Some Aspects of Gabor Analysis on Elementary Locally Compact Abelian Groups

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1 Deutsche Zusammenfassung


In Kapitel 6 dieser Dissertation geben wir explizite Ausdrücke für die metaplektischen Operatoren für endliche zyklische Gruppen an, welche eine Verallgemeinerung der Formeln von A. Weil sind. In seiner bahnbrechende Arbeit gibt A. Weil nur explizite Formeln für metaplektische Operatoren für die erzeugenden Elemente der symplektischen Gruppen $\mathcal{G}$ an, wenn die eine Komponente ein Isomorphismus zwischen $\mathcal{G}$ und $\hat{\mathcal{G}}$
ist, der Gruppe aller irreduziblen Darstellungen von $G$. Folglich wird nur der stetige Fall behandelt, wo die Gruppe $G$ isomorph der dualen Gruppe $\hat{G}$ ist. Die naheliegende Verallgemeinerung der Ergebnisse im stetigen Fall zum diskreten Fall funktioniert leider nicht, daher bilden unsere Formeln die unabhängig von der Kardinalität der endlichen zyklischen Gruppe sind eine Verallgemeinerung der Resultate von A. Weil. Konkret zeigen wir, dass für zwei Komponenten $\alpha$ und $\beta$ einer symplektischen Matrix über einer endlichen zyklischen Gruppe ein $h$ existiert, sodass $\alpha + \beta h$ invertierbar ist. Die letzte Beobachtung war für endlich dimensionale Vektorräume bereits von Igusa gemacht worden.


Auf dem ersten Blick haben die beiden Themenkreise der Dissertation nichts gemein, jedoch die integrierte Darstellung der Zeit-Frequenz Operatoren stellt eine Verbindung her. Diese behandeln wir in Kapitel 5, wo wir allgemeine Begriffe über die integrierte Darstellung angeben und ihre Beziehung zu Gaborframes behandeln.
Chapter 1
2 Acknowledgements

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3 Introduction

In his paper *Theory of Communication* [21], D. Gabor proposed the use of a family of functions obtained from one Gaussian by time and frequency shifts along the lattice \( \mathbb{Z} \times \mathbb{Z} \) to decompose elements from the space of square integrable functions into the series

\[
f(t) = \sum_{n,m \in \mathbb{Z}} c_{m,n} g_{m,n}(t),
\]

where the elementary functions \( g_{m,n} \) are given by

\[
g_{m,n}(t) = M_m T_n g(t) = g(t - n) e^{2\pi i m t} = e^{-(t-n)^2} e^{2\pi i m t}, \quad m, n \in \mathbb{Z},
\]

and \( T_n \) denotes the translation by \( n \) and \( M_m \) the modulation by \( m \). We will write the time frequency shift by an element \( \lambda \) from a lattice \( \Lambda \) as \( \pi(\lambda) \). Since a Gaussian is well concentrated in time and frequency such representation has advantage over the Fourier decomposition, which is not local in both time and frequency at the same time. Unfortunately the setup proposed by Gabor has a disadvantage of not being a frame, which leads to numerical instabilities. The Balian-Low Theorem and its extensions showed that a similar situation occurs if the Gaussian is replaced by any other function that is smooth and localized. To go about this problem one is led to consider higher density time-frequency lattices, namely \( a \mathbb{Z} \times b \mathbb{Z} \) with \( ab < 1 \). Problem of finding the coefficients for the Gabor expansion of \( f \) with atom \( g \) on some given lattice is dual to the problem of recovering \( f \) from the samples of its Short Time Fourier transform with respect to the window \( g \),

\[
V_g f(an, bm) = \langle f, g_{m,n} \rangle,
\]
on the same lattice.

Therefore one is interested in examine all admissible lattices and relations between them. Time-frequency representation of a Gaussian has circular contour lines, hence a better packing of the time-frequency plane is achieved by using a non-separable hexagonal lattice, Fig.3.1, instead of a rectangular one (separable), Fig.3.2.

The so called symplectic transformations of the time-frequency plane are natural ways of transforming standard lattices (a lattice in the time domain × a lattice in the frequency domain) into so called 'symplectic lattices'. The group of symplectic transformations of the group \( \mathcal{G} \times \hat{\mathcal{G}} \) consists of the \( 2 \times 2 \) matrices of homomorphisms

\[
\mathcal{A} = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}, \quad \text{such that} \quad \mathcal{A}^T \mathcal{J} \mathcal{A} = \mathcal{J},
\]
Figure 3.1: Packing of the time-frequency plane by the Gaussian on a separable lattice, rectangular.

where $\mathcal{J} = \begin{pmatrix} 0 & -I_G \\ I_G & 0 \end{pmatrix}$ and $\alpha : \mathcal{G} \rightarrow \mathcal{G}$, $\beta : \hat{\mathcal{G}} \rightarrow \mathcal{G}$, $\delta : \mathcal{G} \rightarrow \hat{\mathcal{G}}$, $\gamma : \hat{\mathcal{G}} \rightarrow \hat{\mathcal{G}}$ are homomorphisms. While in the case of 1-dimensional continuous signals (the phase space can be identified with the complex plane in this case) every lattice is a symplectic lattice, that is not the case in higher dimensions. The celebrated work of A. Weil provides expressions for the intertwining operators, called metaplectic operators, for generating elements of the symplectic group of $\mathcal{G}$, where one component is an isomorphism between $\mathcal{G}$ and $\hat{\mathcal{G}}$, the group of all irreducible representations of $\mathcal{G}$. Hence it deals only with the continuous case when the group $\mathcal{G}$ is isomorphic to its dual $\hat{\mathcal{G}}$. The expressions for the metaplectic operators were investigated by many authors in different settings [39], [29], [17], [51], [28], [37]. The simple minded extension of continuous result to a discrete setting doesn’t work and the obstacle lies when the ‘multiplication by 2’ is not an isomorphism. There are some results in this direction [48], [5], but the authors restrict themselves either to considering groups of prime order [36] or of order power of 2 [44], [19], or to rings [18]. In Chapter 6 we provide compact formulas for the metaplectic operators in the case of finite cyclic groups regardless of the cardinality of a group and also extend the result of A. Weil by showing that for two components $\alpha$ and $\beta$ of a symplectic matrix over finite cyclic group there exists $h$ so that $\alpha + \beta h$ is invertible. Similar result was proved by Igusa [28], [29] only for finite dimensional vector spaces.

As was already observed by K. Gröchenig [23], the metaplectic operators allow the transition between Gabor systems defined on the symplectic and separable lattices. Hence
the explicit formulas for those operators for the finite cyclic group supply a new way for applications to analyse Gabor systems. We present a MATLAB toolbox containing routines that compute the dual window of a Gabor system given any nonseparable lattice. There are results in this direction, namely computing the dual window on a nonseparable lattice, but the authors analyze directly the Gabor frame matrix to arrive to their algorithms [4], [49]. The new method using metaplectic operators is computationally more efficient and allows nonseparable lattices other than hexagonal.

Given the Gabor system \( \{ g_{m,n} \} \), we consider the operator

\[
Sf = \sum_{m,n} \langle f, g_{m,n} \rangle g_{m,n}.
\]

When the above family constitutes a frame then the operator \( S \) is invertible and is called the Gabor frame operator. \( S \) as well as its inverse commute with all time-frequency shifts. The decomposition 3.1 can be written now as

\[
f = SS^{-1}f = \sum_{\lambda \in \Lambda} \langle S^{-1}f, \pi(\lambda)g \rangle \pi(\lambda)g = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)S^{-1}g \rangle \pi(\lambda)g.
\]

Therefore to be able to decompose \( f \) using the building blocks, one needs to know the inverse of the operator \( S \). Twisted convolution arises naturally in the context of time
frequency operators [8], [23]. The frame operators arising in this context have a special representation, the so-called Janssen representation [30] which can be exploited to study invertibility issues. A different, however, equivalent method for studying Gabor frame operators is the well known Zibulski - Zeevi representation [53] based on a generalized Zak-transform. Recently, T. Werther and Y. Eldar [46] showed yet another approach to analyze the invertibility of the Gabor frame operators by investigating the properties of certain function-valued matrices.

In contrast to the standard convolution, the twisted convolution is not commutative. This is opposed to the possibility of applying powerful tools from harmonic analysis, such as Wiener’s Lemma, in order to study twisted convolution operators. The twisted convolution was studied by many authors in different contexts and on various groups, the examples are [9], [20], [33], [3]. The existence of the inverse of the sequence in the twisted convolution algebra associated with Gabor theory was already treated by K. Gröchenig and M. Leinert in [25]. They dealt with the problem in a very abstract manner, using Banach algebras techniques. Their proof however is not constructive. In Chapter 7 we provide a different, constructive proof that works in the continuous and discrete settings equally. We present it in the case of the elementary locally compact abelian group since it covers both settings.

This new approach is based on splitting the twisted convolution with rational parameters into a finite number of weighted convolutions, which can be interpreted as another twisted convolution on a finite cyclic group. In analogy with the twisted convolution of finite discrete signals, we derive an homomorphism between the sequence space and a matrix algebra which preserves the algebraic structure. In this way, the problem reduces to the analysis of finite matrices whose entries are sequences supported on corresponding cosets. Using Cramer’s rule and proving Wiener’s lemma for this special class of matrices, we obtained an invertibility criterium. The inverse also follows easily from Cramer’s rule. This alternative approach gives further insights into Gabor frames for the rational case. In particular, it can be applied for both the continuous (on $\mathbb{R}^d$) and in the finite discrete setting. In the latter case, we obtained algorithms for directly computing the inverse of Gabor frame-type matrices equivalent to those known in the literature. We derive further (known) results in Gabor theory and show the relation to the Zibulski-Zeevi representation as well as to the method of T. Werther and Y. Eldar. Our approach also works for nonseparable lattices, where such a problem can be transformed to a separable case by means of metaplectic transformation. In one dimensional case in fact all lattices are symplectic (an image of some rectangular lattice).

Recently Franz Luef [34] showed the connection of Gabor analysis to the $\ast$-algebras. The algebras in question, are $\ell^1(\Lambda)$ with multiplication being twisted convolution. The integrated representations of such algebras are in 1-to-1 correspondence with the representations of $\Lambda$. Hence when the lattice $\Lambda$ is a subgroup of the finite group $\mathbb{Z}_n \times \mathbb{Z}_n$, then the results from Chapter 6 on the explicit formulas of metaplectic operators give the full characterization of all integrated representations of $\ell^1(\Lambda)$. On the other hand, in the more general setting of elementary locally compact abelian groups, the results from Chapter 7
provide the explicit inverse for the multiplication in $\ell^1(\Lambda)$. 
4 Notations and Definitions

In this chapter we set the stage for our investigation of Gabor analysis on locally compact abelian group. We recall some basic facts about locally compact abelian groups, introduce the notion of time-frequency plane and various types of lattices. We also define a Heisenberg group which plays an important role in Gabor analysis.

4.1 Locally compact abelian groups

For most of the following definitions see and theorems we refer to [42] and [12]. We first collect some definition and then remind some basic facts from the theory of topological groups.

Definition 4.1.1.

(i) A locally compact abelian group is a Hausdorff (there is a disjoint pair of neighborhoods of any pair of distinct points) space $G$ which is also an abelian group, endowed with a locally compact topology such that the group operation $(g_1, g_2) \mapsto g_1 g_2$ and inversion $g \mapsto g^{-1}$ ($g_1, g_2, g \in G$) are continuous.

(ii) A locally compact abelian group $G$ is σ-compact if there exists a countable cover of $G$ by compact sets.

The fundamental feature, whithout which little is possible, is the existance and uniqueness of a translation invariant measure $\mu_G$ on any locally compact group. So any locally compact abelian group admits a Haar measure, that is left and right translation invariant Borel measure $\mu_G(tE) = \mu_G(E)$, for any $t \in G$ and $E$ a Borel subset of $G$, unique up to a scalar multiple. When $G$ is compact the measure is finite and in this case we normalize it so that $\mu_G(G) = 1$. If $G$ is infinite and discrete we normalize it so that $\mu_G(\{e\}) = 1$, where $e$ is the identity element in $G$. We will write $dt$ instead of $d\mu_G(t)$ when there is no ambiguity.

Definition 4.1.2.

(i) The dual group of $G$ is the group $\hat{G}$ of continuous homomorphisms from $G$ to $\mathbb{T}$, the complex numbers whose absolute value is 1,

$$\hat{G} = \left\{ \chi: G \to \mathbb{T} : \chi(xy) = \chi(x)\chi(y) \quad x, y \in G \right\},$$

with the uniform convergence on compact sets.


The elements of $\chi \in \hat{G}$ are the characters of $G$ and we write $\langle g, \chi \rangle = \chi(g)$. Characters can also be seen as the equivalence classes of the irreducible unitary representations of $G$ of degree 1.

The orthogonal group $H^\perp$ of a closed subgroup $H \subseteq G$ is the group

$$H^\perp = \left\{ \chi \in \hat{G} : \chi(x) = 1 \ \forall x \in H \right\},$$

a subgroup of $\hat{G}$. The relation $(H^\perp)^\perp = H$ holds for all closed subgroups of $G$. Likewise, if $\Gamma$ is a closed subgroup of $\hat{G}$, we put

$$\Gamma^\perp = \left\{ x \in G : \chi(x) = 1 \ \forall \chi \in \Gamma \right\},$$

which is a subgroup of $G$.

It can be shown that the characters of $G$ form a locally compact abelian group under multiplication. Also, if $G$ is discrete, then $\hat{G}$ is compact; and if $G$ is compact then $\hat{G}$ is discrete.

A well-known and fundamental result in harmonic analysis is the following

**Proposition 4.1.3** (Pontryagin - van Kampen duality). Any locally compact abelian group $G$ is "reflexive" in the sense that $G = \hat{\hat{G}}$ via the natural embedding $x \in G \mapsto x \in \hat{\hat{G}}$: $\langle \chi, x \rangle := \langle x, \chi \rangle$, for $\chi \in \hat{G}$.

In the Proposition below we gather some more relations between the subgroups of $G$ and their duals.

**Proposition 4.1.4.** Let $H \subset G$ and $H^\perp \subset \hat{G}$ be as in definition. Then the following hold:

(i) the dual group of $G/H$ is $H^\perp$;

(ii) the dual group of $H$ is $\hat{G}/H^\perp$.

For the proofs we refer the reader to [12] or [38].

### 4.2 Representation theory

Let $G$ be a locally compact abelian $\sigma$-compact group and $d\mu_G(t)$ its Haar measure. When there is no ambiguity we will write $dt$ instead of $d\mu_G(t)$. To understand the structure of some abstract group one often uses representation theory, that is tries to understand the action of the group in question on some Hilbert space of functions. The space of square-integrable functions on a locally compact abelian group $G$ plays a distinguished role in the theory of group representations. The Hilbert space

$$L^2(G) = \left\{ f : G \to \mathbb{C} \text{ measurable} : \|f\|_2 = \left( \int_G |f(t)|^2 \, dt \right)^{1/2} < \infty \right\}.$$
is separable, since we have assumed that our locally compact abelian group \( G \) is \( \sigma \)-compact. In the following we recall the main definitions and results on group representation.

A **representation** of a locally compact abelian group \( G \) on a Hilbert space \( H \) is a homomorphism

\[
\pi : G \to \text{End}(H)
\]

from the group \( G \) to the operators on \( H \), such that for every \( f \in H \) the mapping \( x \mapsto \pi(x)f \) is continuous from \( G \) to \( H \). A representation is called **unitary** if \( \pi \) is a homomorphism of \( G \) to the unitary operators on \( H \).

A subspace \( V \) of \( H \) is **\( \pi \)-invariant** if \( \pi(x)V \subseteq V \) for every \( x \in G \). The representation \( \pi \) of \( G \) is said to be **irreducible** when \( H \) and \( \{0\} \) are the only \( \pi \)-invariant closed subspaces of \( H \). A representation \( \pi \) is irreducible if and only if \( \text{span}\{\pi(x)g\}_{x \in G} = H \) for any nonzero \( g \in H \). Therefore if \( \pi \) is irreducible and \( g \) any nonzero element from \( H \), then any \( f \in H \) can be arbitrarily close approximated by finite linear combinations of \( \pi(x)g \).

An important example of a representation, that is admitted by every locally compact group \( G \), is the so-called **left regular representation** of \( G \) on \( L^2(G) \) and is defined as

\[
L_tf(x) = f(t^{-1}x) \quad f \in L^2(G).
\]

Two representations \( (\pi, H) \) and \( (\pi', H') \) are **equivalent** when there exists a continuous isomorphism \( C : H \to H' \), called an **intertwining operator**, with

\[
C \circ \pi(x) = \pi'(x) \circ C \quad \text{for every} \quad x \in G.
\]

The **matrix coefficient** of \( \pi \) is a continuous function \( c^f_g \) on \( G \) of the form \( x \mapsto \langle f, \pi(x)g \rangle \) for any \( f, g \in H \).

A representation \( \pi \) is **square-integrable** if

\[
0 \neq c^f_g \in L^2(G) \quad \text{for some} \quad f, g \in H.
\]

Similarly, we say that \( \pi \) is **integrable** if it has a non-zero integrable coefficient

\[
0 \neq c^f_g \in L^1(G) \quad \text{for some} \quad f, g \in H.
\]

The most important fact about square-integrable representations are the **orthogonality relations** for the representation coefficients

**Theorem 4.2.1** (Schur’s orthogonality relations). Let \( (\pi, H) \) a unitary irreducible square-integrable representation of a locally compact abelian group \( G \). Then there exists a nonzero positive constant \( d \in \mathbb{R} \) such that

\[
\langle c^f_{g_1}, c^f_{g_2} \rangle = \frac{1}{d} \langle f_1, f_2 \rangle \langle g_1, g_2 \rangle.
\]
For the representation $\pi$ that is irreducible and square-integrable, an element $g \in \mathcal{H}$ is called \textbf{admissible} if $0 \neq c_g^g \in L^2(\mathcal{G})$. Let $g \in \mathcal{H}$ be an admissible element and define

$$A_g = \left\{ f \in \mathcal{H} : c_g^f \in L^2(\mathcal{G}) \right\}.$$ 

This is a linear and dense subspace of $\mathcal{H}$ which is moreover invariant under all $\pi(x)$ ($x \in \mathcal{G}$). To an admissible element $g \in \mathcal{H}$ we associate the operator $T_g$ from $\mathcal{H}$ to $L^2(\mathcal{G})$ defined by

$$T_g(f) = c_g^f$$ 

(4.1) with the domain $D(T_g) = A_g$. The above operator admits the following properties

\textbf{Lemma 4.2.2 ([41])}. Let $\pi$ be a unitary irreducible and square-integrable representation of a locally compact abelian group $\mathcal{G}$. Then the following hold

(a) if $g \in \mathcal{H}$ is admissible, then $T_g f \in L^2(\mathcal{G})$ for all $f \in \mathcal{H}$, and the map $T_g$ an isometry between $\mathcal{H}$ and $L^2(\mathcal{G})$,

(b) the adjoint map of $T_g$

$$T_g^* : f \mapsto \int_{\mathcal{G}} f(x)\pi(x)g \, dx$$

is a bounded linear operator from $L^2(\mathcal{G})$ onto $\mathcal{H}$, where the integral converges weakly in $\mathcal{H}$.

As a corollary we get a \textbf{reproducing property} for representations of $\mathcal{G}$

\textbf{Corollary 4.2.3}. Let $\pi$ be a square-integrable, irreducible and unitary representation of a locally compact abelian group $\mathcal{G}$. If $g_1, g_2 \in \mathcal{H}$ are admissible, then

$$\int_{\mathcal{G}} \langle f, \pi(x)g_1 \rangle \pi(x)g_2 \, dx = \frac{1}{d}$$

\subsection*{4.3 Second degree characters}

The notion of second degree characters on a locally compact abelian group was introduced in [51], see also [39] and is strictly connected to the 2-cocycles of the group in question.

Given a locally compact abelian group $\mathcal{G}$, a bicharacter is a continuous mapping $B : \mathcal{G} \times \mathcal{G} \to \mathbb{T}$ such that $B$ is a character when one of the two arguments is fixed. It induces a morphism $\beta_B$ of $\mathcal{G}$ into $\hat{\mathcal{G}}$ in the usual manner:

$$\langle x, \beta_B(y) \rangle = B(x, y).$$

The functional equations satisfied by a bicharacter are sufficiently strong that one can show that any map $B$ of $\mathcal{G} \times \mathcal{G}$ into $\mathbb{T}$ which is a Borel homomorphism in each argument is a bicharacter [35]. A bicharacter is called \textbf{antisymmetric} if it satisfies

$$B(x, y)B(y, x) = 1, \quad B(x, x) = 1.$$
4.3 Second degree characters

for all \( x, y \in \mathcal{G} \), and symplectic if it is antisymmetric and \( \beta_B \) is an isomorphism.

A second degree character associated to \( B \) is a continuous mapping \( \psi: \mathcal{G} \to \mathbb{T} \) such that

\[
\psi(x + x') = \psi(x)\psi(x')B(x, x'), \quad x, x' \in \mathcal{G}.
\] (4.2)

A general existence result for second degree characters is obtained by Mackey’s technique of induced representation, see [1] or [39] for a concise proof. Let \( \mathcal{G} \) be a self-dual group where the doubling of elements \( x \mapsto x + x \) is invertible. A bicharacter is of the form

\[
B(x, x') = \langle x, cx' \rangle,
\]

where \( c \) is a symmetric homomorphism from \( \mathcal{G} \) into the dual group. Then one associated second degree character is constructed by

\[
\psi(x) = \langle x, 2^{-1}cx \rangle,
\]

where \( 2^{-1} \) denotes the inverse of doubling. For example, this construction can be used for

(i) \( \mathcal{G} = \mathbb{R} \), where a bicharacter is of the form \( B(t, t') = e^{2\pi i st t'} \), \( t, t' \in \mathbb{R} \), for some \( s \in \mathbb{R} \). The function

\[
\psi(t) = e^{\pi i st^2}, \quad t \in \mathbb{R}
\]

is an associated second degree character.

(ii) \( \mathcal{G} = \mathbb{Z} \), where the second degree character associated to an element \( \omega \in \mathbb{T} \) is of the form

\[
\psi(k) = e^{\pi i \omega k^2}, \quad k \in \mathbb{Z}.
\]

(iii) \( \mathcal{G} = \mathbb{Z}_n \) when \( n \) is odd (the multiplicative inverse of 2 is \( (n + 1)/2 \) mod \( n \) in this case), and the second degree character associated to some \( c \in \mathbb{Z}_n \) takes the form

\[
\psi(m) = e^{\pi i cm^2}, \quad m \in \mathbb{Z}_n.
\]

For \( \mathbb{Z}_n \) with \( n \) even this construction does not apply since doubling is not an isomorphism.

The notion of second degree characters plays an important role in the construction of the metaplectic representation and it arises in the commutation relations for the Heisenberg group. One useful feature of second degree characters is that, if one second degree character for a given bicharacter is constructed, then all others associated to the same bicharacter are obtained easily as described below.

**Lemma 4.3.1.** Let \( \mathcal{G} \) be a locally compact abelian group and \( B \) its bicharacter. Then the following hold.
(i) If two second degree characters \( \psi_1, \psi_2 \) are associated to the same bicharacter, then
\[
\psi_2 = \chi \psi_1
\]
for some character \( \chi \).

(ii) The second degree characters associated to the trivial bicharacter identical one are just the usual characters.

Proof.

(i) Follows form (4.2), that is the quotient \( \chi := \psi_2/\psi_1 \) is a homomorphism and thus it is a character.

(ii) In this case (6.3) reduces to the homomorphism property.

