_{-2\pi m} P_m f(\xi) \right\} \overline{g(\xi)} d\xi
\]

where \( A_m^* \) is the adjoint of \( A_m \). So it is clear that the operator \( S_X \) should be defined as

\[
(3.3.1) \quad S_X f(\xi) = \sum_{m=-\infty}^{\infty} A_m^* \tau_{-2\pi m} P_m f(\xi).
\]

Note that the mapping \( f \) to \( S_X f \) maps the amalgam space \( l^1(L^2) \) into \( L^2 \) linearly with norm no greater than \( c_1/c_0 \), where the constants \( c_0 \) and \( c_1 \) are the constants associated with the Riesz basis of exponentials \( \{ e^{-ix_n \xi} \}_{n \in \mathbb{Z}} \) via (3.1.1).

\( S_X f(\xi) \) can also be expressed as

\[
(3.3.2) \quad S_X f(\xi) = \sum_{n=-\infty}^{\infty} F(x_n) \phi_n(\xi)
\]

for almost all \( \xi \) in the interval \( Q \), where \( \{ \phi_n \} \) is the Riesz basis dual to \( \{ e^{-ix_n \xi} \} \) in \( L^2(Q) \) and \( F \) and \( f \) are a Fourier transform pair (1.1.1). Identity (3.3.2) follows from

\[
\frac{1}{2\pi} \int_Q S_X f(\xi) e^{ix_n \xi} d\xi = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_Q f(\xi + 2\pi m) e^{ix_n (\xi + 2\pi m)} d\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) e^{ix_n \xi} d\xi = F(x_n)
\]

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and the fact that \(\{e^{-ix_n\xi}\}\) and \(\{\phi_n\}\) are dual Riesz bases for \(L^2(Q)\).

Formulas (3.3.1) and (3.3.2) are essentially our analogue of Poisson’s summation formula (1.1.3) which we summarize as follows.

### 3.4 Proposition

Suppose \(X = \{x_n\}_{n \in \mathbb{Z}}\) is a PWCIS and \(f\) is in \(l^1(L^2)\). Then the following holds:

(i) \(f\) is in \(L^1(\mathbb{R})\) and \(F(x)\) is continuous.

(ii) The summation operator \(S_X\) defined by (3.3.1) is a bounded linear transformation from \(l^1(L^2)\) into \(L^2(Q)\) whose norm depends only on the constants associated with the Riesz basis \(\{e^{-ix_n\xi}\}_{n \in \mathbb{Z}}\).

(iii) For \(n = 0, \pm 1, \pm 2, \ldots\),

\[
F(x_n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) e^{ix_n\xi} d\xi = \frac{1}{2\pi} \int_Q S_X f(\xi) e^{ix_n\xi} d\xi .
\]

(iv) We have the summation formula

\[
(3.4.1) \quad \sum_{m=-\infty}^{\infty} A^*_m \tau_{-2\pi m} P_m f(\xi) = S_X f(\xi) = \sum_{n=-\infty}^{\infty} F(x_n) \phi_n(\xi)
\]

where \(\{\phi_n(\xi)\}_{n \in \mathbb{Z}}\) is the Riesz basis dual to \(\{e^{-ix_n\xi}\}_{n \in \mathbb{Z}}\) so that, in particular, (3.1.2) is valid. Both sums converge in the \(L^2(Q)\) sense and formula is valid for almost all \(\xi\) in \(Q\).

### 3.5 Remarks

a. If \(g\) is in \(L^2(Q)\) there is an explicit formula for \(A^*_m g\) in terms of \(\{\phi_n\}\). Namely, since \(\{e^{-ix_n\xi}\}\) and \(\{\phi_n\}\) are dual Riesz bases for \(L^2(Q)\) and if

\[
A_m(e^{-ix_n\eta})(\xi) = e^{-ix_n2\pi m}e^{-ix_n\xi} \quad \text{then} \quad A^*_m \phi_n(\xi) = e^{ix_n2\pi m} \phi_n(\xi).
\]

So if

\[
g(\xi) = \sum_n b_n \phi_n(\xi)
\]

then

\[
A^*_m g(\xi) = \sum_n e^{ix_n2\pi m} b_n \phi_n(\xi).
\]
b. In the case \( \mathcal{X} = \mathbb{Z} \) it should be clear that \( A_m = A_m^* = \text{the identity} \) and that \( e^{-ix_n \xi} = \phi_n(\xi) = e^{-im\xi} \) so that (3.4.1) reduces to (1.1.3). The fact that the class of PWCISs is sufficiently rich to make identity (3.4.1) interesting should be evident from the examples in Section 4.

c. In general the members of the dual basis, \( \{\phi_n\} \), need not be continuous so that (3.4.1) may not make sense for certain values of \( \xi \). On the other hand there are many cases where these members are continuous and the pointwise interpretation of (3.4.1) is indeed possible.

