TRANSFORMATION, INVERSION AND CONVERSION OF BILINEAR SIGNAL REPRESENTATIONS

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ABSTRACT

Bilinear signal parameters and representations can be described in terms of linear operators. Such a description yields a deeper understanding and a systematic calculus of bilinear time-frequency signal representations such as Wigner distribution, ambiguity function and spectrogram. Basic points of time-frequency representations, like transformations, properties and constraints, inversion and conversion, are discussed in a formal way. The inversion problem leads to the definition of regular representations, where the regularity property makes sure that no essential information about signals is lost.

INTRODUCTION

Hermitian bilinearity is a basic property of important signal parameters such as

- energy densities
  \[ P_{fg}(t) = f(t)g^*(t) \quad \text{and} \quad P_{fg}(\omega) = \frac{1}{2} F(\omega)G^*(\omega), \]

- correlation functions
  \[ R_{fg}(\tau) = \int f(t+\tau)g^*(t) \, dt, \quad R_{fg}(\omega) = \frac{1}{2\pi} \int F(\omega)e^{j\omega t}G^*(\omega) \, d\omega, \]

- energy
  \[ E_{fg} = \int R_{fg}(\tau) \, d\tau, \]

- n-th moments
  \[ m_{fg}^{(n)}(\epsilon) = \int (t-\epsilon)^n R_{fg}(\tau) \, d\tau, \quad M_{fg}^{(n)}(\omega) = \int (\omega-\omega_0)^n P_{fg}(\omega) \, d\omega. \]

Conceptually closely related with these quantities are the time-frequency signal representations which offer a conjoint time-frequency description of signals [1],[2]. Most of these representations also have the bilinearity property, e.g.

- Wigner distribution
  \[ W_{fg}(t,\omega) = \frac{1}{\pi} \int f(t+\tau)g^*(t-\tau) e^{-j\omega \tau} \, d\tau, \]

- ambiguity function
  \[ A_{fg}(\tau,\omega) = \int f(t+\tau)g^*(t-\tau) e^{-j\omega t} \, dt \]

- spectrogram (containing window function \( \eta \))
  \[ S_{fg}(t,\omega) = \frac{1}{2\pi} F(\omega)G^*(\omega) \]

where e.g. \( F(\omega) = \int f(t)e^{j\omega t} \, dt \).

Bilinearity can be expressed as a kind of superposition property. Let \( T_{fg}(\tau,\omega) \) be a representation of two signals \( f, g \), with \( \sigma \) and \( \varepsilon \) denoting time or frequency variables. Signal parameters depending on less than two variables (e.g. \( m_{fg}^{(n)}(\epsilon) \), \( E_{fg} \)) will be considered as special (degenerate) cases of \( T_{fg}(\tau,\omega) \). By definition, then, we call \( T \) a (hermitian) Bilinear Signal Representation (BSR) if it satisfies the following "bilinear superposition principle":

if \( \{ f(t) =\int a(\omega)\eta(t,\omega) \, d\omega \}
\[ g(t) =\int b(\omega)\eta(t,\omega) \, d\omega \]

then

\[ T_{fg}(\tau,\omega) = \int a(\omega)b^*(\omega)T_{fg}(\tau,\omega) \, d\omega. \]

Based on this definition, a description of BSRs can be formulated that characterizes BSRs by means of linear operators.

LINEAR THEORY OF BILINEAR SIGNAL REPRESENTATIONS

We define four elementary BSRs:

- the time signal product \( q \)
  \[ q_{fg}(t,\tau) = f(\tau+\epsilon)g^*(\epsilon), \]

- the frequency signal product \( G \)
  \[ Q_{fg}(\omega,\tau) = \frac{1}{2\pi} F(\omega)G^*(\omega-\tau), \]

- Wigner distribution \( W \) (see (5)),

- ambiguity function \( A \) (see (6)).

These elementary BSRs are mutually connected by one- and two-dimensional Fourier transformation [1]. It can be shown, now,
that any BSR as defined by (8) can be derived from the elementary BSRs $q, Q, W, A$ by linear transformations (comp.[12],[13]):

$$T_{f,g}(s,e) = \int f^s g^e \cdot u_s(e,s;e,t) \, dt \, de =$$

$$= \int f^s g^e \cdot U_s(e,s;e,t) \, dt \, de =$$

$$= \int W_{f,g}(s,e) \cdot V_s(e,e;e,t) \, dt \, de =$$

$$= \int A_{f,g}(s,e) \cdot V_s(e,e;e,t) \, dt \, de.$$  \hspace{1cm} (11)

This is more compactly written using an obvious operator notation:

$$T = u_{q, q} = U_{q, q} - V_{q, W} = V_{q, A}$$  \hspace{1cm} (12)

Eq. (11) - (14) can be considered to be the normal forms of the BSR $T$. According to the normal forms, any BSR $T$ is uniquely characterized by the linear operators $u_T, U_T, V_T$ with kernels $U_T, V_T, W_T, A_T$. These kernels, too, are mutually related by one- and two-dimensional Fourier transformation. While thus any of them alone would suffice to characterize a given BSR, it is yet convenient to consider also the other kernels. $u_T$ bears the interpretation of the BSR's time impulse response ($T = u_T$ if $f$ are impulses). Similarly $U_T$ is the BSR's frequency impulse response ($T = U_T$ if $F, G$ are impulses). $V_T$ and $W_T$ will be called the BSR's Wigner kernel and ambiguity kernel, respectively.