In Chapter 6 we use second degree characters for the product group \( G = \mathbb{Z}_n \times \mathbb{Z}_n \). In this case the bicharacters are of the form
\[
B(\lambda, \lambda') = e^{2\pi i (\lambda, \sigma \lambda')/n}, \quad \lambda, \lambda' \in \mathbb{Z}_n \times \mathbb{Z}_n,
\]
for some symmetric matrix \( \sigma = \begin{pmatrix} p & q \\ q & r \end{pmatrix} \in M_{2,2}(\mathbb{Z}_n) \). Second degree characters for a product group can be constructed from those for the factors. For example, an associated second degree character associated to \( B \) as above is given by
\[
\psi(\lambda) = \psi(k, l) = \psi_1(k)\psi_2(l)e^{2\pi i q k l / n}, \quad \lambda = (k, l) \in \mathbb{Z}_n \times \mathbb{Z}_n,
\]
where \( \psi_1, \psi_2 \) are second degree characters on \( \mathbb{Z}_n \) associated to \( p \) and \( r \), respectively. For \( G = \mathbb{Z}_n \), a bicharacter is of the form
\[
B(m, m') = e^{2\pi i c m m' / n}, \quad m, m' \in \mathbb{Z}_n,
\]
for some \( c \in \mathbb{Z}_n \), and in Chapter 6 we will describe the associated second degree characters.

### 4.4 Gabor frame theory

We denote the dual group (set of continuous, unimodular complex homomorphisms on \( G \)) by \( \hat{G} \), i.e.,
\[
\hat{G} := \left\{ \xi : \xi(x + y) = \xi(x)\xi(y), |\xi(x)| = 1, x, y \in G; x \mapsto \xi(x) \text{continuous} \right\}.
\]

Due to Pontryagin duality theorem (which says, that the character group of \( \hat{G}, \hat{G} \) is topologically isomorphic with \( G \) via the group isomorphism \( x \mapsto \hat{x} \in \hat{G}, \hat{x}(\xi) = \xi(x) \) for \( \xi \in \hat{G} \) we can treat \( \xi(x) \) either as a \( \hat{G} \)-parametrized \( L^\infty(G) \) function or as a \( G \)-parametrized \( L^\infty(\hat{G}) \) function.

The Fourier transform of a function \( f \in L^1(G) \) is defined as
\[
\hat{f}(\xi) = \mathcal{F}_G f(\xi) := \int_G f(x)\overline{\xi(x)}dx,
\]
where $\xi \in \hat{G}$ and $dx$ denotes the translation invariant Haar measure on $G$. When working with characters $\chi : G \times \hat{G} \to \mathbb{T}$ of the product group $G \times \hat{G}$ we shall identify the dual group $\hat{G} \times \hat{G}$ with $G \times \hat{G}$ and we shall keep track of their natural factorization by the usual inner product notation:

$$\chi(\lambda) := \begin{pmatrix} t \\ \nu \end{pmatrix} \cdot \begin{pmatrix} x \\ \xi \end{pmatrix} = \nu(x) \xi(t)$$

with $\lambda := (x, \xi) \in G \times \hat{G}$, $\chi := (t, \nu) \in G \times \hat{G}$.

The unitary time-frequency shift operator is defined as

$$\pi(\lambda) f(t) := \xi(t) f(t - x), \quad \lambda := (x, \xi) \in G \times \hat{G}$$

which act unitarily on $L^2(G)$. It obeys the following commutation rules:

(a) $\pi(\lambda_1) \pi(\lambda_2) = \xi_2(x_1) \pi(\lambda_1 + \lambda_2)$,

(b) $\pi(\lambda_1) \pi(\lambda_2) = \xi_1(x_2) \xi_2(x_1) \pi(\lambda_2) \pi(\lambda_1) = \lambda_2(J \lambda_1) \pi(\lambda_2) \pi(\lambda_1)$,

where

$$\kappa(\lambda_1, \lambda_2) = \lambda_2(J \lambda_1) = \xi_1(x_2) \xi_2(x_1)$$

denotes a symplectic character formulated via the following skew-symmetric matrix

$$J := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad J^* = J^{-1} = -J.$$ 

The symplectic characters induce a new Fourier transform on $L^2(G \times \hat{G})$. The symplectic Fourier transform is an isomorphism $F_s$ of $L^2(G \times \hat{G})$ defined as

$$F_s h(x, \xi) = \int_{G \times \hat{G}} h(t, \omega) \overline{\xi(t) \omega(x)} dt d\omega, \quad h \in L^2(G \times \hat{G}),$$

where $dt$ and $d\omega$ are translation invariant Haar measures on $G$ and $\hat{G}$, respectively. The symplectic Fourier transform is self-inverse, which follows from the skew-symmetry of the symplectic character. We will see that $F_s$ is a more natural choice for the Fourier transformation on the time-frequency plane, rather than the standard one.

Let $H$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$.

**Definition 4.4.1.** A sequence $\{g_j\}_{j \in J} \subset H$ is a frame for $H$ if there exist constants $0 < A \leq B < \infty$ such that

$$A \|f\|^2 \leq \sum_{j \in J} |\langle f, g_j \rangle_H|^2 \leq B \|f\|^2 \quad \text{for all } f \in H.$$  

(4.5)
Chapter 4

The constants $A$ and $B$ are lower and upper frame bounds, respectively. Associated with each frame is the analysis operator $D: \mathcal{H} \to \ell^2(J)$ given by $D(f) = \{(f, g_j)_{\mathcal{H}}\}_{j \in J}$ and its Hilbert space adjoint $D^*: \ell^2(J) \to \mathcal{H}$ which is known as the synthesis operator and is given by $D^* e = \sum_{j \in J} c_j g_j$. The frame operator is the self-adjoint composition $S = D^* \circ D: \mathcal{H} \to \mathcal{H}$ given by $Sf = \sum_{j \in J} \langle f, g_j \rangle_{\mathcal{H}} g_j$. The equation 4.5 may be written as

$$A \|f\|^2 \leq \langle Sf, f \rangle_{\mathcal{H}} \leq B \|f\|^2 \quad \text{i.e.,} \quad AI \leq S \leq BI,$$

so that $S$ is invertible with $B^{-1}I \leq S^{-1} \leq A^{-1}I$. Each $f \in \mathcal{H}$ admits frame expansion

$$f = SS^{-1}f = \sum_{j \in J} \langle S^{-1}f, g_j \rangle_{\mathcal{H}} g_j = \sum_{j \in J} \langle f, \gamma_j \rangle_{\mathcal{H}} g_j$$

(4.6)

with $\gamma_j = S^{-1}g_j$. The collection $\{\gamma_j\}_{j \in J}$ is itself a frame, the dual frame.

When the elements $g_j$ are of the form $g_j = \pi(\lambda_j)g$ for some $g \in L^2(\mathcal{G})$, called a Gabor atom and all $\lambda_j = (x_j, \xi_j) \in \hat{\mathcal{G}} \times \hat{\mathcal{G}}$, then the collection $\{g_j\}$ is called a Gabor frame. From the work of Feichtinger and Gröchenig on irregular sampling we know that under certain conditions on $\{\lambda_j\}_{j \in J}$ the system $\{\pi(\lambda_j)g : \lambda_j \in \mathcal{G} \times \hat{\mathcal{G}}\}$ is a frame for $L^2(\mathcal{G})$ [13]. We restrict our presentation to discrete sets with the structure of a lattice in $\mathcal{G} \times \hat{\mathcal{G}}$.

Definition 4.4.2. A discrete subgroup $\Lambda$ of the locally compact abelian group $\mathcal{G}$ such that $\mathcal{G}/\Lambda$ is compact, is called a lattice. A fundamental domain of the lattice is the measurable set $U \subset \mathcal{G}$ such that every $x \in \mathcal{G}$ can be uniquely written as $x = \lambda + u$ for some $\lambda \in \Lambda$ and $u \in U$. Equivalently, $\mathcal{G}$ is the disjoint union of the translates $\lambda + U$, $\lambda \in \Lambda$. The lattice size is defined as a measure of the fundamental domain of $\Lambda$, and we always assume that the Haar measure on the compact quotient group $\mathcal{G}/\Lambda$ is normalized to a probability measure.

Let $\Lambda$ be a lattice in $\mathcal{G} \times \hat{\mathcal{G}}$ and let $g \in L^2(\mathcal{G})$ be a Gabor atom then we define a Gabor system $\mathcal{G}(g, \Lambda)$ by

$$\mathcal{G}(g, \Lambda) = \left\{ \pi(\lambda)g : \lambda \in \Lambda \right\}.$$

In our case the Gabor frame operator has the following form

$$Sf = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)g \quad f \in L^2(\mathcal{G}).$$

If the operator $S$ is bounded and invertible on $L^2(\mathcal{G})$, then $\mathcal{G}(g, \Lambda)$ is a Gabor frame and $S$ the associated frame operator, cf. [8]. A simple but important feature of Gabor systems is the fact the Gabor frame operator commutes with time-frequency shifts $\pi(\lambda')$,

$$\pi(\lambda')^{-1}S\pi(\lambda')f = \sum_{\lambda \in \Lambda} \langle \pi(\lambda')f, \pi(\lambda)g \rangle \pi(\lambda')^{-1}\pi(\lambda)g$$

$$= \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda - \lambda')g \rangle \pi(\lambda - \lambda')g = Sf.$$
4.5 Lattices in elementary locally compact abelian groups

That allows for the following reconstruction formulas for Gabor frames \( G(g, \Lambda) \),

\[
f = S^{-1} S f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle \pi(\lambda) S^{-1} g
\]

\[
f = S S^{-1} f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) S^{-1} g \rangle \pi(\lambda) g.
\]

Due to its appearance in the reconstruction formulas \( \gamma_0 = S^{-1} g \) is called the (cannonical) dual Gabor atom. Since the time-frequency shifts form a non-orthogonal system, the coefficients in the reconstruction formula are not unique. Therefore there are other dual atoms \( \gamma \in L^2(\mathcal{G}) \) with \( \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle \pi(\lambda) \gamma = I \).

Many studies in Gabor analysis are devoted to the frame operator \([23]\). In what follows we will describe the so-called Janssen representation of the frame operator. First we need a notion of an adjoint lattice.

**Definition 4.4.3.** Let \( \Lambda \) be a lattice in \( \mathcal{G} \times \hat{\mathcal{G}} \). Then the **adjoint lattice** \( \Lambda^\circ \) of \( \Lambda \) is defined as the set of points for which the time-frequency shift operators \( \pi(\lambda^\circ) \) commute with all time-frequency operators \( \pi(\lambda) \), \( \lambda \in \Lambda \),

\[
\Lambda^\circ = \left\{ \lambda^\circ \in \mathcal{G} \times \hat{\mathcal{G}} : \pi(\lambda) \pi(\lambda^\circ) = \pi(\lambda^\circ) \pi(\lambda), \lambda \in \Lambda \right\}.
\]

In \([14, 27, 30]\) it is shown that the frame operator \( S \) satisfies **Janssen representation**,\n
\[
S = \frac{1}{|\Lambda|} \sum_{\lambda^\circ \in \Lambda^\circ} \langle g, \pi(\lambda^\circ) g \rangle \pi(\lambda^\circ).
\]

(4.7)

At this point, the question arises if we can deduce the invertibility of the operator \( S \) from the Janssen coefficients (\( \langle g, \pi(\lambda^\circ) g \rangle \)). It is known from frame theory that if \( S \) is invertible, then its inverse is of the same type, that is, it also has a Janssen representation.

**4.5 Lattices in elementary locally compact abelian groups**

The notion of lattice in a time-frequency plane plays an important role in Gabor analysis, and different lattices specify different Gabor systems. The lattice comes into play when we specify the set of samples for the short time Fourier transform. The density and separability of the lattice can tell a lot about the properties of the Gabor system in question and in practice, specify the techniques, for example, to compute the dual atom.

Let \( \Lambda \) be a lattice of the time-frequency plane \( \mathcal{G} \times \hat{\mathcal{G}} \). Using the standard characters \( \chi \) for the group \( \mathcal{G} \times \hat{\mathcal{G}} \), the **orthogonal lattice** of \( \Lambda \) is defined as the set of points \( \chi \in \mathcal{G} \times \hat{\mathcal{G}} \) such that \( \chi(\lambda) = 1 \) for all \( \lambda \in \Lambda \). We will denote this lattice by \( \Lambda^\perp \). In the previous
section we defined the adjoint lattice of $\Lambda$, as the set of points $\lambda^o \in \Lambda^o$ for which the operators $\pi(\lambda)$ and $\pi(\lambda^o)$ commute for all $\lambda \in \Lambda$. In the commutation relation of time-frequency shifts arises a new character of $G \times \widehat{G}$, namely the symplectic character $\kappa$, (4.4). Therefore, the adjoint lattice can be described as

$$\Lambda^o = \left\{ \lambda^o \in G \times \widehat{G} : \kappa(\lambda^o, \lambda) = 1, \lambda \in \Lambda \right\}.$$  

According to the density/size of the lattice, we can divide them into rational and irrational ones. Another division of lattices is according to separability. We distinguish separable ones, sometimes called rectangular, and nonseparable. A lattice is separable, sometimes also referred to as rectangular if it is a product of a lattice in the time domain times a lattice in the frequency domain, that is $\Lambda = K \times L$, where $K$ is a lattice in $G$ and $L$ is a lattice in $\widehat{G}$. A special example of a separable lattice is $\Lambda = K \times K_\perp$, where $K$ is a lattice of $G$.

**Definition 4.5.1.** A lattice with $\Lambda^o_c \subseteq \Lambda_c$ shall be called co-isotropic. For such a lattice $\Lambda_c/\Lambda^o_c$ is the maximal noncommutative set.

Throughout the exposition we will focus our attention on the so-called elementary locally compact abelian group, namely a group of the form

$$G = \mathbb{R}^d \times \mathbb{T}^a \times \mathbb{Z}^b \times F$$

where $F$ is a finite abelian group of cardinality $m \geq 1$. They are in particular compactly generated and separable. $G$ will be written additively. It can be shown that quotients and closed subgroups of elementary groups are again elementary. From the point of view of Gabor analysis these groups are the most natural choice in many ways. Firstly, they are general enough to cover the classical harmonic analysis and the typical signal processing setups. Secondly, as observed by Rieffel [40] and pointed to the author by F. Luef, they are the only groups in which $\mathbb{Z}^n$ embeds as a lattice.

**Theorem 4.5.2** (Rieffel [40]). Let $\mathcal{G}$ be a locally compact abelian group such that $\Lambda = \mathbb{Z}^n$ embeds as a lattice in $G \times \widehat{G}$. Then $\mathcal{G} = \mathbb{R}^d \times \mathbb{T}^a \times \mathbb{Z}^b \times F$, where $n = 2d + a + b$ and $F$ is a finite abelian group.

**Proof.** Since $\Lambda$ is finitely generated, it is contained in an open compactly generated subgroup $H$ of $G \times \widehat{G}$. Since $(G \times \widehat{G})/\Lambda$ is compact, the same is true of $(G \times \widehat{G})/H$. But $(G \times \widehat{G})/H$ is discrete, so must be finite. It follows that $G \times \widehat{G}$ is compactly generated, and by Theorem 9.8 of [12], $G \times \widehat{G}$ is of the form $\mathbb{R}^k \times \mathbb{Z}^l \times K$, where $K$ is compact. $G \times \widehat{G}$ is self-dual, and so must equally be of the form $\mathbb{R}^k \times \mathbb{T}^l \times \widehat{K}$, where $\widehat{K}$ is discrete. Since $G \times \widehat{G}$ is compactly generated, so must $\widehat{K}$ be. Therefore $\widehat{K}$ is of the form $\mathbb{Z}^c \times F$, where $F$ is a finite abelian group. Thus $G \times \widehat{G}$ is of the form $\mathbb{R}^k \times \mathbb{T}^l \times \mathbb{Z}^c \times F$, that is an elementary locally compact abelian group.

Since $G$ is the summand of $G \times \widehat{G}$ it must be of the form

$$G = \mathbb{R}^d \times \mathbb{T}^a \times \mathbb{Z}^b \times F.$$
Then of course $G \times \hat{G} = \mathbb{R}^{2d} \times \mathbb{T}^{a+b} \times \mathbb{Z}^{a+b} \times F \times \hat{F}$. But for $\Lambda = \mathbb{Z}^n$ to be a lattice of $G \times \hat{G}$, it is easily seen that one must have $n = 2d + a + b$.

Since $G$ is an elementary locally compact group, the Hilbert space $L^2(G)$ can be decomposed in the standard Hilbert spaces on the factor groups (using the Hilbert space tensor product $\otimes$):

$$L^2(G) = L^2(\mathbb{R}^d) \otimes L^2(\mathbb{T}^a) \otimes \ell^2(\mathbb{Z}^b) \otimes \mathbb{C}^m$$

**Definition 4.5.3.** We define a rational lattice as a discrete subgroup $\Lambda \subset G \times \hat{G}$ such that the lattice itself as well as its adjoint lattice contain the adjoint of a co-isotropic lattice (in the sense of a canonical isomorphism) i.e.

$$\Lambda/\Lambda^o = \Lambda_c/\Lambda^o = \mathbb{Z}_q \times \mathbb{Z}_q, \quad \Lambda^o/\Lambda_c = \mathbb{Z}_p \times \mathbb{Z}_p,$$

where $\mathbb{Z}_p$ and $\mathbb{Z}_q$ are finite cyclic groups with cardinality $p$ and $q$ respectively, and $\Lambda^o$ is a co-isotropic lattice with $\Lambda_c/\Lambda^o = \mathbb{Z}_{pq} \times \mathbb{Z}_{pq}$. The density of lattice $\Lambda$ is

$$s(\Lambda) = \frac{p}{q}, \quad \gcd(p, q) = 1.$$  

From the isotropic lattice we obtain $\{\tilde{\gamma}_{k,l}\}_{k,l=0}^{pq-1}$, with $\tilde{\gamma}_{0,0} = 0$, such that

$$\Lambda_c = \bigcup_{k,l=0}^{pq-1} (\tilde{\gamma}_{k,l} + \Lambda^o)$$

We associate each $\tilde{\gamma}_{k,l}$ with an element $(k, l) \in \mathbb{Z}_{pq} \times \mathbb{Z}_{pq}$. Then

$$\pi(\tilde{\gamma}_{m,n})\pi(\tilde{\gamma}_{k,l}) = e^{2\pi i ml/pq} \pi(\tilde{\gamma}_{m,n} + \tilde{\gamma}_{k,l}).$$

Since $\Lambda_c/\Lambda$ is a subgroup of $\Lambda_c/\Lambda^o$ of order $p^2$, the set of representatives $\{\gamma_{k,l}\}_{k,l=0}^{p-1} \subset G \times \hat{G}$ such that

$$\Lambda_c = \bigcup_{k,l=0}^{p-1} (\gamma_{k,l} + \Lambda). \quad (4.8)$$

is given by $\gamma_{k,l} = \tilde{\gamma}_{qk,ql}$. Therefore

$$\pi(\gamma_{m,n})\pi(\gamma_{k,l}) = \omega_p^{lm} \pi(\gamma_{m,n} + \gamma_{k,l}),$$

where $\omega = e^{2\pi i q/p}$.

Periodization of a lattice corresponds to restricting the sublattice on a coarser adjoint group, one has:

$$\Lambda^o = \bigcup_{k,l=0}^{p-1} (\mu_{k,l} + \Lambda_c^o),$$

where the set $\{\mu_{k,l}\}$ is uniquely determined by $\Lambda$ and the set $\{\gamma_{k,l}\}$.
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Examples

Case $\mathcal{G} = \mathbb{R}$: Let the lattice of $\mathcal{G} \times \mathcal{G}$ be $\Lambda = \alpha \mathbb{Z} \times \beta \mathbb{Z}$ with $\alpha \beta = \frac{p}{q} < 1$ and $\gcd(p, q) = 1$. Then the dual lattice is given by $\Lambda^0 = \frac{1}{\beta} \mathbb{Z} \times \frac{1}{\alpha} \mathbb{Z}$. The isotropic lattice of $\Lambda$ in this case is

$$\Lambda_c = \frac{\alpha}{p} \mathbb{Z} \times \frac{\beta}{p} \mathbb{Z}.$$ 

It’s easy to check that $\Lambda^0_c = \frac{p}{\beta} \mathbb{Z} \times \frac{p}{\alpha} \mathbb{Z} \subseteq \Lambda_c$ and that the set of representatives for $\Lambda_c / \Lambda$ is $\gamma_{k,l} = (\frac{k Ln}{p}, \frac{l Nn}{p})$ and for $\Lambda^0 / \Lambda^0_c$ it’s $\mu_{k,l} = (\frac{k L}{\beta}, \frac{l L}{\alpha})$, where $k, l \in \mathbb{Z}_p$.

Case $\mathcal{G} = \mathbb{Z}_L$: We take the lattice $\Lambda = \alpha \mathbb{Z}_L \times \beta \mathbb{Z}_L$, where $\alpha$ and $\beta$ divide $L$, and $\frac{\alpha \beta}{L} = \frac{p}{q} < 1$ with $p$ and $q$ relatively prime. The dual lattice is given by $\Lambda^0 = \frac{L}{\beta} \mathbb{Z}_L \times \frac{L}{\alpha} \mathbb{Z}_L$. Then the isotropic lattice is

$$\Lambda_c = \frac{\alpha}{p} \mathbb{Z}_L \times \frac{\beta}{p} \mathbb{Z}_L.$$ 

Since $p$ divides both $\alpha$ and $\beta$ the above lattice is well defined.
5 Group Algebras and Gabor Analysis

In this chapter we will recall the facts concerning group algebras and their representations and we will mention the $1 - 1$ correspondence of those representations with the representations of the corresponding groups. For the full exposition we refer to [41]. We also show how this topic arises in Gabor analysis.

5.1 Group Algebras

Let $G$ be a locally compact $\sigma$-compact unimodular group and $L^1(G)$ the space of integrable functions on $G$ with respect to a fixed Haar measure on $G$, i.e.

$$L^1(G) = \left\{ f : G \to \mathbb{C} : \|f\|_1 = \int_G |f(x)|dx < \infty \right\}.$$

The space $L^1(G)$ becomes the $\ast$-algebra with the operations of convolution and involution

$$(f \ast g)(x) = \int_G f(y)g(y^{-1}x)dy,$$

$$f^*(x) = \overline{f(x)}.$$

When $G$ is not discrete, $L^1(G)$ has no unit element with respect to convolution, however, it has a norm one approximate identity as a substitute for the unit element. The algebra $L^1(G)$ is unital if and only if $G$ is discrete, and in this case we will write $\ell^1(G)$. Moreover, since

$$\|f \ast g\|_1 < \|f\|_1 \|g\|_1$$

the space $L^1(G)$ is a Banach $\ast$-algebra.

The representations of the group $G$ are tightly connected with the representations of the group algebra $L^1(G)$. Namely, for any unitary strongly continuous representation $\pi$ of $G$ on some Hilbert space $\mathcal{H}$, we can define the operators

$$\pi^1(f) = \int_G f(x)\pi(x)dx$$

for all $f \in L^1(G)$.

One checks without difficulty

$$(i) \quad \pi^1(f \ast g) = \pi^1(f)\pi^1(g),$$

$$(ii) \quad \pi^1(f^*) = \pi^1(f)^*,$$

$$(iii) \quad \|\pi^1(f)\| \leq \|f\|.$$
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for all \( f, g \in L^1(G) \). Therefore we obtain a representation \( \pi^1 : L^1(G) \to \mathcal{U}(\mathcal{H}) \) of \( L^1(G) \), called integrated representation.