3.6 Remarks

To see inequalities (3.1.5) and (3.1.6) suppose \( f \) is in \( L^2(Q) \) and enjoys representation (3.1.3). Then, because \( \{e^{-ix_n \xi}\}_{n \in \mathbb{Z}} \) is a Riesz basis in \( L^2(Q) \),

\[
\| A_m f \|_{L^2(Q)}^2 \leq c_1^2 \sum_{n=-\infty}^{\infty} |a_n e^{-ix_n 2\pi m}|^2 = c_1^2 \sum_{n=-\infty}^{\infty} |a_n|^2 \leq \frac{c_2^2}{c_0^2} \| f \|_{L^2(Q)}^2,
\]

where \( c_0 \) and \( c_1 \) are the constants, from relation (3.1.1), associated with the Riesz basis \( \{e^{-ix_n \xi}\}_{n \in \mathbb{Z}} \). Similarly

\[
\| A_m f \|_{L^2(Q)}^2 \geq c_0^2 \sum_{n=-\infty}^{\infty} |a_n e^{-ix_n 2\pi m}|^2 = c_0^2 \sum_{n=-\infty}^{\infty} |a_n|^2 \geq \frac{c_0^2}{c_1^2} \| f \|_{L^2(Q)}^2.
\]

This proves (3.1.5).

To see (3.1.6) note that if \( \xi \) is in the interval \( Q + 2\pi m \) we may write

\[
\sum_{n=-\infty}^{\infty} a_n e^{-ix_n \xi} = \sum_{n \in \mathbb{Z}} a_n e^{-ix_n 2\pi m} e^{-ix_n (\xi - 2\pi m)} = \tau_{2\pi m} A_m f(\xi).
\]

Thus

\[
\| \sum_{n \in \mathbb{Z}} a_n e^{-ix_n \xi} \|_{L^2(\xi \in Q + 2\pi m)} = \| \tau_{2\pi m} A_m f \|_{L^2(Q + 2\pi m)} = \| A_m f \|_{L^2(Q)}
\]

so that (3.1.6) follows from (3.1.5).
4 Appendix

Basic material on the classical Paley-Wiener space and PWCISs can be found, for example, in [24, 32]. PWCISs have been completely characterized in terms of zeros of entire functions of exponential type in [30], see also [18, 26]. Several examples of specific PWCISs are given in the subsection below.

Formula (3.4.1) has already been implicitly applied to show that certain piecewise polynomial splines converge to entire functions of exponential type, see [25, 27]. Other applications should follow. For example, under appropriate scaling this formula should also be useful in approximating the Fourier transform in terms of irregular samples.

4.1 Examples of PWCISs

a. Any sequence \( \mathcal{X} = \{x_n\}_{n \in \mathbb{Z}} \) such that

\[ |x_n - n| \leq r < \frac{1}{4} \]

is a PWCIS. This was established in [20].

b. If \( N \) is a positive integer, \( 0 \leq \alpha_1 < \cdots < \alpha_N < N \), and

\[ N\mathbb{Z} + \alpha_k = \{Nm + \alpha_k\}_{m \in \mathbb{Z}} \]

then

\[ \mathcal{X} = \bigcup_{k=1}^{\infty} \{N\mathbb{Z} + \alpha_k\} \]

is a PWCIS, see [24].

c. If \( \mathcal{X}_1 \) is a PWCIS and \( \mathcal{X}_2 \) is the sequence which results after any perturbation of a finite number of terms then \( \mathcal{X}_2 \) is also a PWCIS. In particular if

\[ 0 < x_1 < x_2 < \cdots x_N < N + 1 \]

then

\[ \ldots, -2, -1, 0, x_1, x_2, \ldots x_N, N + 1, N + 2, \ldots \]

is a PWCIS. This is an elementary consequence of the characterization mentioned above.
References