We would like to study now under which conditions a representation \( \pi^1 \) of \( L^1(G) \) determines a unique unitary representation of \( G \). We say that a representation \( \rho \) of \( G \) on \( \mathcal{H} \) is non-degenerate if \( \rho(G)\mathcal{H} = \mathcal{H} \). Quite generally, if \( A \) is a \(*\)-algebra, \( \mathcal{H} \) a Hilbert space and \( \rho \) a \(*\)-homomorphism \( A \to \mathcal{U}(\mathcal{H}) \), to the unitary operators on \( \mathcal{H} \), we have that \( \rho(A)\mathcal{H} = \mathcal{H} \). That follows immediately from

\[
\langle x, \rho(a)y \rangle = \langle \rho(a)^*x, y \rangle = \langle \rho(a^*)x, y \rangle
\]

for \( a \in A \) and \( x, y \in \mathcal{H} \). Thus, for any unitary representation of \( G \) on the Hilbert space \( \mathcal{H} \), the \(*\)-representation \( \pi^1 \) of \( L^1(G) \) is non-degenerate. There is a converse to this result.

**Theorem 5.1.1.** Let \( G \) be a unimodular locally compact group, \( \mathcal{H} \) a Hilbert space and \( \rho \) a non-degenerate \(*\)-representation of \( L^1(G) \) on \( \mathcal{H} \). Then there exists a unique unitary representation \( \pi \) of \( G \) on \( \mathcal{H} \) such that \( \rho = \pi^1 \).

The proof relies on the existence of approximate unit for \( L^1(G) \) and a contracting property of \(*\)-homomorphism \( \rho : L^1(G) \to \text{End}(\mathcal{H}) \). For the proof we refer to [41].

The preceding Theorem gives the following important result.

**Corollary 5.1.2.** Let \( G \) be a unimodular locally compact group. Then there exists a \( 1-1 \) correspondence between unitary representations of \( G \) and integrated representations of \( L^1(G) \).

### 5.2 Twisted Group Algebras

In time-frequency analysis we have to deal with a particular locally compact group, the Heisenberg group. It is defined as a direct product \( G \times \mathbb{T} \), where \( G \) is a locally compact abelian group and \( \mathbb{T} \) a group of complex numbers of modulus 1.

**Definition 5.2.1.** Let \( G \) be a locally compact abelian group. Then a 2-cocycle (or representation multiplier) on \( G \) is a continuous function \( c : G \times G \to \mathbb{T} \) that satisfies

\[
(i) \quad |c(x, y)| = 1 \quad \text{for all} \quad x, y \in G; \\
(ii) \quad c(e, x) = c(x, e) = 1; \\
(iii) \quad c(x, y)c(xy, z) = c(y, z)c(x, yz) \quad \text{for all} \quad x, y, z \in G.
\]

We define the multiplication in the Heisenberg group as

\[
(x, \eta)(y, \tau) = (xy, \eta \tau c(x, y))
\]
where \( c \) is a 2-cocycle associated to \( G \). The Heisenberg group is a locally compact group with respect to the product topology, and as a consequence, the Haar measure is the product measure of \( G \) and \( T \).

In Chapter 4 we defined the notion of the representation. A more general type of the representation, called a projective representation, arises while studying for example the Heisenberg group of \( G \). A **projective representation** of a locally compact abelian \( \sigma \)-compact group \( G \) on a Hilbert space \( H \) is a family of unitary operators \( \{ \rho(x) \}_{x \in G} \) that satisfy the following conditions

(i) \( \rho(e) = I_H \), where \( e \) is a unit of \( G \);

(ii) \( \rho(x)\rho(y) = \rho(xy)c(x, y) \), for \( c \) a 2-cocycle of \( G \) and all \( x, y \in G \);

(iii) the mapping \( x \mapsto \langle f, \rho(x)g \rangle \) is a Borel function on \( G \) for all \( x, y \in H \).

As with the standard representations we can define similar notions. A subspace \( V \) of \( H \) is \( \rho \)-invariant if \( \rho(x)V \subseteq V \) for all \( x \in G \). The representation \( \rho \) is irreducible if the only \( \rho \)-invariant and closed subspaces of \( H \) are \( \{0\} \) and \( H \). A **representation coefficient** of \( \rho \) is a function of the form \( g \mapsto \langle f, \rho(x)g \rangle \) for any two \( f, g \in H \).

If \( \rho \) is a projective representation of \( G \) for the 2-cocycle \( c \), then

\[
\pi(x, \eta) = \eta \rho(x)
\]

defines in a natural way a unitary representation of the Heisenberg group \( H = G \times T \). It turns out that the projective representation of \( G \) contains the same information as the unitary representation of the Heisenberg group \( G \times T \). The following results were obtained independently by Christensen, see [7], and Luef [34]. We follow the exposition in [34].

**Lemma 5.2.2.** Let \( G \) be a locally compact abelian group and \( c \) a cocycle of \( G \). Then the following hold:

(i) if the mapping \( x \mapsto \rho(x)f \) is continuous at \( e \), then it is continuous at any group element \( y \);

(ii) if \( x \mapsto \langle f, \rho(x)f \rangle \) is continuous for all \( f \in H \), then \( \rho \) is strongly continuous;

(iii) \( \rho \) is irreducible if and only if for each non-zero \( f \)

\[
\text{span}\{\rho(x)f : x \in G\} = H;
\]
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Proof. Let $x, y \in G$ and $f \in H$. Then

$$||\rho(x)f - \rho(y)f|| = ||\rho(yy^{-1})f - \rho(y)f||$$
$$= ||\rho(y)(c(y, y^{-1})x\rho(y^{-1})f - f)||$$
$$= ||c(y, y^{-1})x\rho(y^{-1})f - f||$$
$$\leq ||c(y, y^{-1})x\rho(y^{-1})f - c(y, y^{-1})f|| + ||c(y, y^{-1})f - f||$$
$$= ||\rho(y^{-1})f - f|| - |c(y, y^{-1})f - 1| \cdot ||f|| \to 0$$

for $x \to y$. Which proves (i). For (ii) we have

$$||\rho(x)f - \rho(e)f||^2 = ||\rho(x)f - f||^2$$
$$= ||\rho(x)||^2 + ||f||^2 - \langle f, \rho(x)f \rangle - \langle \rho(x)f, f \rangle \to 0$$

for $x \to e$, and the irreducibility of $\rho$ follows the same line of reasoning as for unitary representations. \qed

Lemma 5.2.3. Let $G$ be a locally compact abelian group and $c$ a cocycle of $G$. Then the following hold:

(i) the projective representation $\rho$ of $G$ is irreducible if and only if the unitary representation $\pi$ of $H = G \times \mathbb{T}$ is irreducible;

(ii) the projective representation $\rho$ of $G$ is strongly continuous if and only if the unitary representation $\pi$ of $H = G \times \mathbb{T}$ is strongly continuous.

Proof. Let $f \in H$. Then

(i) Follows from the preceding lemma and

$$\text{span}\{\rho(x)f : x \in G\} = \text{span}\{\eta\rho(x)f : \eta \in \mathbb{T}, x \in G\}$$
$$= \text{span}\{\pi(x, \eta)f : (x, \eta) \in G \times \mathbb{T}\}.$$

(ii) Since $H = G \times \mathbb{T}$ is equipped with the product topology, the continuity of $\rho$ follows from that of $\pi$. Now suppose that $\rho$ is continuous. Then

$$||\pi(x, \eta)f - f|| = ||\eta\rho(x)f - f||$$
$$\leq ||\eta\rho(x)f - \rho(x)f|| + ||\rho(x)f - f||$$
$$= |1 - \eta| \cdot ||f|| + ||\rho(x)f - f||,$$

from which the desired result follows. \qed
Similarly, an irreducible continuous projective representation $\rho$ of a locally compact abelian group $G$ is **integrable** if there exists a nonzero $f \in \mathcal{H}$ such that

$$\int_G |\langle f, \rho(x)f \rangle| dx < \infty.$$ 

Furthermore, $\rho$ is called **square-integrable** if there exists a nonzero $f \in \mathcal{H}$ such that

$$\int_G |\langle f, \rho(x)f \rangle|^2 dx < \infty,$$

and we obtain the following equivalence.

**Lemma 5.2.4.** An irreducible continuous projective representation $\rho$ of a locally compact abelian group $G$ is (square) integrable if and only if the unitary representation $\pi$ of the Heisenberg group attached to $G$ and a 2-cocycle $c$ is (square) integrable.

The Lemma follows from the trivial observation, that $|\langle f, \rho(x)f \rangle| = |\langle f, \pi(x, \eta)f \rangle|$. 

Let $\rho$ be a projective representation of $G$ with a 2-cocycle $c$. Then as observed above, $\pi(x, \eta) = \eta \rho(x)$ is a unitary representation of the Heisenberg group $H = G \times \mathbb{T}$. Since the center $Z(H)$ of the Heisenberg group is $\mathbb{T}$, let $L^1(H, Z(H))$ be the space of complex-valued functions $f$ on $H$ such that $f(hz) = zf(h)$ for $h \in H, z \in Z(H)$. 

Since every element $h = (x, \tau)$ of $H$ can be written as a product $(x, \tau) = (x, 1)(e, \tau)$, elements $f$ of $L^1(H, Z(H))$ are determined by their restriction to $H/Z(H) \cong \mathcal{G}$. Normalizing the Haar measure on $H$ so that $Z(H)$ has mass 1, $L^1(H, Z(H))$ becomes an algebra under convolution, given by the usual formula,

$$(f * g)(h') = \int_H f(h)g(h^{-1}h')dh \quad \text{for} \quad f, g \in L^1(H, Z(H)).$$

Since a given $f \in L^1(H, Z(H))$ is determined by its restriction to $\mathcal{G}$, there is an isomorphism $r: L^1(H, Z(H)) \to L^1(\mathcal{G})$. The convolution in $L^1(H, Z(H))$ is expressed in $L^1(\mathcal{G})$ by twisting the usual convolution with a 2-cocycle

$$r(f) \cdot r(g)(x) := r(f * g)(x) = \int_\mathcal{G} r(f)(y)r(g)(y^{-1}x)c(x, y^{-1}x)dy, \quad \text{for} \quad x \in \mathcal{G}.$$ 

We will denote the space $L^1(\mathcal{G})$ with a convolution defined above as $L^1(\mathcal{G}, c)$ to indicate its dependence on the 2-cocycle $c$. $L^1(\mathcal{G}, c)$ together with convolution defined above and involution given by

$$f^*(x) = c(x, x^{-1})f(x^{-1})$$

becomes the twisted convolution algebra. In general,
Definition 5.2.5. Let $G$ be a locally compact abelian group and $c$ a 2-cocycle associated to it. The twisted convolution algebra associated to $(G, c)$ is an algebra of all absolutely integrable functions with the following multiplication and involution

(i) $(f \circ g)(x) = \int_G f(y) g(y^{-1}x) c(x, y^{-1}x) dy$ called twisted convolution;

(ii) $f^*(x) = c(x, x^{-1}) f(x^{-1})$.

As strongly continuous unitary representations of $G$ induce a representation of $L^1(G)$, a continuous projective representation $\rho$ of $G$ on a Hilbert space $H$ induces a representation of $L^1(G, c)$ by integration

$$\rho^1(f) = \int_G f(x) \rho(x) dx,$$

that is $\rho^1$ satisfies

(i) $\rho^1(f \circ g) = \rho^1(f) \rho^1(g)$,

(ii) $\rho^1(f^*) = \rho^1(f)^*$,

(iii) $||\rho^1(f)|| \leq ||f||_1$.

Like in the case of the unitary representations of $G$ and integrated representations of $L^1(G)$ we have the $1-1$ correspondence between the projective representations of $G$ and integrated representations of $L^1(G, c)$.

Theorem 5.2.6. Let $G$ be a unimodular locally compact group. Then there exists a $1-1$ correspondence between continuous projective representations of $G$ and integrated representations of $L^1(G, c)$.

We can sum it up, by noticing that unitary representations of locally compact abelian group $G$ give rise to group $\ast$-algebras that are commutative with multiplication - standard convolution, and the projective representations of $G$ induce twisted group $\ast$-algebras with a non-commutative multiplication - twisted convolution.

5.3 Gabor Analysis and Integrated Representations

In this section we will focus on the representations of the subgroup of time-frequency plane $\mathcal{G} \times \mathcal{G}$, where $\mathcal{G}$ is a locally compact abelian group. Let $\Lambda$ be a time-frequency lattice of $\mathcal{G} \times \mathcal{G}$. Then the time-frequency shifts $\pi(\lambda) = M_{\chi}T_x$, $\lambda = (x, \chi) \in \Lambda$, form a projective representation for $\Lambda$,

$$\pi(\lambda)\pi(\lambda') = c(\lambda, \lambda')\pi(\lambda + \lambda'),$$

where $c(\lambda, \lambda') = \chi'(x)$ and $\lambda = (x, \chi), \lambda' = (x', \chi') \in \mathcal{G} \times \mathcal{G}$. This special form of the 2-cocycle is called a bicharacter, and was already introduced in Chapter 4. Let $a = a(\lambda)$ be an element of $\ell^1(\Lambda)$, then the integrated representation of $\ell^1(\Lambda)$ takes the form

$$\pi^1(a) = \sum_{\lambda \in \Lambda} a(\lambda)\pi(\lambda)$$
Twisted convolution algebras arise naturally in Gabor analysis, and their elements are commonly known under Janssen representation of the Gabor frame operator. Namely, let $g$ be such that $c_g \in L^1(G)$ and $\mathcal{G}(g, \Lambda)$ be the associated Gabor system. Recall from Chapter 4, that the Janssen representation of the Gabor frame operator $Sf = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)g, \ f \in L^2(\mathcal{G})$, is given by

$$S = \sum_{\lambda^0 \in \Lambda^0} \langle g, \pi(\lambda^0)g \rangle \pi(\lambda^0),$$

where $\Lambda^0$ is the adjoint lattice of $\Lambda$. The sequence $a(\lambda^0) = \langle g, \pi(\lambda^0)g \rangle$ is an element of $\ell^1(\Lambda^0)$, because $c_g \in L^1(G)$. Therefore the Janssen representation of $S$ is nothing more but the integrated representation of $\ell^1(\Lambda^0)$

$$S = \pi^1(a) = \sum_{\lambda^0 \in \Lambda^0} a(\lambda^0) \pi(\lambda^0).$$

In frame theory we are interested in inverting the frame operator $S$ in order to obtain a decomposition of a function $f \in L^2(\mathcal{G})$ in terms of the shifted and modulated, along some lattice $\Lambda$, atom $g$. That is, if $S$ is invertible, then the dual window to $g$ is $\gamma = S^{-1}g$ and

$$f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\gamma \rangle \pi(\lambda)g.$$

An important consequence of the fact that Janssen representation is merely the integrated representation of $(\ell^1(\Lambda^0), \hat{z})$ is that knowing that two integrated representations are equivalent will give us the information which two Gabor systems are equivalent. That was already observed, in a different context, by Gröchenig in [23]. Namely, if $\pi$ and $\mu$ are two projective representations of $\Lambda^0$, then, by Theorem 5.2.6, the integrated representations $\pi^1$ and $\mu^1$ of $\ell^1(\Lambda^0)$ are also equivalent. Let $U$ be the intertwining operator for $\pi$ and $\mu$, then $U \pi U^{-1} = \mu$, and $S$ Gabor frame operator associated to $g$ in the Janssen representation. Then

$$USU^{-1} = U \sum_{\lambda^0 \in \Lambda^0} \langle g, \pi(\lambda^0)g \rangle \pi(\lambda^0)U^{-1} = \sum_{\lambda^0 \in \Lambda^0} \langle g, \pi(\lambda^0)g \rangle U \pi(\lambda^0)U^{-1} = \sum_{\lambda^0 \in \Lambda^0} \langle g, U^{-1} \mu(\lambda^0)Ug \rangle \mu(\lambda^0) = \sum_{\lambda^0 \in \Lambda^0} \langle Ug, \mu(\lambda^0)Ug \rangle \mu(\lambda^0).$$

Therefore the Gabor system associated to $g$ and time-frequency shifts $\pi(\lambda)$ is equivalent to the Gabor system associated to $Ug$ and time-frequency shifts $\mu(\lambda)$.

Later, in Chapter 6 we construct explicitly all the intertwining operators for the representations of the time-frequency plane characterizing at the same time all representations of the twisted group algebra $(\ell^1(\Lambda^0), \hat{z})$.

The invertibility of Gabor frame operators in this context was studied already by Gröchenig and Leinert [25], where they proved, using the techniques of Banach algebras.
the inverse closeness of the twisted convolution algebra \((\ell^1(\Lambda), \circledast)\). Their proof however was not constructive. In Chapter 7 we will present another proof of their result, that gives the explicit construction of the inverse of a sequence with respect to twisted convolution. The proof works in continuous \((\mathbb{R}^d)\) as well as in discrete finite case \((\mathbb{Z}_m)\), therefore providing an algorithm for inverting Gabor frame operators.

In the following two chapters we will study two issues concerning integrated representations of \(L^1(\mathcal{G})\), arising from the projective representations of \(\mathcal{G}\). One, treated in Chapter 6, is to characterize all integrated representations of \(L^1(\mathcal{G})\), \(\mathcal{G}\) being finite cyclic group, which is achieved through metaplectic operators. Another issue is to find the explicit inverse of the operator \(\pi_1(f)\), which boils down to finding an inverse of \(f\) in the twisted convolution algebra \(L^1(\mathcal{G})\).
6 Metaplectic Operators

The metaplectic representation is concerned with a class of automorphisms of the Heisenberg group $H = H(G)$, where $G$ is a locally compact abelian group [39], [51]. The case $G = \mathbb{R}^d$ has been deeply investigated for example in [17], [48], [37]. We focus on the case $G = \mathbb{Z}_n$, finite cyclic group, where we construct explicitly the metaplectic operators. The metaplectic operators for finite fields were studied in [36], [47], and in the case when the group has an order of power of 2 in [19], [2], and [50]. The important new features of our approach are:

- It works for general $n$, even and odd.
- We present a general and explicit construction, while the results found in the literature provide explicit formulas only for special cases;
- We use chirp functions defined for general $n$ (even or odd), see Section 6.2.

Moreover, the explicit knowledge of metaplectic operators gives the characterization of equivalent representations of twisted group algebra $\ell^1(\mathbb{Z}_n, c)$, which we mentioned in Chapter 5.

This chapter consists of the results obtained in the cooperation with H.G. Feichtinger, M. Hazewinkel, N. Kaiblinger, and M. Neuhauser, [26].

6.1 Notations

We consider functions on the cyclic group $\mathbb{Z}_n = \{0, 1, \ldots, n - 1\}$. The translation and modulation operators on $\mathbb{C}^n$ are defined by

$$Tkg(m) = g(m - k), \quad k \in \mathbb{Z}_n,$$
$$M_lg(m) = e^{2\pi i lm/n}g(m), \quad l \in \mathbb{Z}_n, \quad g \in \mathbb{C}^n,$$

where all computations of the indices are modulo $n$. As before, in the case of locally compact abelian groups, we will denoted the time-frequency shift operator by

$$\pi(\lambda) = M_lT_k, \quad \lambda = (k, l) \in \mathbb{Z}_n \times \mathbb{Z}_n.$$

Let $\mathbb{T} = \{\tau \in \mathbb{C} : |\tau| = 1\}$ denote the circle group. The commutation relation

$$M_lT_k = e^{2\pi i kl/n}T_kM_l, \quad k, l \in \mathbb{Z}_n, \quad (6.1)$$

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implies that the composition of time-frequency shift operators is given by
\[ \pi(\lambda)\pi(\lambda') = c(\lambda, \lambda')\pi(\lambda + \lambda'), \quad \text{where} \quad c(\lambda, \lambda') = e^{2\pi i (\lambda, \kappa \lambda')/n}, \quad \kappa = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \]
and the inverse is \( \pi(\lambda)^{-1} = e^{2\pi i (\lambda, \kappa \lambda)/n} \pi(-\lambda) \).

### 6.2 Second degree characters

For \( Z_n \) with \( n \) even, the second degree characters can only be constructed by the induced representation approach. The explicit form of the second degree characters were first introduced by N. Kaiblinger in [31], where he pointed out that simple extension of the definition from the continuous case doesn’t work in the case of \( n \) odd. Therefore he distinguished between the case of \( n \) even and \( n \) odd and defined

(i) the high-frequency second degree character associated to \( c \in \mathbb{Z}_n \) for both \( n \) even and \( n \) odd, as
\[
\psi_c(m) = e^{\pi i c m^2 / (n+1)} / n, \quad m \in \mathbb{Z}_n.
\]
(ii) the low-frequency second degree character associated to \( c \in \mathbb{Z}_n \) that exists only for \( n \) even, as
\[
\psi_c(m) = e^{\pi i c m^2 / n}, \quad m \in \mathbb{Z}_n.
\]
The other references concerning second degree characters, or also refer to as chirp functions, are [5], [6]. Here we gather the above results by providing a unified formula for \( n \) even and \( n \) odd when the group is \( \mathbb{Z}_n \times \mathbb{Z}_n \).

For \( G = \mathbb{Z}_n \), a bicharacter is of the form
\[
B(m, m') = e^{2\pi i c m m' / n}, \quad m, m' \in \mathbb{Z}_n,
\]
for some \( c \in \mathbb{Z}_n \).

**Lemma 6.2.1.** Given \( c \in \mathbb{Z}_n \), fix a representative \( [c] \in \mathbb{Z} \) for \( c \) and define
\[
\psi_{[c]}(m) = e^{\pi i [c] m^2 (n+1)} / n, \quad m \in \mathbb{Z}_n.
\]
Then \( \psi = \psi_{[c]} \) is a second degree character associated to \( c \), that is,
\[
\psi(m + m') = \psi(m) \psi(m') e^{2\pi i c m m' / n}, \quad m, m' \in \mathbb{Z}_n.
\]

Before we proceed with the proof, we give the explicit formula for the second degree characters on \( \mathbb{Z}_n \times \mathbb{Z}_n \). It was already mentioned in Chapter 4 that the second degree characters on the product group can be constructed from the ones on the factors. Therefore, let \( \sigma = \begin{pmatrix} p & q \\ q & \sigma \end{pmatrix} \) be a symmetric matrix with elements in \( \mathbb{Z}_n \) and \( B \) a bicharacter,
\[
B(\lambda, \lambda') = e^{2\pi i (\lambda, \sigma \lambda') / n}, \quad \lambda, \lambda' \in \mathbb{Z}_n \times \mathbb{Z}_n
\]
associated to \( \sigma \). Then the second degree character on \( \mathbb{Z}_n \times \mathbb{Z}_n \) associated to \( B \) takes the form

\[
\psi(\lambda) = \psi[p](k)\psi[r](l)e^{2\pi i qkl/n} = e^{\pi i [p]k^2 (n+1)/n}e^{\pi i [r]l^2 (n+1)/n}e^{2\pi i qkl/n},
\]

where \( \lambda = (k, l) \in \mathbb{Z}_n \times \mathbb{Z}_n \).

**Proof of Lemma 6.2.1.** First we point out that the choice of representatives for \( m \) in the definition of \( \psi[c] \) is irrelevant. Indeed,

\[
\psi(m + n) = e^{\pi i [c](m+n)^2 (n+1)/n} = e^{\pi i [c]m^2 (n+1)/n}e^{\pi i [c]n(n+1)}e^{2\pi i [c]m(n+1)} = \psi(m), \quad m \in \mathbb{Z}_n.
\]

The above computation holds for any given \( n \), be it even or odd. Thus \( \psi \) is well-defined and we calculate

\[
\psi(m + m') = e^{\pi i [c](m+m')^2 (n+1)/n} = e^{\pi i [c]m^2 (n+1)/n}e^{\pi i [c]m'^2 (n+1)/n}e^{2\pi i [c]mm' (n+1)/n} = \psi(m) \psi(m') e^{2\pi i cm'm'/n}, \quad m, m' \in \mathbb{Z}_n.
\]

**Remark 6.2.2.** For \( n \) odd, \( \psi[c] \) is uniquely defined, independent of the choice of \([c]\). On the other hand, by Lemma 4.3.1 in Chapter 4, for \( n \) even, there are two possible vectors \( \psi[c]_1, \psi[c]_2 \) depending on the choice of \([c]\) and they differ by the modulation

\[
\psi[c]_2(m) = (-1)^m \psi[c]_1(m), \quad m \in \mathbb{Z}_n.
\]

By \( M_{2,2}(\mathbb{Z}_n) \) we denote the set of \( 2 \times 2 \) matrices with elements in \( \mathbb{Z}_n \). The special linear group \( SL_2(\mathbb{Z}_n) \) consists of all matrices in \( M_{2,2}(\mathbb{Z}_n) \) with determinant \( 1 \) (computed modulo \( n \)). A matrix \( A = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in M_{2,2}(\mathbb{Z}_n) \) that satisfies \( a^T c = c^T a \), \( b^T d = d^T b \), and \( a^T d - c^T b = 1 \) is called **symplectic**. An equivalent way to express that \( A \) is symplectic is by the condition \( A^T JA = J \), with \( J = (\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}) \). For the case we consider here, that is for \( \mathbb{Z}_n \), we have that \( A \) is symplectic if and only if \( \text{det} A = 1 \). In other words, the symplectic group over \( \mathbb{Z}_n \) is the special linear group \( SL_2(\mathbb{Z}_n) \).

**Definition 6.2.3.**

(i) Given \( A = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in M_{2,2}(\mathbb{Z}_n) \), we define \( \sigma = \sigma_A \in M_{2,2}(\mathbb{Z}_n) \) by

\[
\sigma_A := A^T \kappa A - \kappa = \begin{pmatrix} -ac & 1 - ad \\ -bc & -bd \end{pmatrix} = -\begin{pmatrix} ac & ad - 1 \\ bc & bd \end{pmatrix}, \quad (6.2)
\]

with \( \kappa \) given above.
(ii) Let $\sigma \in M_{2,2}(\mathbb{Z}_n)$ be symmetric, e.g. $\sigma = \sigma^T$, where $\sigma^T$ is the transposed of $\sigma$. A function $\psi: \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{T}$ that satisfies
\[
\psi(\lambda + \lambda') = \psi(\lambda)\psi(\lambda')e^{2\pi i(\lambda,\sigma\lambda')/n}, \quad \lambda, \lambda' \in \mathbb{Z}_n \times \mathbb{Z}_n,
\] (6.3)
is called a second degree character associated to $\sigma$. Given a symmetric matrix $\sigma$, we denote by $\Psi(\sigma)$ the set of all second degree characters associated to $\sigma$.

The following lemma describes relations between second degree characters associated to different $\sigma$’s.

**Lemma 6.2.4.** Given $A, B \in \text{SL}_2(\mathbb{Z}_n)$, let $\psi_A \in \Psi(\sigma_A)$ and $\psi_B \in \Psi(\sigma_B)$.

(i) The complex conjugate or reciprocal $\overline{\psi}_A = \psi_A^{-1}$ belongs to $\Psi(-\sigma_A)$.

(ii) $\psi(\lambda) := \overline{\psi}_A(A^{-1}\lambda)$ belongs to $\Psi(\sigma_A^{-1})$.

(iii) $\psi(\lambda) := \psi_A(B\lambda)\psi_B(\lambda)$ belongs to $\Psi(\sigma_{AB})$.

**Proof.**

(i) By direct computation, $\overline{\psi}_A = \psi_A^{-1}$ satisfies (6.3) with $\sigma$ replaced by $-\sigma$.

(ii) $\psi$ satisfies (6.3) with $\sigma$ replaced by
\[-A^{-T}\sigma_AA^{-1} = -(\kappa - A^{-T}\kappa A^{-1}) = \sigma_{A^{-1}}.\]

(iii) $\psi$ satisfies (6.3) with $\sigma$ replaced by
\[B^T\sigma_AB + \sigma_B = (B^T A^T A B - B^T \kappa B) + (B^T \kappa B - \kappa) = B^T A^T \kappa AB - \kappa = \sigma_{AB}.\]

6.3 Heisenberg group and its automorphisms

We defined the Heisenberg group $H_n = H(\mathbb{Z}_n)$ in the form of the Schrödinger representation, i.e., by time-frequency shift operators
\[H_n := \{ \tau \pi(\lambda) : \lambda \in \mathbb{Z}_n \times \mathbb{Z}_n, \ \tau \in \mathbb{T} \}\]

This is a group under composition, the (reduced) Heisenberg group over $\mathbb{Z}_n$. We are concerned with automorphisms of $H_n$ that are based on the transformation
\[\pi(\lambda) \mapsto \pi(A\lambda), \quad \lambda \in \mathbb{Z}_n \times \mathbb{Z}_n,\]
for some $A = (a \ b \ c \ d) \in M_{2,2}(\mathbb{Z}_n)$, i.e., on the modification of time-frequency shift parameters $(k,l) \mapsto (ak + bl, ck + dl)$. This class of automorphisms of $H_n$ is important in the modifications of Gabor frames. It can be shown that this class consists exactly of those automorphisms that leave the center $\{ \tau Id : \tau \in \mathbb{T} \}$ of $H_n$ pointwise fixed.
6.3 Heisenberg group and its automorphisms

Proposition 6.3.1. Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2,2}(\mathbb{Z}_n)$, suppose that the mapping

$$\tau \pi(\lambda) \mapsto \tau \psi(\lambda) \pi(A \lambda), \quad \tau \in \mathbb{T}, \quad \lambda \in \mathbb{Z}_n \times \mathbb{Z}_n,$$

is an automorphism of $H_n$, for some $\psi : \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{T}$. Then $A$ belongs to $\text{SL}_2(\mathbb{Z}_n)$ and $\psi$ is a second degree character associated to $\sigma_A$.

Proof. Since the given mapping is a homomorphism, the image of the operator product

$$\pi(\lambda)\pi(\lambda') = e^{2\pi i (\lambda \cdot \lambda')/n} \pi(\lambda + \lambda') \mapsto e^{2\pi i (\lambda \cdot \lambda')/n} \psi(\lambda + \lambda') \pi(A \lambda + A \lambda')$$

must coincide with the composition of the images

$$\psi(\lambda) \pi(A \lambda) \psi(\lambda') \pi(A \lambda') = e^{2\pi i (A \lambda \cdot A \lambda')/n} \psi(\lambda) \psi(\lambda') \pi(A \lambda + A \lambda').$$

Therefore $\psi$ satisfies

$$\frac{\psi(\lambda + \lambda')}{\psi(\lambda) \psi(\lambda')} = e^{2\pi i (A \lambda \cdot A \lambda')/n} e^{-2\pi i (\lambda \cdot \lambda')/n}, \quad \lambda, \lambda' \in \mathbb{Z}_n \times \mathbb{Z}_n,$$  \hspace{1cm} (6.4)

i.e., (6.3) holds with $\sigma = \sigma_A$ given in (6.2). In particular, $\sigma$ is necessarily symmetric, since the left-hand side of (6.4) is invariant when $\lambda$ and $\lambda'$ are interchanged. As observed above if $\sigma$ is symmetric it implies $A \in \text{SL}_2(\mathbb{Z}_n)$. \hfill $\Box$

It is known that the Schrödinger representation is irreducible [22] and by Schur’s lemma this fact is equivalent to the following result.

Lemma 6.3.2. Suppose that an operator $Q$ satisfies

$$Q \pi(\lambda) = \pi(\lambda) Q, \quad \text{for all } \lambda \in \mathbb{Z}_n \times \mathbb{Z}_n.$$  

Then $Q$ is a scalar multiple of the identity operator, $Q = c \cdot \text{Id}$ for some $c \in \mathbb{C}$.

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Metaplectic operators on $\mathbb{C}^n$ are unitary operators $U$ that intertwine the automorphisms of the Heisenberg group $H_n$ described in Proposition 6.3.1. That is, they satisfy

$$U \pi(\lambda) U^{-1} = \psi(\lambda) \pi(A \lambda), \quad \lambda \in \mathbb{Z}_n \times \mathbb{Z}_n,$$ \hspace{1cm} (6.5)

for given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}_n)$ and a second degree character $\psi$ associated to $\sigma = \sigma_A$. We use the following consequence of Lemma 6.3.2.

Lemma 6.3.3. Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}_n)$, the following holds. Two unitary operators $U = U_1, U_2$ satisfy (6.5) for the same $\psi \in \Psi(\sigma_A)$ if and only if $U_2 = \tau U_1$ for some $\tau \in \mathbb{T}$.

Proof. By assumption $U_1 \pi(\lambda) U_1^{-1} = U_2 \pi(\lambda) U_2^{-1}$ for all $\lambda \in \mathbb{Z}_n \times \mathbb{Z}_n$. In other words the operator $Q := U_1^{-1} U_2$ satisfies $Q \pi(\lambda) Q^{-1} = \pi(\lambda)$. Hence Lemma 6.3.2 implies $Q = c \text{Id}$, for some $c \in \mathbb{C}$. Since $Q$ is unitary $c \in \mathbb{T}$. \hfill $\Box$
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The next lemma describes the inner automorphisms of $H_n$. Note that for $A = (\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})$ we have $\sigma_A = 0$ and by Lemma 4.3.1(ii) the associated second degree characters are just the usual characters. That is, if $\chi \in \Psi(\sigma_A) = \Psi(0)$, then $\chi(\lambda) = e^{2\pi i (\mu, \lambda)/n}$, for some $\mu \in \mathbb{Z}_n \times \mathbb{Z}_n$.

**Lemma 6.3.4.** Given $\lambda' = (k', l') \in \mathbb{Z}_n \times \mathbb{Z}_n$, the operator $W = \pi(\lambda')$ satisfies (6.5) for $A = Id = (\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})$,

$$W \pi(\lambda) W^{-1} = e^{2\pi i (J \lambda', \lambda)/n} \pi(\lambda),$$

where $J := \kappa - \kappa^T = (\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})$.

**Proof.** By using (6.1),

$$(M_{l'}T_{k'})^{-1}M_lT_kM_{l'}T_{k'} = M_{l'}T_{-k'}M_{l}T_{k}M_{l'} = e^{2\pi i (k'[,k')/n}M_{l}T_{k}.$$

The next result describes the relation between different intertwiners that correspond to the same $A \in \text{SL}_2(\mathbb{Z}_n)$.

**Lemma 6.3.5.** Given $A \in \text{SL}_2(\mathbb{Z}_n)$, suppose that a unitary operator $U = U_1$ satisfies (6.5), for some $\psi = \psi_1 \in \Psi(\sigma_A)$. Then a unitary operator $U = U_2$ satisfies (6.5), for some, possibly different $\psi = \psi_2 \in \Psi(\sigma_A)$, if and only if $U_2 = U_1W$ for some $W \in H_n$.

**Proof.** ($\Rightarrow$) Since by assumption

$$U_1 \pi(\lambda)U_1^{-1} = \psi_1(\lambda)\pi(A \lambda) \quad \text{and} \quad U_2 \pi(\lambda)U_2^{-1} = \psi_2(\lambda)\pi(A \lambda)$$

we conclude that the operator $W := U_1^{-1}U_2$ satisfies

$$W \pi(\lambda) W^{-1} = \frac{\psi_2(\lambda)}{\psi_1(\lambda)} \pi(\lambda). \quad (6.6)$$

By Lemma 4.3.1 the quotient $\chi(\lambda) := \psi_2(\lambda)/\psi_1(\lambda)$ is a character. By Lemma 6.3.4 there is an element $W = W_0 \in H_n$ such that (6.6) holds. By Lemma 6.3.3 the choice of $W$ is unique up to a scalar $\tau \in \mathbb{T}$, i.e., $W = \tau W_0$. Therefore also $W$ belongs to $H_n$.

($\Leftarrow$) By Lemma 6.3.4 the operator $U_2 = U_1W$ satisfies

$$U_1W \pi(\lambda) W^{-1}U_1^{-1} = \chi(\lambda) U_1 \pi(\lambda) U_1^{-1} = \chi(\lambda) \psi_1(\lambda)\pi(A \lambda),$$

and we note that by Lemma 4.3.1 the function $\psi_2 := \chi \psi_1$ indeed belongs to $\Psi(\sigma_A)$.

Next, we show how to obtain intertwiners that correspond to inverses or products of matrices $A, B \in \text{SL}_2(\mathbb{Z}_n)$.

**Lemma 6.3.6.** Given $A, B \in \text{SL}_2(\mathbb{Z}_n)$, let $U = U_A U_B$ satisfy (6.5) for $\psi_A \in \Psi(\sigma_A)$, $\psi_B \in \Psi(\sigma_B)$, respectively.
(i) $U := (U_A)^{-1}$ satisfies (6.5) for $A^{-1} \in \text{SL}_2(\mathbb{Z}_n)$ and some $\psi \in \Psi(\sigma_{A^{-1}})$.

(ii) $U := U_A U_B$ satisfies (6.5) for $AB \in \text{SL}_2(\mathbb{Z}_n)$ and some $\psi \in \Psi(\sigma_{AB})$.

Proof.

(i) By assumption we have

$$U_A \pi(\lambda) U_A^{-1} = \psi_A(\lambda) \pi(A \lambda)$$

and hence the substitution $\lambda \mapsto A^{-1} \lambda$ allows us to write

$$U_A^{-1} \pi(\lambda) U_A = \overline{\psi_A(A^{-1} \lambda)} \pi(A^{-1} \lambda).$$

Since by Lemma 6.2.4(ii) the function $\lambda \mapsto \overline{\psi_A(A^{-1} \lambda)}$ belongs to $\Psi(\sigma_{A^{-1}})$ we thus conclude that $U = U_A^{-1}$ satisfies the intertwining identity (6.5) with $A$ replaced by $A^{-1}$.

(ii) Here we have

$$U_A U_B \pi(\lambda) U_B^{-1} U_A^{-1} = \psi_B(\lambda) U_A \pi(B \lambda) U_A^{-1} = \psi_A(B \lambda) \psi_B(\lambda) \pi(AB \lambda).$$

By Lemma 6.2.4(iii) the function $\lambda \mapsto \psi_A(B \lambda) \psi_B(\lambda)$ belongs to $\Psi(\sigma_{AB})$ and hence $U = U_A U_B$ satisfies the intertwining identity (6.5) with $A$ replaced by $AB$.

\[\Box\]

6.4 Metaplectic operators on $\mathbb{Z}_n$: explicit formulas

The set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}_n)$ such that an element $a$ is invertible is generated by three types of matrices

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where $a, c \in \mathbb{Z}_n$ and $a$ is invertible in $\mathbb{Z}_n$. The decomposition of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ into the product of matrices above is called the Weil decomposition and reads

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & a^{-1} + a^{-1} bc \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ ca^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & a^{-1} b \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ ca^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a^{-1} b & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (6.7)$$

The following Lemma allows us to apply the Weil decomposition to arbitrary $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}_n)$

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Lemma 6.4.1. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}_n)$. Factorize $n = \prod_i p_i^{e_i}$, its prime decomposition, and define $\vartheta := \prod_{\text{prime } p_i \mid a} p_i$. Then $a_0 := a + \vartheta b$ is invertible in $\mathbb{Z}_n$ and

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_0 & b \\ c_0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\vartheta & 1 \end{pmatrix},
$$

(6.8)

where $c_0 = c + \vartheta d$.

Proof. We show that $a_0 \not\equiv 0 \pmod{p_i}$, for any $i$. There are two cases.

First, if $p_i \nmid a$, then $p_i \nmid \vartheta$ and thus $a + \vartheta b \equiv a \not\equiv 0 \pmod{p_i}$.

Secondly, if $p_i \mid a$, then $p_i \nmid \vartheta$. Since $\det A = 1$, we cannot have $p_i \mid a$ and $p_i \mid b$ at the same time. Hence, $p_i \mid b$ and thus $p_i \mid \vartheta b$. Consequently, $a + \vartheta b \equiv \vartheta b \not\equiv 0 \pmod{p_i}$.

Equation (6.8) follows by straightforward computations. Observe that the first and third matrix in (6.8) belong to $\text{SL}_2(\mathbb{Z}_n)$ and hence so does the second. \qed

Before we state the main result we introduce the three operators on $\mathbb{C}^n$ that we will use

(i) The Fourier transform $\mathcal{F}$, normalized so that it is a unitary operator, is given by

$$
\mathcal{F}g(m) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} g(k) e^{-2\pi i km/n}, \quad g \in \mathbb{C}^n.
$$

(ii) Given an invertible element $a$ in $\mathbb{Z}_n$, we write $D_a$ for the index permutation

$$
D_ag(m) = g(a^{-1}m), \quad g \in \mathbb{C}^n.
$$

(iii) Given $c \in \mathbb{Z}_n$, the chirp multiplication is denoted by $R_c$,

$$
R_cg(m) = \psi_{[c]}(m) g(m), \quad g \in \mathbb{C}^n,
$$

where $\psi_{[c]} \in \mathbb{C}^n$ is the second degree character constructed in Lemma 6.2.1, with $[c] \in \mathbb{Z}$ denoting some representative for $c \in \mathbb{Z}_n$.

Lemma 6.4.2.

(i) $U = \mathcal{F}$ satisfies (6.5) for $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,

$$
\mathcal{F}M_lT_k\mathcal{F}^{-1} = e^{2\pi ik/l} M_{-k}T_l.
$$

(ii) $U = D_a$ satisfies (6.5) for $A = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$,

$$
D_a M_lT_k D_{a^{-1}} = M_{a^{-1}l}T_{ak}.
$$

(iii) $U = R_c$ satisfies (6.5) for $A = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$,

$$
R_c M_lT_k R_{c^{-1}} = \tau_k M_{ck+l} T_k.
$$

with $\tau_k = e^{-\pi i |c| k^2 (n+1)/n}$.
6.5 Gabor Frames and Representation Theory

Proof.

(i) By (6.1), since \( F M_l = T_l F \) and \( F F g(k) = g(-k) \).

(ii) \( D_a T_k = T_{ak} D_a \) and \( M_l D_a = D_a M_{al} \).

(iii) \( R_c M_l = M_l R_c \) and \( R_c T_k = \tau_k T_{ck} R_c \), cf. the derivations in the proof of Lemma 6.2.1.

The next theorem is the explicit construction of metaplectic operators on \( \mathbb{C}^n \) for an arbitrary matrix \( A \in \text{SL}_2(\mathbb{Z}_n) \).

Theorem 6.4.3. Let \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}_n) \). Let \( \vartheta \) be given by Lemma 6.4.1 and let \( a_0 = a + \vartheta b \) and \( c_0 = c + \vartheta d \). Define the operator \( U_A \) by

\[
U_A := R_{c_0 a_0^{-1}} \cdot D_{a_0} \cdot F^{-1} \cdot R_{-a_0^{-1} b} \cdot F \cdot R_{-\vartheta}.
\]

Then \( U_A \) is unitary and satisfies (6.5) for some \( \psi \in \Psi(\sigma_A) \).

Proof. Since \( a = a_0 - \vartheta b \) and \( c = c_0 - \vartheta d \),

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_0 & b \\ c_0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\vartheta & 1 \end{pmatrix}.
\]

(6.9)

Since \( a_0 \) is invertible by Lemma 6.4.1, we obtain \( d = a_0^{-1} + a_0^{-1} b c_0 \) and thus we can make use of the Weil decomposition

\[
\begin{pmatrix} a_0 & b \\ c_0 & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c_0 a_0^{-1} & 1 \end{pmatrix} \begin{pmatrix} a_0 & 0 \\ 0 & a_0^{-1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a_0^{-1} b & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

(6.10)

Combining (6.9) and (6.10),

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c_0 a_0^{-1} & 1 \end{pmatrix} \begin{pmatrix} a_0 & 0 \\ 0 & a_0^{-1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a_0^{-1} b & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\vartheta & 1 \end{pmatrix}.
\]

Thus (6.5) follows from Lemma 6.4.2 and the repeated use of Lemma 6.3.6, stated and proved in the previous section. \( \square \)

6.5 Gabor Frames and Representation Theory

The metaplectic representation is an important technique in time-frequency analysis and it is often used as a tool both in continuous-time and finite settings, such as in [32]. We describe the use of the above results for Gabor frames in \( \mathbb{C}^n \).

Metaplectic operators are also an important tool in representation theory of twisted group algebras. There is a bijective correspondence between the projective representations of the group \( \mathcal{G} \) and integrated representations of \( L^1(\mathcal{G}, c) \), Theorem 5.2.6. Below we describe the use of our results to characterize those representations.
Gabor frames

Given a Gabor window $g \in \mathbb{C}^n$ and a time-frequency lattice $\Lambda \subseteq \mathbb{Z}_n \times \mathbb{Z}_n$, the family
\[
G(g, \Lambda) := \{ \pi(\lambda)g : \lambda \in \Lambda \}
\]
is called a Gabor system. The interesting systems are those that span $\mathbb{C}^n$, they are exactly the regular Gabor frames for $\mathbb{C}^n$. The system $G(g, \Lambda)$ is a Gabor frame if and only if the frame operator
\[
S_{g,\Lambda}f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)g, \quad f \in \mathbb{C}^n,
\]
is invertible. The focus of many questions and results in Gabor analysis is the dual window $\gamma$, defined for a Gabor frame $G(g, \Lambda)$ by
\[
\gamma = S_{g,\Lambda}^{-1}g.
\]
Most importantly, the dual window $\gamma$ allows us to reconstruct any vector $f \in \mathbb{C}^n$ from its Gabor coefficients $\langle f, \pi(\lambda)g \rangle$ with respect to $g$. Indeed one has the frame expansions
\[
f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)g = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\gamma \rangle \pi(\lambda)g, \quad \text{for all } f \in \mathbb{C}^n.
\]
The standard time-frequency lattices are rectangular lattices $\Lambda_0 = r\mathbb{Z}_n \times s\mathbb{Z}_n$, with $r,s$ divisors of $n$. For this case there are fast algorithms for computing $\gamma$ by making use of the FFT (fast Fourier transform), see [43], and also now a method presented in Chapter 7. An important application of the metaplectic operators is that they allow us to apply the techniques designed for the case of a rectangular lattice also to more general time-frequency lattices. The method is formulated in the corollary below, it is analogous to the continuous-time case in [23]. We use the following lemma.

Lemma 6.5.1. A general lattice $\Lambda \subseteq \mathbb{Z}_n \times \mathbb{Z}_n$ can always be written $\Lambda = A\Lambda_0$, where $\Lambda_0$ is a rectangular lattice and $A \in \text{SL}_2(\mathbb{Z}_n)$.

Proof. We identify the finite lattice $\Lambda$ in $\mathbb{Z}_n \times \mathbb{Z}_n$ with an infinite lattice $\bar{\Lambda}$ of $\mathbb{Z} \times \mathbb{Z}$, its preimage $\tilde{\Lambda}$ under the canonical epimorphism $\mathbb{Z} \to \mathbb{Z}_n$. Denote by $M_{2,2}(\mathbb{Z})$ the set of $2 \times 2$ matrices with integer entries, and by $\text{SL}_2(\mathbb{Z})$ denote the corresponding special linear group of matrices with determinant 1. Let $\tilde{L} \in M_{2,2}(\mathbb{Z})$ be a generator matrix for $\tilde{\Lambda}$, i.e., the columns of $\tilde{L}$ are generators of the lattice $\tilde{\Lambda}$,
\[
\tilde{\Lambda} = \tilde{L}(\mathbb{Z} \times \mathbb{Z}).
\]
The Smith normal form of integer matrices [11] allows us to write
\[
\tilde{L} = \tilde{A} \tilde{L}_0 \tilde{B},
\]
where $\tilde{A}, \tilde{B}$ belong to $\text{SL}_2(\mathbb{Z})$ and $\tilde{L}_0 \in M_{2,2}(\mathbb{Z})$ is diagonal. Hence,
\[
\Lambda = \tilde{A} \Lambda_0 \tilde{B}, \tag{6.11}
\]
with $\Lambda_0$ generated by $\tilde{L}_0$. Note that $\tilde{B}$ leaves $\mathbb{Z} \times \mathbb{Z}$ invariant, i.e., $\tilde{B}(\mathbb{Z} \times \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}$, as well as $n(\mathbb{Z} \times \mathbb{Z})$. Hence we can consider all integers modulo $n$ and thus the splitting (6.11) of $\tilde{\Lambda}$ in $\mathbb{Z} \times \mathbb{Z}$ yields the desired splitting $\Lambda = A\Lambda_0$ in $\mathbb{Z}_n \times \mathbb{Z}_n$. □
6.5 Gabor Frames and Representation Theory

Theorem 6.4.3 and Lemma 6.5.1 allow us to use the metaplectic operators for time-frequency analysis in $\mathbb{C}^n$ in a similar way as they are used in the continuous-time setting [23]. The next result illustrates this fact.

**Corollary 6.5.2.** Let $\Lambda \subseteq \mathbb{Z}_n \times \mathbb{Z}_n$ be a general lattice and $g \in \mathbb{C}^n$. Write $\Lambda = A \Lambda_0$ as described in Lemma 6.5.1; let $U = U_A$ be given by Theorem 6.4.3; and let $g_0 := U^{-1}g \in \mathbb{C}^n$. Then the following are equivalent:

(i) $\mathcal{G}(g, \Lambda)$ is a Gabor frame, with dual window $\gamma$,

(ii) $\mathcal{G}(g_0, \Lambda_0)$ is a Gabor frame, with dual window $\gamma_0$, and the dual windows are related to each other by $\gamma = U\gamma_0$.

**Proof.** By Theorem 6.4.3 we have for $f \in \mathbb{C}^n$,

\[
US_{g_0,\Lambda_0}U^{-1}f = \sum_{\lambda \in \Lambda_0} \langle U^{-1}f, \pi(\lambda)g_0 \rangle \ U\pi(\lambda)g_0 \\
= \sum_{\lambda \in \Lambda_0} \langle f, U\pi(\lambda)U^{-1}g \rangle \ U\pi(\lambda)U^{-1}g \\
= \sum_{\lambda \in \Lambda_0} \langle f, \psi_A(\lambda) \pi(\lambda)g \rangle \\
= \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle = S_{g,\Lambda}f
\]

and hence $\gamma = S_{g_0,\Lambda_0}^{-1}g = (US_{g_0,\Lambda_0}U^{-1})^{-1}g = US_{g_0,\Lambda_0}^{-1}g_0 = U\gamma_0$. \(\square\)

The corollary allows us to obtain the dual window for $\mathcal{G}(g, \Lambda)$ by computing the dual window for $\mathcal{G}(g_0, \Lambda_0)$. Since in this computation the general lattice $\Lambda$ is replaced by the rectangular lattice $\Lambda_0$, the fast algorithms mentioned above can be used.

**Twisted convolution algebras**

Let $c$ be a cocycle associated to the projective representation of $\mathbb{Z}_n \times \mathbb{Z}_n$, $c(\lambda, \lambda') = e^{2\pi i (\kappa \lambda + \lambda')/n}$, where $\kappa = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$, arising from the commutation relations of time-frequency shifts.

The group of all the automorphisms of $\mathbb{Z}_n \times \mathbb{Z}_n$ is the group of all $2 \times 2$ matrices of determinant $1$, that is $\text{SL}_2(\mathbb{Z}_n)$. In the previous sections we characterized all the irreducible projective representations of $\mathbb{Z}_n \times \mathbb{Z}_n$ equivalent to the Schrödinger representation $\pi(k, l) = M_l T_k$, by the means of metaplectic operators. Since, by Theorem 5.2.6, the irreducible projective representations of $\mathbb{Z}_n \times \mathbb{Z}_n$ are in bijective correspondence with all the irreducible $*$-representations of $\ell^1(\mathbb{Z}_n \times \mathbb{Z}_n)$, we also characterized the equivalent $*$-representations of $\ell^1(\mathbb{Z}_n \times \mathbb{Z}_n)$. The same is true for the subgroup of $\mathbb{Z}_n \times \mathbb{Z}_n$, namely the lattice $\Lambda$. 

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Definition 6.5.3. Let \( c_1 \) and \( c_2 \) be two cocycles on \( \mathbb{Z}_n \times \mathbb{Z}_n \). We say that they are cohomologous if there exists some \( \tau : \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{T} \) such that for all \( \lambda, \lambda' \in \mathbb{Z}_n \times \mathbb{Z}_n \)

\[
c_1(\lambda, \lambda') = c_2(\lambda, \lambda')\tau(\lambda)^{-1}\tau(\lambda')^{-1}\tau(\lambda + \lambda').
\]

(6.12)

The definition is the same in a more general setting, namely for locally compact groups.

Theorem 6.5.4. Let \( c \) be a cocycle defined above and \( c' \) another cocycle for \( \mathbb{Z}_n \times \mathbb{Z}_n \). Then \( \ell^1(\mathbb{Z}_n \times \mathbb{Z}_n, c) \) is isomorphic to \( \ell^1(\mathbb{Z}_n \times \mathbb{Z}_n, c') \) if and only if \( c \) and \( c' \) are cohomologous. Moreover, if \( \ell^1(\mathbb{Z}_n \times \mathbb{Z}_n, c) \) and \( \ell^1(\mathbb{Z}_n \times \mathbb{Z}_n, c') \) are isomorphic, then the 2-cocycle \( c' \) is associated to some symplectic matrix \( A \in \mathbb{M}_{2,2}(\mathbb{Z}_n) \) and the mapping \( \tau \), in the definition of the cohomologous cocycles, is a second degree character associated to \( \sigma_A \).

**Proof.** If we replace \( c' \) by a cohomologous cocycle \( c \), then we get an involutive isomorphism between \( \ell^1(\mathbb{Z}_n \times \mathbb{Z}_n, c') \) and \( \ell^1(\mathbb{Z}_n \times \mathbb{Z}_n, c) \) by the following computations:

\[
(a^*c'b)(\lambda') = \sum_{\lambda \in \mathbb{Z}_n \times \mathbb{Z}_n} a(\lambda)b(\lambda' - \lambda)c(\lambda, \lambda' - \lambda)\tau(\lambda)^{-1}\tau(\lambda' - \lambda)^{-1}\tau(\lambda')
\]

\[
= \tau(\lambda') \sum_{\lambda \in \mathbb{Z}_n \times \mathbb{Z}_n} \tau(\lambda)^{-1}a(\lambda)\tau(\lambda' - \lambda)^{-1}b(\lambda' - \lambda)c(\lambda, \lambda' - \lambda)
\]

\[
= \tau(\lambda')(\tau^{-1}a^*(c^{-1}b)(\lambda'),
\]

and

\[
a^*(\lambda) = \frac{c(\lambda, -\lambda)\tau(\lambda)^{-1}\tau(-\lambda)^{-1}\tau(0)a(-\lambda)}{a(\lambda)\tau(\lambda)^{1}(\tau^{-1}a)(\lambda)}
\]

\[
= \tau(\lambda)(\tau^{-1}a^*(\lambda))
\]

Let \( \pi \) be a representation of \( \mathbb{Z}_n \times \mathbb{Z}_n \) through time-frequency shifts and \( A \in \mathbb{M}_{2,2}(\mathbb{Z}_n) \) with determinant equal to 1. Let \( U_A \) be given by Theorem 6.4.3. Then \( U_A \) intertwines the operators \( \pi(\lambda) \) and \( \pi_A(\lambda) := \pi(\lambda A) \) for all \( \lambda \in \mathbb{Z}_n \times \mathbb{Z}_n \). Since there is a bijective correspondence between the representations of \( \mathbb{Z}_n \times \mathbb{Z}_n \) and representations of \( \ell^1(\mathbb{Z}_n \times \mathbb{Z}_n) \), Theorem 5.2.6, the operator \( U_A \) also intertwines the integrated representations \( \pi^1 \) and \( \pi^1_A \),

\[
\pi^1_A(a) = \sum_{\lambda \in \mathbb{Z}_n \times \mathbb{Z}_n} a(\lambda)\pi_A(\lambda) = \sum_{\lambda \in \mathbb{Z}_n \times \mathbb{Z}_n} a(\lambda)\psi_A(\lambda)^{-1}U_A\pi(\lambda)U_A^{-1}
\]

\[
= U_A\left(\sum_{\lambda \in \mathbb{Z}_n \times \mathbb{Z}_n} a(\lambda)\psi_A(\lambda)^{-1}\pi(\lambda)\right)U_A^{-1} = U_A\pi^1(\psi_A^{-1}a)U_A^{-1}.
\]

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The integrated representation $\pi_A^1$ induces a new multiplication $\sharp_{c_A}$ in $\ell^1(\Lambda)$, namely

$$\pi_A^1(a)\pi_A^1(b) = \sum_{\lambda,\lambda' \in \mathbb{Z}_n \times \mathbb{Z}_n} a(\lambda)\pi_A(\lambda) \sum_{\lambda' \in \mathbb{Z}_n \times \mathbb{Z}_n} b(\lambda')\pi_A(\lambda')$$

$$= \sum_{\lambda,\lambda' \in \mathbb{Z}_n \times \mathbb{Z}_n} a(\lambda)b(\lambda')\pi(4\lambda)\pi(4\lambda')$$

$$= \sum_{\lambda,\lambda' \in \mathbb{Z}_n \times \mathbb{Z}_n} a(\lambda)b(\lambda')c(4\lambda, 4\lambda')\pi(4\lambda + 4\lambda')$$

$$= \sum_{\lambda,\lambda' \in \mathbb{Z}_n \times \mathbb{Z}_n} a(\lambda)b(\lambda' - \lambda)c(4\lambda, 4(\lambda' - \lambda))\pi(4\lambda')$$

$$= \sum_{\lambda' \in \mathbb{Z}_n \times \mathbb{Z}_n} \pi_A(\lambda') \sum_{\lambda \in \mathbb{Z}_n \times \mathbb{Z}_n} a(\lambda)b(\lambda' - \lambda)c(4\lambda, 4(\lambda' - \lambda))$$

$$= \sum_{\lambda' \in \mathbb{Z}_n \times \mathbb{Z}_n} (a\sharp_{c_A} b)(\lambda')\pi_A(\lambda') = \pi_A^1(a\sharp_{c_A} b),$$

where $\sharp_{c_A}$ is a twisted convolution with respect to the cocycle $c_A$ defined as $c_A(\lambda, \lambda') := c(4\lambda, 4\lambda')$. Therefore $\ell^1(\mathbb{Z}_n \times \mathbb{Z}_n, c)$ and $\ell^1(\mathbb{Z}_n \times \mathbb{Z}_n, c_A)$ are isomorphic and $c$ and $c_A$ are cohomologous. From the definition of $c$ follows that

$$c_A(\lambda, \lambda') = c(4\lambda, 4\lambda') = e^{2\pi i (\lambda, \lambda')/n} = e^{2\pi i (\lambda A^T, \lambda')/n}$$

$$= e^{2\pi i (\lambda, A^T \lambda')/n} e^{-2\pi i (\lambda, \lambda')/n} e^{2\pi i (\lambda, \lambda')/n}$$

$$= c(\lambda, \lambda') e^{2\pi i (\lambda, \sigma_A \lambda')/n}$$

$$= c(\lambda, \lambda') \psi_A(\lambda)^{-1} \psi_A(\lambda')^{-1} \psi_A(\lambda + \lambda'),$$

and indeed, $\tau = \psi_A$ in equation 6.12. \qed

### 6.6 MATLAB Implementations

As mentioned in the previous section, the metaplectic representation establishes an equivalence between Gabor system on the rectangular lattice and the Gabor system (with respectively different Gabor atom) on the nonseparable lattice. This transition from one system to another helps finding the dual atom. In applications one usually avoids nonseparable lattice due to complications in finding the dual atom. Nevertheless they exhibit better properties and are more suitable if one uses Gaussians as Gabor atoms - better packing of the time-frequency plane is achieved that way. There exist ways to deal with certain nonseparable lattices, more precisely with hexagonal ones, where the authors analyze the structure of a Gabor matrix associated to such a lattice. However, using the metaplectic representation one is not restricted to hexagonal lattices, but can take any symplectic one. Below we present MATLAB codes to compute the dual Gabor atom on the nonrectangular lattice. We call it a Metalectic Toolbox and it can be found on the webpage of NuHAG, [http://www.univie.ac.at/nuhag-php/matlab/](http://www.univie.ac.at/nuhag-php/matlab/). To fully be able to run it one also need a basic Time-Frequency (Gabor) Toolbox, also found on the same webpage.
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We present our toolbox on an example where the Gabor window is a Gaussian, that is \( g(k) = e^{-0.5k^2} \), on \( \mathbb{Z}_n \) (\( n = 256 \)) and the lattice is \( \Lambda = L(\mathbb{Z}_n \times \mathbb{Z}_n) \), where \( L = \begin{pmatrix} 16 & 8 \\ 0 & 8 \end{pmatrix} \). This is a nonrectangular lattice and we use a symplectic mapping \( A = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \) to transform it to a rectangular one \( \Lambda_r = L_0(\mathbb{Z}_n \times \mathbb{Z}_n) \), \( L_0 = \begin{pmatrix} 8 & 0 \\ 0 & 16 \end{pmatrix} \). The atom \( g \) and its time-frequency concentration are pictured in Fig 6.1 and Fig. 6.2. The lattice \( \Lambda \) is pictured in Fig 6.2 and the dual atom to \( g \) in Fig 6.1. We see that since the short time Fourier transform of the Gaussian has a circular contour in the time-frequency plane, the hexagonal lattice \( \Lambda \) gives the more optimal packing of the plane, rather then the rectangular one \( \Lambda_r \). The next two figures present a transformed Gabor window \( g \) under the mapping \( U_A, U_A g \) (Fig. 6.3) and its dual (Fig. 6.3).
Here we present few routines that make the core of the Metaplectic Toolbox. First one computes all second degree characters associated to the matrix $A$, or more precisely to $\sigma_A$.

```matlab
function AP = allsecdegchar(A,n);

% all second degree characters (as a matrix) associated to sigma(A)

a = A(1,1);
b = A(1,2);
c = A(2,1);
d = A(2,2);
sigma =mod([a*c b*c; b*c b*d],n);
AP = zeros(n,n,n,n);

for r=0:n-1
  for s=0:n-1
    for k=0:n-1
      for l=0:n-1
        AP(k+1,l+1,r+1,s+1)=
          exp(2*pi*i*k*r/n)*exp(2*pi*i*l*s/n)*exp(pi*i*([k,l]*([k;l])*(n+1)/n));
      end
    end
  end
end
```

Figure 6.1: Gabor atom $g$ on $\mathbb{Z}_{256}$ (on the left), and its dual (on the right) computed using the metaplectic toolbox.
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Figure 6.2: A nonrectangular lattice $\Lambda$ given by the matrix $L$ and a time-frequency localization of the Gaussian $g$, the short-time Fourier transform.

function $U = \text{allmeta}_u(A,n,\text{AP},x,y,g)$;

% Input:
% $n$ = group order;
% $A$ = symplectic matrix $(a,b;c,d)$
% $g$ = $nx1$ vector to which we apply metaplectic operator
% $\text{AP}$ = set of all possible sdc associated to $A$
% $x,y$ = specific choice of a sdc
% Output:
% $U$ = applying the metaplectic transformation to $g$

% the summation is taken over $l$ in $\{0,1,...,n/\gcd(b,n)-1\}$

a = A(1,1);
b = A(1,2);
c = A(2,1);
d = A(2,2);
v = [v; v];
sigma =mod( [a*c b*c; b*c b*d],n);

for $k = 0:n-1$
    $U(k+1) = 0;$
6.6 MATLAB Implementations

Figure 6.3: The Gabor atom \( g \) after transformation by the operator \( U_A \) (left) and its dual (right).

\[
\text{for } l = 0:n-1 \\
U(k+1) = U(k+1) + g(\mod(a*k + b*l,n)+n+1) \ast AP(k+1,l+1,x,y); \\
\text{end}; \\
U(k+1) = 1/\sqrt{n \ast \gcd(b,n)} \ast U(k+1); \\
\text{end}; \\
U = \text{transpose}(U);
\]

\[
\text{function } t = \text{theta}(A,n);
\]

\% Computes \( t \) so that \( a + t \ast b \) is invertible in \( \mathbb{Z}_n \)
\%
\%
\% Input :
\%
\% A = symplectic matrix \([a b; c d]\) over \( \mathbb{Z}_n \)
\%
\% n = order of the group
\%

\text{a = A(1,1);} \\
\text{f=factor(n);} \\
\text{p=zeros(1,length(f));} \\
\text{s = find(f);} \\
\text{for } k = 1:length(f) \\
\quad \text{if } s \neq 0 \\
\quad \quad p(k) = f(s(length(s)));
Figure 6.4: A rectangular lattice $\Lambda_r$ associated to the matrix $L_0$ and the time-frequency localization of the atom $U_{ag}$, the short time Fourier transform.

```matlab
s = find(f(s) < p(k));
end
p=p(find(p));
t = 1;
for l = 1:length(p)
    if mod(a,p(l)) ≠ 0
        t = t * p(l);
    end
end
```

Here, we decompose a metaplectic operator $U$ as in Theorem 6.4.3,

```matlab
function [U,psi] = meta_decomp(A,n);
% Decomposition of an operator U as in Thm 5
% A = symplectic matrix
% n = order of the group
% acts on column vectors nx1
% Output:
```
Figure 6.5: Comparison of the time necessary to compute the dual Gabor atom on a nonseparable lattice $\Lambda$ using the metaplectic toolbox (green plot) and the standard Gabor toolbox (red plot). The experiment was performed when $n$ was a multiple (up to 10th) of 256.

```matlab
% U = metaplectic operator
% psi = second degree character obtained as
% psi(k,l) = U^{-1} \cdot (M_{\text{LT}_k} \cdot U \cdot (M_\{ck+dl\}^T \cdot (ak+bl))
%
% uses: theta.m, invmod.m, chirp.m
% permmod.m, ufft.m, mdtran.m

a = A(1,1);
b = A(1,2);
c = A(2,1);
d = A(2,2);

t = theta(A,n);

a0 = mod(a + t*b,n);
c0 = mod(c + t*d,n);
a0_inv = invmod(a0,n);

R_c = chirp(-mod(c0*a0_inv,n),n).
R_a = chirp(mod(a0_inv*b,n),n);
R_t = chirp(t, n);
D_a = permmod(a0_inv,n);
```
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\[
F = \text{u fft}(n);
U = R_{ca} \ast D_{a} \ast F \ast R_{ab} \ast F' \ast R_t;
\]

\[
X = \text{eye}(n,n);
\text{for} \ k=0:n-1
\quad \text{for} \ l=0:n-1
\quad \quad e = \text{inv}(U) \ast \text{modtran}(X,k,l) \ast U \ast \text{inv} (\text{modtran}(X,\text{mod}(a \ast k+b \ast l,n),\text{mod}(c \ast k+d \ast l,n)));
\quad \text{chk} = \text{norm}(e-e(1,1) \ast \text{eye}(n,n),\text{inf});
\quad \text{if} \ \text{chk} > 1e^{-6}
\quad \quad \text{warning}('\text{matrix } e \text{ is not scalar}');
\quad \quad \psi(k+1,l+1) = e(1,1);
\quad \text{end}
\quad \text{end}
\text{end}
\]

\[
\text{function} \ [U,V,L] = \text{smith.normal.form}(A);
\]

% returns matrices U,V in SL_2(Z) and L = diagonal matrix,
% such that A = U\ast L \ast V
% uses: redrow.m
% redcol.m

\[
X = \text{redrow}(A);
B = A \ast X;
U = \text{eye}(2);
V = X;
\text{while} \ \text{rem}(B(2,1),B(1,1)) > 0
\quad Y = \text{redcol}(B);
\quad C = Y \ast B;
\quad X = \text{redrow}(C);
\quad B = C \ast X;
\quad U = Y \ast U;
\quad V = V \ast X;
\quad \text{end}
\]

\[
k = B(2,1)/B(1,1);
U = [1 \ 0; -k \ 1] \ast U;
L = U \ast A \ast V;
U = \text{inv}(U);
V = \text{inv}(V);
\]

\[
\text{function} \ Y = \text{redcol}(A);
\]

% A has to be an invertible 2x2 matrix
% the output is matrix Y, such that YA = [* *; 0 *]
% (used in the SNF decomposition)
a = A(1,1);  
b = A(1,2);  
c = A(2,1);  
d = A(2,2);  

[e, x, y] = gcd(a, c);  
alpha = a/e;  
beta = c/e;  
Y = [x y; -beta alpha];

function X = redrow(A);

% A has to be an invertible 2x2 matrix  
% the output is matrix X, such that AX = [* 0; * *]  
% (used in the SNF decomposition)

a = A(1,1);  
b = A(1,2);  
c = A(2,1);  
d = A(2,2);  

[e, x, y] = gcd(a, b);  
alpha = a/e;  
beta = b/e;  
X = [x -beta; y alpha];
Chapter 6
We tackle the problem of studying the invertibility of twisted convolution operators. Non-commutativity is the main subtle point in this problem. In fact, the question when the mapping

\[ C_b : a \in \ell^1 \rightarrow a \circledast b \in \ell^1 \]

for some \( b \in \ell^1 \) is invertible and how we can compute the inverse is more difficult than for a commutative setting. In particular, Wiener’s Lemma which deals with the problem that if, for some \( b \in \ell^1 \), \( C_b \) is invertible on \( \ell^2 \) then the inverse is generated from an element again in \( \ell^1 \), has to be proven separately. An abstract and more general, but not constructive, proof of Wiener’s Lemma for twisted convolution is given in [25].

The important new features of our approach are:

- Our approach works for general elementary locally compact abelian group, not limited to the case of \( \mathbb{R} \).

- The results found in the literature provide solutions for special cases, \( \mathbb{R} \) and \( \mathbb{Z}_m \), and the approaches are different in both cases. We present a general and explicit construction.

- We base our solution on the twisted convolution on the finite cyclic group, see Section 7.1.

This chapter consists of the results obtained in the cooperation with Y. Eldar and T. Werther, [45] and [52].

## 7.1 Twisted Convolution

In the following subsection we study the twisted convolution in a finite setting and draw analogies for approaching the problem of invertibility of \( C_b \) in the general case.

**Twisted convolution on \( \mathbb{Z}_p \times \mathbb{Z}_p \)**

In what follows we describe the twisted convolution on the finite group \( \mathbb{Z}_p \times \mathbb{Z}_p \). The standard (commutative) convolution of two elements \( f, g \in \mathbb{C}^{p \times p} \) is defined by

\[ (f \ast g)_{m,n} = \sum_{k,l=0}^{p-1} f_{k,l} g_{m-k,n-l} \]
where operations on indices is performed modulo \( p \).

In analogy to the infinite case, we define the twisted convolution \( f \circ g \) of two elements \( f, g \in \mathbb{C}^{p \times p} \) by

\[
(f \circ g)_{m,n} = \sum_{k,l=0}^{p-1} f_{k,l}g_{m-k,n-l}\omega^{(m-k)l}
\]

with \( \omega = e^{2\pi i q/p} \). For a fixed \( g \), the twisted convolution can be seen as a linear mapping \( \mathbb{C}^g : f \rightarrow f \circ g \) whose matrix \( G \) is block circulant with \( p \) blocks, i.e.,

\[
G = C(G_0, G_{p-1}, \ldots, G_1) = \begin{pmatrix}
G_0 & G_{p-1} & \cdots & G_1 \\
G_1 & G_0 & \cdots & G_2 \\
\vdots & \vdots & \ddots & \vdots \\
G_{p-1} & G_{p-2} & \cdots & G_0
\end{pmatrix}.
\]

Each block has entries of the form

\[
(G_j)_{kl} = \omega^{jl}g_{j,k-l}.
\]

Note that for the regular convolution each block is itself circulant. For the invertibility of block circulant matrices we apply a well known result from Fourier analysis.

**Lemma 7.1.1.** [10] The matrix \( G = C(G_0, G_{p-1}, \ldots, G_1) \) is invertible if and only if every \( \hat{G}_s = \sum_{r=0}^{p-1} e^{-2\pi isr/p}G_r \), \( s = 0, \ldots, p-1 \), is invertible. In this case

\[
G^{-1} = C(H_0, H_{p-1}, \ldots, H_1)
\]

where \( H_r = \frac{1}{p} \sum_{s=0}^{p-1} e^{2\pi isr/p}(\hat{G}_s)^{-1} \).

By analyzing \( \hat{G}_s \), we see that all blocks are unitary equivalent, in the sense that

\[
T_r\hat{G}_sT_r^* = \hat{G}_{s-qr},
\]

where \( T_r \) denotes the unitary matrix with entries

\[
(T_r)_{kl} = \begin{cases} 
1 & \text{if } p - r = l - k, \\
0 & \text{else}.
\end{cases}
\]

Since \( p \) and \( q \) are relatively prime, we obtain all blocks by such a unitary transformation. This implies that showing that if \( \hat{G}_0 \) is invertible, then all \( \hat{G}_s \) are invertible for \( s = 1, \ldots, p-1 \). In other words, the \( p \times p \) matrix \( \hat{G}_0 \) contains all the information about the invertibility of \( C_g \). An easy computation shows that the entries of \( \hat{G}_0 \) are given by

\[
(\hat{G}_0)_{n,l} = \sum_{k=0}^{p-1} \omega^{nl}g_{k,n-l}.
\]

We will later see that this observation motivates the matrix algebra that we introduce to study the invertibility of the twisted convolution.

Now, also all \( \hat{G}_s^{-1} \) satisfy the same unitary equivalence. It follows that we can read from \( \hat{G}_0^{-1} \) the element \( g^{-1} \) which inverts the twisted convolution \( f \rightarrow f \circ g \), i.e., \( g^{-1} \circ g = g \circ g^{-1} = \delta \).

The twisted convolution on \( \mathbb{Z}_p \times \mathbb{Z}_p \) serves as analogy for modelling the twisted convolution for the continuous and the finite dimensional case.
Twisted Convolution on $\mathcal{G} \times \hat{\mathcal{G}}$

Let $\mathcal{G}$ be an elementary locally compact abelian group

$$\mathcal{G} = \mathbb{R}^d \times \mathbb{Z}^p \times \mathbb{T}^q \times F \quad (d \geq 0, p \geq 0, q \geq 0, m = |F| \geq 1),$$

and $\hat{\mathcal{G}}$ a group of its characters, $\hat{\mathcal{G}} = \mathbb{R}^d \times \mathbb{T}^p \times \mathbb{Z}^q \times F$. Let also $\Lambda$ be a separable rational lattice of the group $\mathcal{G} \times \hat{\mathcal{G}}$, with density $s(\Lambda) = \frac{q}{q'}$, $p$ and $q$ relatively prime. That is, by the definition from Chapter 4, $\Lambda$ and its adjoint $\Lambda^o$ contain the co-isotropic lattice $\Lambda_c$ in the sense

$$\Lambda/\Lambda_c = \Lambda_c/\Lambda = \mathbb{Z}_q \times \mathbb{Z}_q, \quad \Lambda^o/\Lambda_c = \Lambda_c/\Lambda = \mathbb{Z}_p \times \mathbb{Z}_p.$$ 

We define the **twisted convolution** for sequences $a, b \in \ell^1(\Lambda^o)$ by

$$(a \ast b)(\tau^o) = \sum_{\lambda^o \in \Lambda^o} a(\lambda^o)b(\tau^o - \lambda^o)\kappa(\lambda^o, \tau^o - \lambda^o),$$

where $\kappa$ is a 2-cocycle associated with the group $\mathcal{G} \times \hat{\mathcal{G}}$. Notice that since $\Lambda$ is a rational lattice, and $\Lambda^o$ can be decomposed into cosets of $\Lambda_c^o$

$$\Lambda^o = \bigcup_{k,l=0}^{p-1} (\mu_{k,l} + \Lambda_c^o),$$

the cocycle admits a decomposition. Namely, let $\lambda^o = \mu_{k,l} + \lambda_c^o$ and $\tau^o = \mu_{m,n} + \tau_c^o$ for some $\mu_{k,l}, \mu_{m,n} \in \Lambda^o/\Lambda_c^o$ and $\lambda_c^o, \tau_c^o \in \Lambda_c^o$, then

$$\kappa(\lambda^o, \tau^o - \lambda^o) = \kappa(\mu_{k,l} + \lambda_c^o, \mu_{m,n} + \tau_c^o - \mu_{k,l} - \lambda_c^o) = \kappa(\mu_{k,l}, \mu_{m,n} - \mu_{k,l}) \cdot \kappa(\lambda_c^o, \tau_c^o - \lambda_c^o)$$

$$= \kappa(\mu_{k,l}, \mu_{m,n} - \mu_{k,l}) = \kappa(\mu_{k,l}, \mu_{m-k,n-l}).$$

The middle equality follows from the fact that $\Lambda_c^o \subseteq \Lambda_c$ and $\Lambda^o \subseteq \Lambda_c$ (because $\Lambda_c^o \subseteq \Lambda$). Therefore the twisted convolution reduces to

$$(a \ast b)(\tau^o) = (a \ast b)(\mu_{m,n} + \tau_c^o) = \sum_{\lambda^o \in \Lambda^o} a(\lambda^o)b(\tau^o - \lambda^o)\kappa(\lambda^o, \tau^o - \lambda^o)$$

$$= \sum_{k,l \in \mathbb{Z}_p} \sum_{\lambda_c^o \in \Lambda_c^o} a(\mu_{k,l} + \lambda_c^o)b(\mu_{m,n} + \tau_c^o - \mu_{k,l} - \lambda_c^o)\kappa(\mu_{k,l}, \mu_{m-k,n-l})$$

$$= \sum_{k,l \in \mathbb{Z}_p} \sum_{\lambda_c^o \in \Lambda_c^o} a(\mu_{k,l} + \lambda_c^o)b(\mu_{m,n} + \tau_c^o - \mu_{k,l} - \lambda_c^o)\omega^{(m-k)p},$$

where $\omega = e^{2\pi i / p}$. Although the twisted convolution depends on $p, q$ we do not specify this dependence because $p, q$ will always be given and fixed beforehand, by specifying the lattice of $\mathcal{G} \times \hat{\mathcal{G}}$. Later, we show how the twisted convolution is related to a class of operators with a special time-frequency representation.
Chapter 7

In contrast to the conventional convolution with symbol $\ast$, in which $\omega = 1$, the twisted convolution is not commutative, and turns $\ell^1(\Lambda^\circ)$ into a non-commutative algebra with the delta-sequence $\delta$ as its unit element.

To simplify the expression for $a \circledast b$ even more, we introduce new sequences based on the decomposition of $\Lambda^\circ$ into cosets. Let $c \in \ell^1(\Lambda^\circ)$, then for every $k, l = 0, \ldots, p - 1$ define a new sequence

$$c^{\mu_{k,l}}(\lambda^\circ) := \begin{cases} c(\lambda^\circ) & \lambda^\circ \in \mu_{k,l} + \Lambda_c^\circ, \\ 0 & \text{otherwise}, \end{cases}$$

(7.3)

where $\mu_{k,l} \in \Lambda^\circ/\Lambda_c^\circ$. Obviously, $c^{\mu_{k,l}}$ is supported on the coset $\mu_{k,l} + \Lambda_c^\circ$ and $c = \sum_{\mu_{k,l} \in \Lambda^\circ/\Lambda_c^\circ} c^{\mu_{k,l}}$. We write $c^{\mu_{k,l}}$ for a sequence supported on $\bigcup_{k \in \mathbb{Z}_p} (\mu_{k,l} + \Lambda_c^\circ)$, and $c^{\mu_{k,l}}$ for a sequence supported on $\bigcup_{k \in \mathbb{Z}_p} (\mu_{k,l} + \Lambda_c^\circ)$. The idea of splitting into a sum of sequences supported on cosets has first been introduced by K. Gröchenig and W. Kozek in [24].

According to this splitting, the twisted convolution can be rewritten as

$$(a \circledast b)(\tau^\circ) = \sum_{k,l \in \mathbb{Z}_p} \sum_{\lambda^\circ_c \in \Lambda_c^\circ} a(\mu_{k,l} + \lambda^\circ_c) b(\mu_{m,n} + \tau^\circ_c - \mu_{k,l} - \lambda^\circ_c) \omega^{(m-k)l}$$

$$= \sum_{k,l \in \mathbb{Z}_p} \sum_{\lambda^\circ \in \Lambda^\circ} a^{\mu_{k,l}}(\lambda^\circ) b^{\mu_{m,n} - \mu_{k,l}}(\tau^\circ - \lambda^\circ) \omega^{(m-k)l}$$

$$= \sum_{k,l \in \mathbb{Z}_p} (a^{\mu_{k,l}} \ast b^{\mu_{m,n} - \mu_{k,l}})(\tau^\circ) \omega^{(m-k)l}$$

where $\tau^\circ \in \mu_{m,n} + \Lambda_c^\circ$. Therefore

$$(a \circledast b)^{\mu_{m,n}} = \sum_{k,l \in \mathbb{Z}_p} (a^{\mu_{k,l}} \ast b^{\mu_{m,n} - \mu_{k,l}}) \omega^{(m-k)l}.$$  

(7.4)

We observe now that the upper indices in (7.4) behave like a twisted convolution in $\mathbb{Z}_p \times \mathbb{Z}_p$. What changes is that we have sequences as elements and standard convolution instead of multiplication.

Before proceeding further we gather here some properties of the sequences defined in (7.3).

**Lemma 7.1.2.** Let $a, b, c$ be in $\ell^1(\Lambda^\circ)$.

(a) For $r, s, u, v \in \mathbb{Z}_p$, $a^{\mu_{r,s}} \ast b^{\mu_{u,v}}$ is a sequence supported on the coset $\mu_{r+u, s+v} + \Lambda_c^\circ$.

(b) If $c = c^{\mu_{0,0}}$ is invertible in $(\ell^1(\Lambda^\circ), \ast)$, then $c^{-1}$ is also supported on the same coset as $c$, namely: $\bigcup_{r \in \mathbb{Z}_p} (\mu_{r,0} + \Lambda_c^\circ)$.

**Proof.** Let $a^{\mu_{r,s}}$, $b^{\mu_{u,v}}$ be sequences in $\ell^1(\Lambda^\circ)$ and $\mu_{k,l} \in \Lambda^\circ/\Lambda_c^\circ$, $\tau^\circ_c \in \Lambda_c^\circ$. Then

$$(a^{\mu_{r,s}} \ast b^{\mu_{u,v}})(\mu_{k,l} + \tau^\circ_c) = \sum_{\lambda^\circ \in \Lambda^\circ} a^{\mu_{r,s}}(\lambda^\circ) b^{\mu_{u,v}}(\mu_{k,l} + \tau^\circ_c - \lambda^\circ)$$

$$= \sum_{\mu_{m,n} \in \Lambda^\circ/\Lambda_c^\circ} \sum_{\lambda^\circ_c \in \Lambda_c^\circ} a^{\mu_{r,s}}(\mu_{m,n} + \lambda^\circ_c) b^{\mu_{u,v}}(\mu_{k,l} + \tau^\circ_c - \mu_{m,n} - \lambda^\circ_c).$$
Since $a^{\mu_{r,s}}$ is nonzero only for $\mu_{m,n} = \mu_{r,s}$, and $b^{\mu_{u,v}}$ for $\mu_{k-m,l-n} = \mu_{u,v}$, we obtain that $\mu_{k,l} = \mu_{a+r,v+s}$ for $a^{\mu_{r,s}} \ast b^{\mu_{u,v}}$ to be nonzero.

To show (b), let $c = c^{\mu,0}$ be invertible and $e$ be its inverse. Then

$$\delta = c \ast e = c^{\mu,0} \ast \left( \sum_{s=0}^{p-1} e^{\mu,s} \right) = \sum_{s=0}^{p-1} c^{\mu,0} \ast e^{\mu,s},$$

where, by previous calculations, $c^{\mu,0} \ast e^{\mu,s}$ is a sequence supported on $\bigcup_{r \in \mathbb{Z}_p} (\mu_{r,s} + \Lambda_0^0)$ for each $s \in \mathbb{Z}_p$. Since $\delta = \sum_{s \in \mathbb{Z}_p} \delta^{\mu,s}$, and elements of the sum have disjoint supports, $c^{\mu,0} \ast e^{\mu,s} = \delta^{\mu,s}$. But since $\delta^{\mu,s} = 0$ for $s \neq 0$ and $\delta^{\mu,0} = \delta$, we conclude that

$$c^{\mu,0} \ast e^{\mu,s} = \begin{cases} \delta & s = 0 \\ 0 & s \neq 0 \end{cases}$$

and therefore $e = e^{\mu,0}$.

**7.2 Wiener’s Lemma**

Our aim is to find a way to describe those sequences that have an inverse in $(\ell^1(\Lambda^0), \sharp)$. Having decomposed the twisted convolution into a finite sum of weighted normal convolutions of sequences that have disjoint support, we now introduce a new matrix algebra that is one which is isomorphic to $(\ell^1(\Lambda^0), \sharp)$.

Let $(\mathcal{M}, \oplus)$ be an algebra of $p \times p$ matrices whose entries are $\ell^1$ sequences and multiplication of two elements $A, B \in \mathcal{M}$ is given by

$$(A \oplus B)_{k,l} = \sum_{m \in \mathbb{Z}_p} A_{k,m} \ast B_{m,l} \quad k, l \in \mathbb{Z}_p.$$ 

The identity element $Id$ is a matrix with $\delta$ sequences on the diagonal.

**Theorem 7.2.1.** Let

$$\mathcal{M}_0 := \left\{ A \in \mathcal{M} : A_{i,j} = \sum_{m \in \mathbb{Z}_p} \omega^{mj} a^{\mu_{m,i-j}}, a \in \ell^1_v \text{ and } i, j \in \mathbb{Z}_p \right\}.$$ 

Then $\mathcal{M}_0$ is a subalgebra of $\mathcal{M}$.

**Proof.** Motivated by the twisted convolution structure on finite cyclic group we define a mapping $\phi: (\ell^1, \sharp) \rightarrow (\mathcal{M}, \oplus)$ by

$$(\phi(a))_{i,j} = \sum_{m \in \mathbb{Z}_p} \omega^{mj} a^{\mu_{m,i-j}}.$$ 

(7.5)
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Then \( \phi \) is linear, \( (\phi(\delta))_{i,j} = \sum_{m \in \mathbb{Z}_p} \omega^{mj} \delta^{mu_{m,i,j}} = \delta \) if \( i = j \) and zero otherwise. So \( \phi(\delta) = \text{Id} \). For \( i, j \in \mathbb{Z}_p \),

\[
\left( \phi(a \triangledown b) \right)_{i,j} = \sum_{m \in \mathbb{Z}_p} \omega^{mj} (a \triangledown b)_{m,i,j} = \sum_{m \in \mathbb{Z}_p} \omega^{mj} \sum_{l,s \in \mathbb{Z}_p} \omega^{(m-l)j} a_{l,i,j} b_{m-l,i,j-s} = \sum_{m,l,s \in \mathbb{Z}_p} \omega^{m(j+s)} \omega^{-l} a_{l,i,j} b_{m-l,i,j-s} = \sum_{m,l,s \in \mathbb{Z}_p} \omega^{l(m-l)} a_{l,i,j} b_{m-l,i,j-s} = \sum_{s \in \mathbb{Z}_p} \left( \sum_{l \in \mathbb{Z}_p} \omega^{lj} a_{l,i,j} \right) * \left( \sum_{m \in \mathbb{Z}_p} \omega^{s(m-l)} b_{m-l,i,j-s} \right) = \sum_{n \in \mathbb{Z}_p} \left( \sum_{m \in \mathbb{Z}_p} \omega^{mn} b_{m,i,j} \right) * \left( \sum_{l \in \mathbb{Z}_p} \omega^{lj} a_{l,i,j} \right) = \sum_{n \in \mathbb{Z}_p} \phi(b)_{i,n} * \phi(a)_{n,j} = \left( \phi(b) \otimes \phi(a) \right)_{i,j}.
\]

Therefore \( \phi \) is an anti-homomorphism, it is linear and satisfies \( \phi(a \triangledown b) = \phi(b) \otimes \phi(a) \).

Hence \( \mathcal{M}_0 \) is an algebra, being an image of an anti-homomorphism. \( \square \)

Before stating the main theorem, we explore properties of elements of \( \mathcal{M}_0 \). For \( i, j \in \mathbb{Z}_p \) and a matrix \( A \in \mathcal{M}_0 \) we define a new matrix \( A(j, i) \) obtained from \( A \) by substituting the \( j \)th row of \( A \) with a vector of zeros having \( \delta \) on the \( i \)th position, and the \( i \)th column with a column of zeros having \( \delta \) on the \( j \)th position.

**Lemma 7.2.2.** Let \( A \in \mathcal{M}_0 \). Then

(a) \( \det(A) \) is a sequence supported on \( \bigcup_{r \in \mathbb{Z}_p} (\mu_r, 0) + \Lambda_0^c \).

(b) \( \det(A(0, i)) \) is a sequence supported on \( \bigcup_{r \in \mathbb{Z}_p} (\mu_r, i) + \Lambda_0^c \) for \( i \in \mathbb{Z}_p \).

**Proof.** Let \( S_p \) be a group of permutations of the set \( \{0, 1, \ldots, p-1\} \). Then

\[
\det(A) = \sum_{\sigma \in S_p} (-1)^{\sigma} \prod_{i=0}^{p-1} A_{\sigma(i), i} = \sum_{\sigma \in S_p} (-1)^{\sigma} \prod_{i=0}^{p-1} \left( \sum_{m_i \in \mathbb{Z}_p} \omega^{m_i} a_{l_i, \sigma(i)-i} \right) = \sum_{\sigma \in S_p} (-1)^{\sigma} \sum_{m_0, \ldots, m_{p-1} \in \mathbb{Z}_p} \omega^{\sum_{i=0}^{p-1} m_i \cdot i} a_{m_0, 0} \ast \cdots \ast a_{m_{p-1}, (p-1) \cdot (p-1)} = G_{m_0, \ldots, m_{p-1}}.
\]

Since \( \sigma \) is a permutation of \( \{0, 1, \ldots, p-1\} \),

\[(\sigma(0) - 0) + (\sigma(1) - 1) + \cdots + (\sigma(p-1) - (p-1)) = 0.\]
7.2 Wiener’s Lemma

Therefore, by Lemma 7.1.2, \( G_{m_0, \ldots, m_{p-1}} \) is a sequence supported on the coset \( \bigcup_{r \in \mathbb{Z}_p} (\mu_r, 0 + \Lambda_c^o) \), i.e., \( \det(A) = \det(A)^{\mu, o} \).

In order to compute the support of \( \det(A(0, i)) \) for \( i \in \mathbb{Z}_p \), let \( S_{p-1} \) denote a group of permutations of \( \{1, \ldots, p-1\} \). Then \( \det(A) \) is invertible in \( M \), and there exists a matrix \( \tilde{B} \in M \) such that \( A \otimes \tilde{B} = \text{Id} \). By Lemma 7.2.2, \( \det(A) = \det(A)^{\mu, o} \) and by Lemma 7.1.2 its inverse, \( \mathbf{e} = \det(A)^{-1} \), is also supported on the same coset, hence \( \mathbf{e} = \mathbf{e}^{\mu, o} \). By Cramer’s rule the inverse of \( A \) is given by

\[
\tilde{B}_{i,j} = \det(A(j, i)) \ast \mathbf{e}.
\]

We see that by Lemma 7.2.2 (b), \( \tilde{B}_{i,0} \) is a sequence supported on \( \bigcup_{r \in \mathbb{Z}_p} (\mu_r, i + \Lambda_c^o) \). Let \( \mathbf{b} \) be a sequence defined by

\[
\mathbf{b} = \tilde{B}_{0,0} + \tilde{B}_{1,0} + \ldots + \tilde{B}_{p-1,0}.
\]

Then \( \tilde{B}_{i,0} = \sum_{j \in \mathbb{Z}_p} \mathbf{b}^{\mu, i,j} \). Define a new matrix, denoted by \( B \), as

\[
B_{i,j} = \sum_{m \in \mathbb{Z}_p} \omega^{mj} \mathbf{b}^{\mu, i-j}.
\]

Then \( B \in M_0 \) and we will show that \( B = \tilde{B} \), that is, \( B \) is the inverse of \( A \).
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Since $\tilde{B}$ is the inverse of $A$,

$$
\text{Id}_{i,0} = (A \circledast \tilde{B})_{i,0} = \sum_{j \in \mathbb{Z}_p} A_{i,j} * \tilde{B}_{j,0}
$$

$$
= \sum_{j \in \mathbb{Z}_p} \sum_{m \in \mathbb{Z}_p} \omega^{mj} a^{\mu_{m,i-j}} * \tilde{B}_{j,0}
$$

$$
= \sum_{j \in \mathbb{Z}_p} \sum_{m \in \mathbb{Z}_p} \omega^{mj} a^{\mu_{m,i-j}} * \left( \sum_{n \in \mathbb{Z}_p} b^{\mu_{n,j}} \right)
$$

$$
= \sum_{j \in \mathbb{Z}_p} \sum_{m \in \mathbb{Z}_p} \omega^{mj} \sum_{n \in \mathbb{Z}_p} a^{\mu_{m,i-j}} * b^{\mu_{n,j}}
$$

$$
= \sum_{m \in \mathbb{Z}_p} \sum_{j \in \mathbb{Z}_p} \sum_{n \in \mathbb{Z}_p} \omega^{(m-n)j} a^{\mu_{m-n,i-j}} * b^{\mu_{n,j}}
$$

$$
= \sum_{m \in \mathbb{Z}_p} G(m,i),
$$

where $G(m,i) = \sum_{j \in \mathbb{Z}_p} \sum_{n \in \mathbb{Z}_p} \omega^{(m-n)j} a^{\mu_{m-n,i-j}} * b^{\mu_{n,j}}$ is a sequence supported on $\mu_{m,i} + \Lambda_c^c$. Therefore, $G(0,0) = \delta$ and $G(m,i) = 0$ for $m \neq 0$ and $i \neq 0$. Using the above identity we will show that $A \circledast B = \text{Id}$, and by the uniqueness of the inverse we will conclude that $B = \tilde{B}$:

$$
(A \circledast B)_{i,j} = \sum_{s \in \mathbb{Z}_p} A_{i,s} * B_{s,j} =
$$

$$
= \sum_{s \in \mathbb{Z}_p} \left( \sum_{m \in \mathbb{Z}_p} \omega^{ms} a^{\mu_{m,i-s}} \right) * \left( \sum_{n \in \mathbb{Z}_p} \omega^{nj} b^{\mu_{n,s-j}} \right)
$$

$$
= \sum_{m \in \mathbb{Z}_p} \sum_{s \in \mathbb{Z}_p} \sum_{n \in \mathbb{Z}_p} \omega^{ms} \omega^{nj} a^{\mu_{m,i-s}} * b^{\mu_{n,s-j}}
$$

$$
= \sum_{m \in \mathbb{Z}_p} \sum_{s \in \mathbb{Z}_p} \sum_{n \in \mathbb{Z}_p} \omega^{(s+j)m} \omega^{nj} a^{\mu_{m,i-j-s}} * b^{\mu_{n,s}}
$$

$$
= \sum_{m \in \mathbb{Z}_p} \sum_{s \in \mathbb{Z}_p} \sum_{n \in \mathbb{Z}_p} \omega^{(s+j)(m-n)} \omega^{nj} a^{\mu_{m-n,i-j-s}} * b^{\mu_{n,s}}
$$

$$
= \sum_{m \in \mathbb{Z}_p} \sum_{s \in \mathbb{Z}_p} \sum_{n \in \mathbb{Z}_p} \omega^{(m-n)s} a^{\mu_{m-n,i-j-s}} * b^{\mu_{n,s}}
$$

$$
= \sum_{m \in \mathbb{Z}_p} \omega^{mj} G(m,i-j) = \begin{cases} 
\delta & i = j; \\
0 & i \neq j;
\end{cases}
$$

Hence, $A \circledast B = I$. □

Theorem 7.2.3 provides the key result to study invertibility of twisted convolution. Indeed, for a given sequence $a$ in $\ell^1$ we look at the corresponding matrix $A = \phi(a)$ as defined in (7.5). If $A$ is invertible in $(\mathcal{M}, \circledast)$, which can be checked showing that the determinant is invertible in $(\ell^1, *)$, then its inverse $A^{-1}$ is of the form $\phi(b)$ for another element $b$ in $\ell^1$. This element $b$, in turn, provides the inverse of $a$ in $(\ell^1, \zeta)$.
7.3 Gabor Frames and Twisted Convolution Algebra

The approach is constructive in the sense that algebraic methods such as Cramer’s Rule can be applied to find the inverse of $A$. Then, the sequence $b$ can simply be read from the entries of $A^{-1}$ according to the mapping $\phi$. In particular for small $p$ and $d$ this method leads to fast inversion schemes for the twisted convolution operator.

7.3 Gabor Frames and Twisted Convolution Algebra

Twisted convolution arises naturally in the context of Gabor analysis, namely in the product of two Gabor frame operators in their Janssen representations. Hence studying the twisted convolution is a natural way to obtain conditions on the invertibility of Gabor frame-type operators and the explicit inverse.

Gabor analysis deals with the problem of decomposing and reconstructing signals according to a special basis system which consists of regular time-frequency shifts of a single so-called window function [15, 16]. Let $\Lambda$ be a time-frequency lattice, i.e., a discrete subgroup of the time-frequency plane $G \times \hat{G}$, and let $g$ be in $L^2(G)$. Then we define a Gabor system $G(g, \Lambda)$ by

$$G(g, \Lambda) = \{ \pi(\lambda)g : \lambda \in \Lambda \}.$$ 

We associate with this Gabor system the positive operator

$$S : f \in L^2 \to Sf = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)g.$$ 

If the operator $S$ is bounded and invertible on $L^2(G)$, then $G(g, \Lambda)$ is called a frame and $S$ the associated frame operator, cf. [8].

Many studies in Gabor analysis are devoted to the frame operator [23]. In [14, 27, 30] it is shown that the frame operator $S$ satisfies Janssen representation,

$$S = \sum_{\lambda^o \in \Lambda^o} \langle g, \pi(\lambda^o)g \rangle \pi(\lambda^o).$$

At this point, the question arises if we can deduce the invertibility of the operator $S$ from the Janssen coefficients ($\langle g, \pi(\lambda^o)g \rangle$). It is known from frame theory that if $S$ is invertible, then its inverse is of the same type, that is, it also has a Janssen representation.

As was already mentioned in Chapter 5 frame operator $S$ in the Janssen representation is nothing more than an element of the $C^*$-algebra

$$A(\Lambda^o) = \left\{ S \in B(L^2(G)) : S = \sum_{\lambda^o \in \Lambda^o} a(\lambda^o) \pi(\lambda^o) , \ a = (a(\lambda^o)) \in \ell^1(\Lambda^o) \right\}.$$ 

The elements of this algebra are integrated representation of $(\ell^1(\Lambda^o), \sharp)$, hence

$$\pi^1(a)\pi^1(b) = \pi^1(a \sharp b)$$
and \( \pi^1(\delta) = \text{Id} \) where \( \delta \) and \( \text{Id} \) denote the Dirac sequence and the identity operator, respectively. Both represent the unit element of the corresponding algebra. Since \( \pi^1 \) is an algebra homomorphism from \((\ell^1(\Lambda^o), \hat{\cdot})\) to \( A(\Lambda^o) \), the invertibility of an element in \( A(\Lambda^o) \) can be transferred to the invertibility of the associated \( \ell^1 \)-sequence with respect to the twisted convolution.

It is important to observe, that all the results go through also for weighted \( \ell^1 \)-spaces. These facts are used to design dual Gabor windows of a special type, cf. [25].

**Remark 7.3.1.** We briefly describe how the presented inversion scheme applies to Gabor frame operators in a one-dimensional setting. A more detailed discussion also for finite dimensional signals is described in the last section.

Assume \( d = 1, p = q = m = 0 \). Let the rational lattice of \( \mathbb{R}^2 \) be \( \alpha \mathbb{Z} \times \beta \mathbb{Z} \) for some constants \( \alpha, \beta \) such that \( \alpha \beta = \frac{p}{q} < 1 \) (with \( p, q \) relative prime), then the isotropic lattice \( \Lambda_c \) is \( \Lambda_c = \frac{p}{\beta} \mathbb{Z} \times \frac{p}{\alpha} \mathbb{Z} \), whose adjoint takes the from

\[
\Lambda^o_c = \frac{p}{\beta} \mathbb{Z} \times \frac{p}{\alpha} \mathbb{Z}.
\]

Let \( \mathbf{c} \) be in \( \ell^1(\Lambda^o_c) \). Set

\[
\pi^1(\mathbf{c}) = \sum_{k, l \in \mathbb{Z}} c(\beta^{-1}k, \alpha^{-1}l) \pi(\beta^{-1}k, \alpha^{-1}l).
\]

We define a new sequence \( \mathbf{a} \) as \( a_{k,l} = c(\beta^{-1}k, \alpha^{-1}l) \). Then \( \mathbf{a} \in \ell^1(\mathbb{Z}^2) \) and we can rewrite the above map as \( \pi^1 : \ell^1(\mathbb{Z}^2) \to A \)

\[
\pi^1(\mathbf{a}) = \sum_{k, l \in \mathbb{Z}} a_{k,l} \pi(\beta^{-1}k, \alpha^{-1}l).
\]

In order to verify if \( \pi^1(\mathbf{a}) \) is invertible on \( L^2(\mathbb{R}) \) we simply look at the coefficient sequence \( \mathbf{a} \) and check whether \( \mathbf{a} \) is invertible in \( (\ell^1(\mathbb{Z}^2), \hat{\cdot}) \). To this end, we apply the above results and switch to the matrix \( A \) whose entries are defined by

\[
A_{i,j} = \sum_{m=0}^{p-1} \omega^{mj} a^{m,i-j},
\]

with \( \omega = e^{2\pi i q/p} \). Next, we need to show that the matrix \( A \) is invertible in \( (\mathcal{M}, \otimes) \). For example, we can calculate the determinate which is a sequence in \( \ell^1 \) and show that it is invertible in \( (\ell^1, \ast) \).

Assume that the determinant of \( A \) is invertible. We denote its inverse by \( \mathbf{e} \). By Cramer’s Rule, we compute the first column of the inverse matrix \( B \) of \( A \) as

\[
B_{k,0} = \det A(0,k) \ast \mathbf{e},
\]
for \( k = 0, \ldots, p - 1 \). Then \( b = \sum_{k=0}^{p-1} B_{k,0} \) provides the inverse sequence of \( a \) which, in turns, gives \( \pi^1(a)^{-1} = \pi^1(b) \).

Note that for \( p = 1 \), the twisted convolution turn into normal convolution and we can simply apply the standard Fourier inversion scheme of sequences in \( (\ell^1(\mathbb{Z}^2), \ast) \) since in this case the matrix \( A \) reduces to the sequence \( a \).

As we saw in Chapter 5 the expression
\[
\sum_{\lambda \in \Lambda} a(\lambda) \pi(\lambda) \quad \lambda \in \Lambda,
\]
where \( \Lambda \) is the lattice of the time-frequency plane \( \mathcal{G} \times \hat{\mathcal{G}} \), is a faithful \( \ast \)-representation of the group algebra \( (\ell^1(\Lambda), \sharp) \) and all the representations of \( (\ell^1(\Lambda), \sharp) \) are in 1-1 correspondence with the representations of the group \( \Lambda \).

In Chapter 6 we characterized all the irreducible representations of \( \mathcal{G} \times \hat{\mathcal{G}} \) that keep the symplectic form invariant, by providing explicit formulas of the intertwining operators for the representations of the finite Heisenberg group. Hence in the case when \( \mathcal{G} = \mathbb{Z}_n \), finite cyclic group, we also have the characterization of all the \( \ast \)-representations of \( (\ell^1(\Lambda), \sharp) \) on \( L^2(\mathcal{G}) \).

### 7.4 Equivalence to Other Known Methods

In this section we focus on the case when \( \mathcal{G} = \mathbb{R} \), describe two known methods dealing with the characterization of invertibility of frame-type operators \( S_{g, \gamma} \), where \( g, \gamma \in L^2(\mathbb{R}) \), and their equivalence to the method presented in the previous section. We consider only those functions \( g \) and \( \gamma \) such that \( \langle \gamma, M_{m/\alpha} T_{n/\beta} g \rangle \), \( m, n \in \mathbb{Z} \), is an \( \ell^1 \) sequence. Furthermore, since we assume that \( \mathcal{G}(g, \alpha, \beta) \) and \( \mathcal{G}(\gamma, \alpha, \beta) \) form Gabor frames, \( \alpha \beta = p/q \leq 1 \).

The first approach uses sophisticated methods from Gabor analysis, and the second makes use of the vector-valued Zak transform and its properties [23].

#### 7.4.1 The approach using Gabor analysis tools

This approach was developed in [46], and is based on time-frequency methods. We assume that \( \alpha \beta = p/q \) and that \( g, \gamma \in S_0 \). Define the bi-infinite matrix valued function \( G(x) \) by the correlation functions
\[
G_{j,l}(x) = \sum_{k \in \mathbb{Z}} g(x - l/\beta - ak) \gamma(x - j/\beta - ak), \quad j, l \in \mathbb{Z}.
\]

Because \( g, \gamma \in S_0 \), \( T_{l/\beta} g \cdot T_{j/\beta} \gamma \) is in \( S_0 \), since \( S_0 \) is closed under translation and under pointwise multiplication [23]. Moreover, the periodization \( G_{j,l}(x) \) of period \( \alpha \) is
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continuous. As shown in [23, Chapter 6] by virtue of Schur’s test, the bi-infinite matrix \( G(x) \) is a bounded operator on \( \ell^2(\mathbb{Z}) \) for all \( x \). We can also define \( G(x) \) by the Fourier series

\[
G_0,n(x) = \alpha^{-1} \sum_{m,n \in \mathbb{Z}} a_{m,n} e^{2\pi i m x/\alpha},
\]

where \( a_{m,n} = \langle \gamma, M_{m/\alpha} T_{n/\beta} g \rangle \), and use the fact that \( G_{j,l}(x) = G_{0,l-j}(x - j/\beta) \) to extend it to \( G_{j,n}(x) \).

The correlation functions \( G_0,n(x) \) provide an important representation of the frame-type operator \( S_{g,\gamma} \), the so-called \textit{Walnut representation}

\[
S_{g,\gamma} f = \alpha \sum_{n \in \mathbb{Z}} G_{0,n} T_{n/\beta} f, \quad f \in L^2,
\]

that follows directly from the Janssen representation (4.7) of \( S_{g,\gamma} \). Since \( g, \gamma \in S_0 \), the series of the Walnut representation converges unconditionally in \( L^2 \) [23]. Next, we define the entries of the \((p \times p)\)-matrix valued function \( \Psi(\xi, x) = (\psi_{r,s}(\xi, x))_{r,s=0}^{p-1} \) by

\[
\psi_{r,s}(\xi, x) = \alpha^{-1} \sum_{m,n \in \mathbb{Z}} a_{m-s-r-m} e^{-2\pi i m r / \alpha \beta} e^{2\pi i (m x / \alpha + n \xi / \beta)}.
\]

As shown in [46], the frame operator \( S_{g,\gamma} \) is invertible if and only if \( \det(\Psi(\xi, x)) \neq 0 \) for all \( (\xi, x) \).

In the case of integer oversampling, the matrix \( \Psi \) simply becomes

\[
\psi(\xi, x) = \alpha^{-1} \sum_{m,n \in \mathbb{Z}} a_{m,n} e^{2\pi i (m x / \alpha + n \xi / \beta)}.
\]

That is \( \psi \) is the inverse Fourier transform of the sequence \( a \). This implies that \( S_{g,\gamma} \) is invertible if and only if the sequence \( a \) is invertible in \( (\ell^1(\mathbb{Z}^2), \ast) \), and this approach coincides with our method. Theorem 7.4.1 below shows that not only in integer oversampling, but also in the general rational case, the matrix \( \Psi(\xi, x) \) is equivalent to the matrix \( A \), described in the previous section, via the Fourier transform:

**Theorem 7.4.1.** Let \( g, \gamma \in S_0 \), \( \alpha \beta = p/q \) and \( a_{m,n} = \langle \gamma, M_{m/\alpha} T_{n/\beta} g \rangle \). Then

\[
\mathcal{F} \psi_{r,s} = \beta A_{r,s} \quad \text{for} \quad r, s = 0, \ldots, p - 1,
\]

where \( A \) is a matrix with entries given by \( A_{r,s} = \sum_{k=0}^{p-1} \omega^{k s} a^{k r-s} \). Moreover, \( \mathcal{F} \det \Psi = \beta^p \det A \).
7.4 Equivalence to Other Known Methods

Proof. Let \( k, l \in \mathbb{Z} \),

\[
(F \psi_{r,s})(k, l) = \int_0^\alpha \int_0^\beta \psi_{r,s}(\xi, x) e^{-2\pi ikx/\alpha} e^{-2\pi il\xi/\beta} dxd\xi \\
= \frac{1}{\alpha} \int_0^\alpha \int_0^\beta \sum_{m,n \in \mathbb{Z}} a_{m,s-r-pn} e^{-2\pi i m r \alpha} e^{2\pi i (m-k)x/\alpha} dxd\xi \\
= \frac{1}{\alpha} \sum_{m,n \in \mathbb{Z}} a_{m,s-r-pn} e^{-2\pi i m r \alpha} \delta_{m,k} \delta_{n,l} \\
= \beta e^{-2\pi i k r / \alpha} a_{k,s-r-pn}.
\]

Therefore,

\[
F(\psi_{r,s}) = \beta \sum_{l=0}^{p-1} \omega^{lr} a^{l-r} = \beta A_{s,r}.
\]

Let \( k, l \in \mathbb{Z} \) and \( S_p \) a group of permutations of the set \( \{0, \ldots, p-1\} \), then by the definition of the determinant and previous calculations

\[
(F \det \Psi)(k, l) = (F \sum_{\sigma \in S_p} (-1)^\sigma \psi_{0,\sigma(0)} \psi_{1,\sigma(1)} \cdots \psi_{p-1,\sigma(p-1)})(k, l) \\
= (\sum_{\sigma \in S_p} (-1)^\sigma F \psi_{0,\sigma(0)} \ast \cdots \ast F \psi_{p-1,\sigma(p-1)})(k, l) \\
= (\sum_{\sigma \in S_p} (-1)^\sigma \beta A_{0,\sigma(0)} \ast \cdots \ast \beta A_{p-1,\sigma(p-1)})(k, l) \\
= \beta^p (\sum_{\sigma \in S_p} (-1)^\sigma A_{0,\sigma(0)} \ast \cdots \ast A_{p-1,\sigma(p-1)})(k, l) \\
= \beta^p (\det A)(k, l).
\]

Theorem 7.4.1 provides an equivalence between two invertibility characterizations, in the sense that \( \det(\Psi(\xi, x)) \neq 0 \) if and only if \( \det A \) is invertible in \( (\ell^1, \ast) \). However, whereas our approach applies also to the finite dimensional discrete setting, this is not the case in [46]. There the authors considered this problem not from the point of view of Janssen representation but taking advantage of the structure of the Gabor matrix.

7.4.2 The Zibulski-Zeevi representation

Another characterization of invertibility of the frame operator \( S_{g,\gamma} \) is due to Zibulski and Zeevi and relies on the notion of the Zak transform. Given \( g, \gamma \in L^2(\mathbb{R}) \), the Zak transform of \( g \) is defined as

\[
Z_\alpha g(x, \xi) = \sum_{k \in \mathbb{Z}} g(x - \alpha k) e^{2\pi i \alpha k \xi}.
\]
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The Zak transform is a unitary mapping from $L^2(\mathbb{R})$ into $L^2([0,\alpha]^2)$, that is
\[
\alpha(\mathcal{Z}_\alpha \gamma, \mathcal{Z}_\alpha g)_{L^2([0,\alpha]^2)} = \langle \gamma, g \rangle_{L^2(\mathbb{R})},
\]
and satisfies the following relation with respect to time-frequency shifts, for $k, l \in \mathbb{Z}$
\[
\mathcal{Z}_\alpha(M_{k/\alpha}T_{l/\beta}g)(x,\xi) = e^{2\pi i x k/\alpha} e^{-2\pi i l/\beta} \mathcal{Z}_\alpha g(x,\xi).
\]

For $j \in \{0, \ldots, p-1\}$, $x \in [0, \alpha/p]$ and $\xi \in [0, 1/\alpha]$ we consider the vector-valued Zak transform
\[
(\mathcal{Z}_\alpha g(x,\xi))_j = \mathcal{Z}_\alpha g(x + aj/p,\xi),
\]
and introduce the matrix-valued function $\mathcal{A}(x,\xi)$ over the index set $\{0, \ldots, p-1\}$ defined by the entries
\[
\mathcal{A}_{r,s}(x,\xi) = \alpha \sum_{j=0}^{q-1} \mathcal{Z}_\alpha g(x + \frac{as}{p},\xi - \beta j) \mathcal{Z}_\alpha \gamma(x + \frac{ar}{p},\xi - \beta j) e^{2\pi i j (r-s)/q}.
\]

Then Zibulski-Zeevi representation states that
\[
S_{g,\gamma} = \mathcal{Z}_\alpha^{-1} \mathcal{A} \mathcal{Z}_\alpha,
\]
and hence $S_{g,\gamma}$ is invertible if and only if the matrix-valued function $\mathcal{A}(x,\xi)$ is invertible for almost all $(x,\xi) \in [0, \alpha/p] \times [0, 1/\alpha]$.

In the case of the integer oversampling, $p = 1$ and $\alpha q = 1/\beta$, the matrix-valued function $\mathcal{A}$ becomes a complex valued function
\[
\mathcal{A}(x,\xi) = \alpha \sum_{j=0}^{q-1} \mathcal{Z}_\alpha g(x + \frac{as}{p},\xi - \beta j) \mathcal{Z}_\alpha \gamma(x + \frac{ar}{p},\xi - \beta j) e^{2\pi i j (r-s)/q},
\]
on $[0, \alpha] \times [0, 1/\alpha q]$. By taking the Fourier transform of $\mathcal{A}$ and using (7.9) and (7.10) we obtain
\[
\mathcal{F} \mathcal{A}(k,l) = \alpha \int_0^{\alpha} \int_0^{\alpha} \sum_{j=0}^{q-1} \mathcal{Z}_\alpha g(x,\xi - \beta j) \mathcal{Z}_\alpha \gamma(x,\xi - \beta j) e^{2\pi i x k/\alpha} e^{-2\pi i l/\beta} dx d\xi
\]
\[
= \alpha \int_0^{\alpha} \int_0^{1/\alpha} \mathcal{Z}_\alpha g(x,\xi) \mathcal{Z}_\alpha \gamma(x,\xi) e^{2\pi i x k/\alpha} e^{-2\pi i l/\beta} dx d\xi
\]
\[
= \alpha \int_0^{\alpha} \int_0^{1/\alpha} \mathcal{Z}_\alpha(M_{k/\alpha}T_{l/\beta}g)(x,\xi) \mathcal{Z}_\alpha \gamma(x,\xi) dx d\xi
\]
\[
= \alpha \langle \mathcal{Z}_\alpha \gamma, \mathcal{Z}_\alpha(M_{k/\alpha}T_{l/\beta}g) \rangle_{L^2([0,\alpha]^2)} = \langle \gamma, M_{k/\alpha}T_{l/\beta}g \rangle_{L^2(\mathbb{R})}
\]
\[
= a_{k-l}.
\]
Thus the inversion of the frame operator reduces to inverting $a$ in $(\ell^1, *)$.

In the more general setting, that is for any rational oversampling, the Zibulski-Zeevi matrix is equivalent to the matrix $\Psi$ described in the previous section.
7.4 Equivalence to Other Known Methods

Theorem 7.4.2. Let \( g, \gamma \in L^2(\mathbb{R}) \) and \( \alpha \beta = p/q \). Then for all \((x, \xi) \in [0, \alpha/p] \times [0, 1/\alpha] \) and \( r, s = 0, \ldots, p-1 \)
\[
A_{r,s}(x, \xi) = \alpha q e^{2\pi i k_0 (r-s) \alpha \xi} \psi_{-rl_0-sl_0}(p\xi, x),
\]
where \( \psi_{r,s}(\xi, x) \) is defined in (7.8), and \( k_0, l_0 \in \mathbb{Z} \) are such that \( pk_0 + ql_0 = 1 \). Moreover, \( \det A(x, \xi) \neq 0 \) if and only if \( \det \Psi(p\xi, x) \neq 0 \) for almost all \((x, \xi)\).

Before proving Theorem 7.4.2, we come back to a matrix-valued function \( G \) introduced in Section 5.1 and state its two important properties:
\[
G_{j,l}(x) = G_{j+pk,l+pk}(x), \quad (7.11)
\]
\[
G_{j,l}(x) = G_{0,l-j}(x - j/\beta). \quad (7.12)
\]
The first property is based on the fact that \( q\alpha = p/\beta \). The second relation shows that \( G_{j,l}(x) \) can be derived from \( G_{0,l}(x) \). Next, we define the entries of the \((p \times p)\)-matrix valued function \( \hat{G}(\xi, x) \) by
\[
\hat{G}_{r,s}(\xi, x) = \sum_{k \in \mathbb{Z}} G_{r,s+pk}(x) e^{-2\pi ik\xi/\beta}, \quad r, s = 0, \ldots, p-1. \quad (7.13)
\]
As shown in [23, Chapter 13] (replace one \( g \) by \( \gamma \)), \( \hat{G}(\xi, x) \) has an absolutely converging Fourier series expansion and is therefore continuous and of period \((\beta, \alpha)\). Moreover, \( G(x) \) is invertible on \( \ell^2(\mathbb{Z}) \) if and only if the \((p \times p)\)-matrix \( \hat{G}(\xi, x) \) is invertible for every \((\xi, x)\).

Lemma 7.4.3. The matrix-valued function \( \Psi(\xi, x) \) defined in (7.8) coincides with the matrix-valued function \( \hat{G}(\xi, x) \) defined in (7.13).

Proof. Using (7.6),(7.11) and (7.12), we compute
\[
\begin{align*}
\hat{G}_{r,s}(\xi, x) &= \sum_{n \in \mathbb{Z}} G_{r,s+pn}(x) e^{-2\pi in\xi/\beta} = \sum_{n \in \mathbb{Z}} G_{0,s-r+pn}(x - r/\beta) e^{-2\pi in\xi/\beta} \\
&= \frac{1}{\alpha} \sum_{m,n \in \mathbb{Z}} a_{m,s-r+pn} e^{2\pi im(x-r/\beta)/\alpha} e^{-2\pi in\xi/\beta} \\
&= \frac{1}{\alpha} \sum_{m,n \in \mathbb{Z}} a_{m,s-r-pn} e^{2\pi im(mz + n\xi)/\alpha} e^{2\pi i(mz + n\xi)/\alpha} = \psi_{r,s}(\xi, x).
\end{align*}
\]
Now we are in the position to prove Theorem 7.4.2.
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Proof. Let \((x, \xi) \in [0, \alpha/p] \times [0, 1/\alpha]\). Then

\[
\mathcal{A}_{r,s}(x, \xi) = \alpha \sum_{j=0}^{q-1} \mathcal{Z}_\alpha g(x + \frac{\alpha s}{p}, \xi - \beta j) \mathcal{Z}_\alpha \gamma(x + \frac{\alpha r}{p}, \xi - \beta j)e^{2\pi i(r-s-q)j/q}
\]

\[
= \alpha \sum_{m,n \in \mathbb{Z}} g(x + \frac{\alpha s}{p} - am)\gamma(x + \frac{\alpha r}{p} - an)e^{-2\pi i\alpha(m-n)}e^{2\pi i(r-s-p(m-n))j/q}
\]

\[
= \alpha \sum_{m,k \in \mathbb{Z}} g(x + \frac{\alpha s}{p} - am)\gamma(x + \frac{\alpha r}{p} - am + \alpha k)e^{-2\pi i\alpha(m-k)}e^{2\pi i(r-s+pk)j/q}
\]

Since \(\gcd(p, q) = 1\), there exist \(k_0, l_0 \in \mathbb{Z}\) such that \(pk_0 + ql_0 = 1\). Then \(r-s+pk \in q\mathbb{Z}\) if and only if \(k \in -k_0(r-s) + q\mathbb{Z}\). Therefore, by changing summation over \(m\) to \(m + k_0 s\) and noticing that for every \(s \in \mathbb{Z}_p\), \(\frac{p}{p} s - \alpha k_0 s = \frac{d}{p}\), we obtain that

\[
\mathcal{A}_{r,s}(x, \xi) = \alpha q \sum_{m,l \in \mathbb{Z}} \mathcal{Z}_\alpha g(x + sl_0/\beta - am) \times \gamma(x + \frac{rl_0 + pl}{\beta} - am)e^{2\pi i(k_0(r-s)\alpha - l_0q\xi)}
\]

\[
= \alpha q \sum_{l \in \mathbb{Z}} \mathcal{G}_{-rl_0 - pl - sl_0}(x)e^{-2\pi il_0q\xi}e^{2\pi ik_0(r-s)\alpha\xi}
\]

\[
= \alpha q \sum_{l \in \mathbb{Z}} \mathcal{G}_{-rl_0 - sl_0 + pl}(x)e^{-2\pi il_0q\xi}e^{2\pi ik_0(r-s)\alpha\xi}
\]

\[
= \alpha q e^{2\pi ik_0(r-s)\alpha\xi} \mathcal{G}_{-rl_0 - sl_0}(p\xi, x).
\]

By Lemma 7.4.3, \(\Psi(\xi, x)\) coincides with the matrix-valued function \(\mathcal{G}(\xi, x)\). Hence

\[
\mathcal{A}_{r,s}(x, \xi) = \alpha q e^{2\pi ik_0(r-s)\alpha\xi} \Psi_{-rl_0 - sl_0}(p\xi, x).
\]

By Cramer’s rule,

\[
\det \mathcal{A}(x, \xi) = \sum_{\sigma \in S_p} (-1)^\sigma \mathcal{A}_{\sigma(0),0}(x, \xi) \cdots \mathcal{A}_{\sigma(p-1),p-1}(x, \xi)
\]

\[
= (\alpha q)^p \sum_{\sigma \in S_p} (-1)^\sigma \prod_{i \in \mathbb{Z}_p} e^{2\pi ik_0(\sigma(i)-i)\alpha\xi} \mathcal{G}_{-\sigma(i)l_0 - sl_0}(p\xi, x)
\]

\[
= (\alpha q)^p \sum_{\sigma \in S_p} (-1)^\sigma \prod_{i \in \mathbb{Z}_p} e^{2\pi ik_0\alpha\xi} \prod_{i=1}^{p-1} \mathcal{G}_{-\sigma(i)l_0 - il_0}(p\xi, x)
\]

\[
= (\alpha q)^p \sum_{\tau \in S_p} (-1)^\tau \mathcal{G}_{\tau(0),0}(p\xi, x) \cdots \mathcal{G}_{\tau(p-1),p-1}(p\xi, x)
\]

\[
= (\alpha q)^p \det \mathcal{G}(p\xi, x) = (\alpha q)^p \det \Psi(p\xi, x)
\]

which completes the proof. \(\square\)
Theorem 7.4.2 implies that the Zibulski-Zeevi condition on invertibility of $S_{g,\gamma}$ is equivalent to the one described in Section 4, by Theorem 7.4.1.

The benefit of our approach is that the inverse of a matrix $A$ gives us back in a constructive manner an inverse of the sequence $a$ in $(\ell^1, \natural)$. Secondly, this new approach works for the continuous as well as the discrete case, with no need to distinction as it was in [46].

7.5 MATLAB Implementations

In this section we will illustrate in more detail how the method described in the previous sections, more precisely the division of the sequence $a$ into smaller sequences, looks like when we consider Gabor analysis on a space of finite dimensional signals. We will also perform numerical analysis and present a MATLAB code to invert sequences in $(\ell^1, \natural)$, or more precisely to invert Gabor frame operators.

7.5.1 Gabor frame-type operator for finite-length signals

Thus, we now consider Gabor analysis of sequences of length $L$. The goal is to determine if the frame-type operator $S_{g,\gamma}$, for some window functions $g$ and $\gamma$ from $\mathbb{C}$ and a lattice $\Lambda \subseteq \mathbb{Z}_L \times \mathbb{Z}_L$, which we denote by $S$, is invertible and if so then to construct the inverse. We can describe $S$ by its Janssen representation associated to vectors $g, \gamma \in \mathbb{C}^L$. Let $\Lambda = \alpha \mathbb{Z}_L \times \beta \mathbb{Z}_L$ be a lattice of the time-frequency plane $\mathbb{Z}_L \times \mathbb{Z}_L$ where $\alpha$ and $\beta$ divide $L$ and $\alpha \beta / L = p/q \leq 1$, with $p$ and $q$ relatively prime. Since the adjoint of $\Lambda$ is $\Lambda^\circ = \beta \mathbb{Z}_L \times \alpha \mathbb{Z}_L$, the Janssen representation of the operator $S$ is

$$S = \sum_{(m,n) \in \mathbb{Z}_\alpha \times \mathbb{Z}_\beta} a_{m,n} M_{mL/\alpha} T_{nL/\beta},$$

where $a_{m,n} = \langle \gamma, M_{mL/\alpha} T_{nL/\beta} g \rangle$ for all $(m,n) \in \mathbb{Z}_\alpha \times \mathbb{Z}_\beta$. We define the set of sequences with index set $\mathbb{Z}_\alpha \times \mathbb{Z}_\beta$ by $\ell^1(\mathbb{Z}_\alpha \times \mathbb{Z}_\beta)$. As discussed in the previous sections, if $S$ is invertible, then there is a sequence $b \in \ell^1(\mathbb{Z}_\alpha \times \mathbb{Z}_\beta)$ such that $a \natural b = \delta$, and the inverse of $S$ is the operator $T = \sum_{(m,n) \in \mathbb{Z}_\alpha \times \mathbb{Z}_\beta} b_{m,n} M_{mL/\alpha} T_{nL/\beta}$. First we present the method for general $p$ and then show an explicit example when $p = 2$.

Let $a, b \in \ell^1(\mathbb{Z}_\alpha \times \mathbb{Z}_\beta)$. Then the standard convolution between $a$ and $b$ is defined as

$$(a \ast b)_{m,n} = \sum_{k,l \in \mathbb{Z}_\alpha \times \mathbb{Z}_\beta} a_{k,l} b_{m-k,n-l} \quad m,n \in \mathbb{Z}_\alpha \times \mathbb{Z}_\beta,$$

where operation on indices is performed modulo $(\alpha, \beta)$. By analogy with the infinite case, we define the twisted convolution $a \natural b$ of two elements $a, b \in \ell^1(\mathbb{Z}_\alpha \times \mathbb{Z}_\beta)$ by

$$(a \natural b)_{m,n} = \sum_{k,l \in \mathbb{Z}_\alpha \times \mathbb{Z}_\beta} a_{k,l} b_{m-k,n-l} \omega^{(m-k)l}$$

$$\omega = \exp(-2\pi i/p)$$

where $\omega$ is the $p$-th root of unity.
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with \( m, n \in \mathbb{Z}_\alpha \times \mathbb{Z}_\beta \), where again the operation on indices is performed modulo \((\alpha, \beta)\), and \( \omega = e^{-2\pi ip/q} \), with \( p, q \) relatively prime and \( p \) dividing \( L, \alpha \) and \( \beta \).

Let \( \mathbf{a} \) be a sequence of Janssen coefficients representing \( S \). We want to see if \( S \) is invertible, and if so, then to construct the inverse. Invertibility of \( S \) is equivalent to the invertibility of the sequence \( \mathbf{a} \) with respect to twisted convolution. Hence we use the techniques introduced in the last two sections.

Since \( p \) divides \( \alpha \) and \( \beta \), \( \mathbb{Z}_\alpha \times \mathbb{Z}_\beta \) admits a decomposition into \( p^2 \) disjoint subsets \((r, s) + (p\mathbb{Z}_\alpha \times p\mathbb{Z}_\beta)\), where \( r, s \in \mathbb{Z}_p \). Namely,

\[
\mathbb{Z}_\alpha \times \mathbb{Z}_\beta = \bigcup_{r,s=0}^{p-1} (r,s) + (p\mathbb{Z}_\alpha \times p\mathbb{Z}_\beta) .
\]

Therefore we can define subsequences of \( \mathbf{a} \) in a similar manner as in (7.3):

\[
(a^{r,s})_{m,n} = \begin{cases} 
  a_{m,n} & \text{if } (m, n) \equiv_p (r, s) \\
  0 & \text{else.}
\end{cases}
\]

The construction for \( p = 2 \) is pictured in Figure 1.2 for a random sequence \( \mathbf{a} \in \ell^1(\mathbb{Z}_8 \times \mathbb{Z}_8) \) (Figure 1.1).

![Figure 7.1: A random 2-dimensional sequence \( \mathbf{a} \) of size 8 by 8.](image)
We now consider an explicit example. Let $L = 2^9 \cdot 3$, $\alpha = \beta = 2^5$ and let $g, \gamma$ be two Gaussians of different spread, that is
\[
g[k] = e^{-k^2}, \quad \text{and} \quad \gamma[k] = e^{-k^2/2}.
\] (7.14)
Since $\alpha\beta/L = 2/3$, $p = 2$ and the matrix $A = \phi(a)$ corresponding to a Janssen representation sequence $a_{m,n} = \langle \gamma, M_{3 \cdot 2^m}T_{3 \cdot 2^n}g \rangle$ of a Gabor frame-like operator $S_{g,\gamma}$ is a $2 \times 2$ matrix of the form
\[
\phi(a) = \begin{pmatrix}
a^{0,0} + a^{1,0} & a^{0,1} + \omega a^{1,1} \\
a^{0,1} + a^{1,1} & a^{0,0} + \omega a^{1,0}
\end{pmatrix}
\]
where we used the fact that $\omega = e^{-2\pi i q/p} = -1$. Note that summing up the elements of the first column gives us back the sequence $a$. The entries of this matrix are finite length $\alpha\beta = 2^{10}$ sequences. The two dimensional Fourier transform of $\det A$ (Figure 1.3),
\[
\det A = (a^{0,0} + a^{1,0}) \ast (a^{0,0} - a^{1,0}) - (a^{0,1} + a^{1,1}) \ast (a^{0,1} - a^{1,1}).
\]
has no zeros, therefore $\det A$ is an invertible sequence in $(\ell^1(Z_\alpha \times Z_\beta), \ast)$. That implies, by Wiener’s Lemma (Theorem 7.2.3) that the matrix $A$ is invertible and the inverse can

Figure 7.2: A decomposition of signal $a$ into four ($p = 2$) subsequences: a) $a^{00}$, b) $a^{10}$, c) $a^{01}$ and d) $a^{11}$. 
be computed using the Cramer’s Rule,

\[ A^{-1} = (\det A)^{-1} \ast \begin{pmatrix} a^{0,0} - a^{1,0} & -a^{0,1} + a^{1,1} \\ -a^{0,1} - a^{1,1} & a^{0,0} + a^{1,0} \end{pmatrix}. \]

From here we obtain the inverse of a sequence \( a \) in the twisted convolution algebra \((\ell^1(\mathbb{Z}_\alpha \times \mathbb{Z}_\beta), \ast)\), namely a sequence

\[ b = (\det A)^{-1} \ast (a^{0,0} - a^{1,0} - a^{0,1} - a^{1,1}), \]

which is a sum of the elements from the first column of \( A^{-1} \).

This shows that inverting a sequence \( a \) in \((\ell^1(\mathbb{Z}_\alpha \times \mathbb{Z}_\beta), \ast)\) essentially boils down to inverting a sequence, namely \( \det A \), in \((\ell^1(\mathbb{Z}_\alpha \times \mathbb{Z}_\beta), \ast)\), for which fast Fourier transform schemes can be applied.

We presented a method of inverting Gabor frame operators given by the Janssen representation. There exist many other methods of inverting Gabor frame operators in the case of finite-length signals that take advantage of a highly structured matrix form that the operator \( S \) admits ([43], [46]). However, the previous methods apply only when considering finite discrete models and there is no analogue of them in the continuous setting. The method that we presented here covers both environments (since it is not based on \( S \) directly, but rather on the Janssen coefficient sequence associated to \( S \)), thus giving a smooth transition between the continuous and discrete settings.
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Figure 7.3: 2-D DFT of $\det A$, where $L = 2^9 \cdot 3$, $\alpha = \beta = 2^5$ and the windows $g, \gamma$ are Gaussians given by (7.14). The transform is performed on the group $\mathbb{Z}_\alpha \times \mathbb{Z}_\beta$, and the axis $x$ and $y$ are the numbers $0 - 31$, that is representatives of the group $\mathbb{Z}_\alpha = \mathbb{Z}_\beta$.

MATLAB Code

The approach described above is constructive in the sense that algebraic methods such as Cramer’s Rule can be applied to find the inverse of $A$. To matrix $A^{-1}$, according to Theorem 7.2.3, corresponds another element $b$ in $\ell^1$, that is $A^{-1} = \phi(b)$. This element, in turn, provides the inverse of $a$ in $(\ell^1, \natural)$. The sequence $b$ can simply be read from the entries of $A^{-1}$ according to the mapping $\phi$, that is we obtain $b$ by summing up the elements from the first column of $A^{-1}$. Therefore, it is enough to compute only the first column of the inverse matrix $A^{-1}$. For small $p$ this method leads to fast inversion schemes for the twisted convolution operator.

We can translate all these derivations and results into a computer language to produce a MATLAB code for inverting Gabor frames. Here we present a toolbox for computing twisted convolution of two dimensional sequences and also a twisted convolution inverse if such exists for a given sequence. This code is then used to invert a Gabor system. The code below was written for the case when $p = 2$. 

81
function a_inv = twc_inv(a);

% Computes the inverse of the 2-dimensional sequence a
% in the twisted convolution algebra l^1(Z_r \times Z_c)
% where the parameter theta defining the twisted convolution
% equals 2/q, where q is some odd number.

[r,c] = size(a);
%% divides the sequence a into p^2 subsequences of the same length

a00 = zeros(r,c);
for m=0:r/2-1
    for n=0:c/2-1
        a00(2*m+1,2*n+1) = a(2*m+1,2*n+1);
    end
end

a01 = zeros(r,c);
for m=0:r/2-1
    for n=0:c/2-1
        a01(2*m+1,2*n+2) = a(2*m+1,2*n+2);
    end
end

a10 = zeros(r,c);
for m=0:r/2-1
    for n=0:c/2-1
        a10(2*m+2,2*n+1) = a(2*m+2,2*n+1);
    end
end

a11 = zeros(r,c);
for m=0:r/2-1
    for n=0:c/2-1
        a11(2*m+2,2*n+2) = a(2*m+2,2*n+2);
    end
end

%%

det = convol(a00+a10,a00-a10) - convol(a01+a11,a01-a11);
fdet = fft2(det);

% checks if the sequence has the inverse with respect to twisted
% convolution
if size(find(fdet == 0)) > 0
    display('The sequence doesn't have the twisted convolution inverse')
else
    idet = ifft2(1./fdet);
    a_inv = convol(idet,a00-a10-a01-a11);
end
The twisted convolution MATLAB code is the following:

```matlab
function t = twc(a,b);

% Computes the twisted convolution of two sequences in Z_M x Z_M
% with parameter p/q where p=2, and p divides N and M

[N,M] = size(a);

a00 = zeros(N,M);
b00 = zeros(N,M);
for m=0:N/2-1
    for n=0:M/2-1
        a00(2*m+1,2*n+1) = a(2*m+1,2*n+1);
b00(2*m+1,2*n+1) = b(2*m+1,2*n+1);
    end
end

a01 = zeros(N,M);
b01 = zeros(N,M);
for m=0:N/2-1
    for n=0:M/2-1
        a01(2*m+1,2*n+2) = a(2*m+1,2*n+2);
b01(2*m+1,2*n+2) = b(2*m+1,2*n+2);
    end
end

a10 = zeros(N,M);
b10 = zeros(N,M);
for m=0:N/2-1
    for n=0:M/2-1
        a10(2*m+2,2*n+1) = a(2*m+2,2*n+1);
b10(2*m+2,2*n+1) = b(2*m+2,2*n+1);
    end
end

a11 = zeros(N,M);
b11 = zeros(N,M);
for m=0:N/2-1
    for n=0:M/2-1
        a11(2*m+2,2*n+2) = a(2*m+2,2*n+2);
b11(2*m+2,2*n+2) = b(2*m+2,2*n+2);
    end
end


\[ t_{00} = \text{convol}(a_{00},b_{00}) + \text{convol}(a_{01},b_{01}) + \text{convol}(a_{10},b_{10}) - \text{convol}(a_{11},b_{11}) ; \]
\[ t_{01} = \text{convol}(a_{00},b_{01}) + \text{convol}(a_{01},b_{00}) + \text{convol}(a_{10},b_{11}) - \text{convol}(a_{11},b_{10}) ; \]
\[ t_{10} = \text{convol}(a_{00},b_{10}) - \text{convol}(a_{01},b_{11}) + \text{convol}(a_{10},b_{00}) + \text{convol}(a_{11},b_{01}) ; \]
\[ t_{11} = \text{convol}(a_{00},b_{11}) - \text{convol}(a_{01},b_{10}) + \text{convol}(a_{10},b_{01}) + \text{convol}(a_{11},b_{00}) ; \]

\[ t = t_{00} + t_{01} + t_{10} + t_{11} ; \]
```
The function `convol.m` computes the convolution of two 2-dimensional sequences,

\begin{verbatim}
function c = convol(a,b)

% convolution of two 2 dimensional sequences in Z_N x Z_M,
% and the result stays in Z_N x Z_M
% both sequences must be of the same size, that is N x M

p = conv2(a,b);
[N,M] = size(a);

cc = zeros(N,M+M-1);
cc(N,:) = p(N,:);
for k=1:N-1
    cc(k,:) = p(k,:) + p(k+N,:);
end

c = zeros(N,M);
c(:,M) = cc(:,M);
for l=1:M-1
    c(:,l) = cc(:,l) + cc(:,l+M);
end

end
\end{verbatim}

Applying it to Gabor theory, we can efficiently compute the inverse Gabor frame operator. Suppose, that \( \mathcal{G}(g, \Lambda) \) is a Gabor frame with atom \( g \) and rational, separable lattice \( \Lambda = \alpha \mathbb{Z}_L \times \beta \mathbb{Z}_L \), where \( \alpha \) and \( \beta \) divide \( L \) and are such that \( \alpha \beta / L = 2/q < 1 \). Then the Janssen coefficient sequence corresponding to the Gabor frame operator associated with this system is

\[ a_{m,n} = \langle g, \pi(\lambda^o)g \rangle \text{ for } (m, n) \in \mathbb{Z}_\alpha \times \mathbb{Z}_\beta, \]

and

\[ S = \sum_{(m,n)\in \mathbb{Z}_\alpha \times \mathbb{Z}_\beta} a_{m,n} M_{mL/\alpha} T_{nL/\beta}. \]

As we know by the previous computations the inverse operator \( T \) of \( S \) in the Janssen representation has the coefficient sequence given by the twisted convolution inverse \( b \) of \( a \) and then

\[ T = \sum_{(m,n)\in \mathbb{Z}_\alpha \times \mathbb{Z}_\beta} b_{m,n} M_{mL/\alpha} T_{nL/\beta}. \]

Therefore, in MATLAB we have
function T = inv_gab_op(g1,g2,a,b);

% Provides the inverse T to a gabor frame-like operator S with
% Janssen coefficients <g1,M_kT_l g2>, (k,l) are elements from
% the dual lattice L/b Z_L x L/a Z_L, where the density ab/L = 2/q,
% for some q relatively prime to 2. Then I = ST.
%
% Input:
% a = time gap
% b = frequency gap
% g1 = window;
% g2 = window;
% if g1=g2 the we have the standard Gabor frame operator
%
% Output:
% T = (a x b) matrix of Janssen coefficients for the inverse of the Gabor
% frame operator S

stf = stft(g,g,L/b,L/a); % Janssen coefficient sequence for the Gabor
T = twc_inv(stf); % frame operator associated with G(g,a,b)

where the function stft.m is a standart short time Fourier transform of g1 with the
window g2

function stf = stft(x,w,a,b);

% STFT.M    H.G.Feichtinger, 5-23/25-1990 , Aug. 92
% determines the STFT of a vector over a lattice
% with lattice constants a (time) and b (frequ)
% STFT(x,a,b) is the output
% USAGE: STFT = STFT(signal,window,a,b)
% See also: IISTFT, TFFILT, PERBAS

if nargin < 3; a = 1; b = 1; end;

n = length(x);

res = zeros(n/a,n/b);
    w1 = [ w zeros(1,n-length(w))];
    wwl = [w1,w1];

for jj = 1 : n/a;
    y = x.*wwl( (n+1-(jj-1)*a) : (2*n - (jj-1)*a) ) ;
    yl = perbas(y,b);
    v = fft(yl);
    res(jj,:) = v;
end;

stf = res.';
function perx = perbas(x,k);

% PERBAS
% Function perx = perbas(x,k)
% Description: PERBAS(x,k) gives the periodic version
% (with k periods) from a vector x.
% Example: perbas([1 2 3 4 5 6 7 8 9],3)=[1+4+7,2+5+8,3+6+9]
% Input: x: vector
% k: integer, which divides the length of x
% Output: perx=a vector of length n/k
% Converse: periodz (not yet done)
% Usage: y = perbas(x,k);
% See also: STFT (important use!)
% Author(s): H.G. FEICHTINGER, 05/1990, revised W.Reuter, 1999
% Copyright: (c) NUHAG, Dept.Math., University of Vienna, AUSTRIA
% http://nuhag.mat.univie.ac.at/
% Permission is granted to modify and re-distribute this
% code in any manner as long as this notice is preserved.
% All standard disclaimers apply.

l = length(x);
if rem(l,k) ≠ 0
    error('k does not divide the length of l');
end
m = l/k;
u = zeros(m,k);
u(:,) = x;
perx = sum(u.);
if k == 1
    perx = x;
end
8 Bibliography


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