Example 1. To find the time impulse response of the degenerate BSR $m^{(t)}(t)$, we write $m^{(t)}$ (dropping the indices $f, g$)

$$m^{(t)}(t) = \frac{1}{i} \int (t-t)^{p} q(t,0) \, dt = \frac{1}{i} \int q(t,e) \cdot (t-t)^{p} \delta(t) \, dt \, de.$$  \hspace{1cm} (13)

It follows

$$u_{m}(t,s;e) = (t-t)^{p} \delta(t).$$  \hspace{1cm} (14)

Similarly, the frequency impulse response of $M^{(t)}$ is seen to be

$$U_{m}(w_0, s, w) = (w-w_0)^{p} \delta(w).$$

From $m^{(t)} (U_{m})$, the Wigner kernel $m^{(w)} (W_{m})$ can be derived by Fourier transformation:

$$m^{(w)}(t,t;e) = (w-t)^{p} , \quad m^{(w)}(w,w;e) = (w-w_0)^{p}.$$  \hspace{1cm} (15)

Insertion into the normal form (13) yields the Wigner distribution's "moment properties",

$$m^{(w)}(t) = \int f^{(t-t)^{p} w(t,t;e) \, dt \, de = \hspace{1cm} (16)$$

$$M^{(w)}(w) = \int f^{(w-w_0)^{p} W(t,t;e) \, dt \, de = \hspace{1cm} (17)$$

Note the simple and systematic way in which this result has been obtained.

Example 2. By the same reasoning, the spectrogram's Wigner kernel is found to be

$$v_{f}(t,t;e) = W_{f}(t,t;e, \omega) \hspace{1cm} (18)$$

i.e. the Wigner distribution of the window used. Thus we obtain the well-known relation [1]

$$S(t, \omega) = \int f^{(t-t)^{s} W_{f}(t,t;e, \omega) \, dt \, de = \hspace{1cm} (19)$$

Describing a BSR by linear operators offers a theoretical framework in which some questions of basic interest can be conveniently studied. The following sections give examples.

TRANSFORMATION OF BSRs

Linear transformation of a BSR $T$ by an operator $L$ with kernel $L,$

$$(LT)(s,e) = \int \int T(t,s;e) \cdot L_{t,s;e} (t,s;e) \, dt \, de = \hspace{1cm} (18)$$

again results in a BSR $T$, for we have

$$L_{t,s;e} = (LT)_{t,s;e}.$$  \hspace{1cm} (19)

The operators $u_T, U_T$ of the original and the new BSR $T, L$ are related by the composition

$$u_L = L u_T \hspace{1cm} (19)$$

$u_T (s,t;e) = \int \int u_T (s,t;e) \cdot L_{t,s;e} (t,s;e) \, dt \, de = \hspace{1cm} (19)$$

Corresponding relations exist for $U_T, V_T.$

Example 3. If $T(t, \omega)$ is time shifted by $t_0,$ we have $L(t,t_0;w) = \delta(t-t_0)$ and the time impulse response of the new (i.e. shifted) BSR $T(t,t_0;w)$ is, by virtue of (19),

$$u_T(t,t_0;w) = u_T (t-t_0,t_0;w).$$

TRANSFORMATION OF SIGNALS

If the signals $f, g$ (i.e., the BSR's "input") are linearly transformed by operators $x, y$ with kernels $x, y,$

$$f = xf, \hspace{1cm} f(t) = \int \int f(t') x(t', t) \, dt,' \hspace{1cm} (20)$$

$$g =yg, \hspace{1cm} g(t) = \int \int g(t') y(t', t) \, dt,' \hspace{1cm} (20)$$

the signal product $z$ of the new signals $f, g$ is itself a transformed version of the original signal product,

$$z = xz, \hspace{1cm} z(t,t') = \int \int z(t', t') x(t', t') \, dt,' \hspace{1cm} (21)$$

where $z(t,t') = x(t', t') y(t', t'). \hspace{1cm} (22)$
The BSR of \( \tilde{f}, \tilde{g} \) is therefore

\[
T_{f,g} = \mathbf{u}_T \tilde{g} = \mathbf{u}_T(\mathbf{e}(q)) = \mathbf{u}_T(\mathbf{e}(q')) = \mathbf{u}_T q' = \tilde{T}_{f,g}.
\]

It is itself a new BSR of the original signals, with operator

\[
\mathbf{u}_T = \mathbf{u}_T q = \tilde{T}_f q.
\]

A dual derivation applies to \( \mathbf{v}_T \), and there exists also a formulation using \( \mathbf{v}_T \).

**Example 4.** If the signals are time shifted by \( t_0 \), we have \( \chi(t+z) = \tilde{\phi}(t-(t-t_0)) \).

Evaluation of (22) and (23) yields for the new BSR \( T(\omega) \) (BSR of shifted signals)

\[
\mathbf{u}_T(t,\omega,t',\omega') = \mathbf{u}_T(t,\omega,t'+t_0,\omega').
\]

**BSR PROPERTIES AND CONSTRAINTS**

There are essentially two types of desirable BSR properties:

1. **Signal parameter properties**: we desire that a given (bilinear) signal parameter \( a(q) \) can be derived from the BSR by a given transformation \( L^{(a)} \):

\[
a = L^{(a)} \mathbf{u}_T \]

\[
a(q) = \int \mathbf{T}(q',q) L^{(a)}(q',q') dq'dq'.
\]

It is easily seen that (24) is equivalent to

\[
\mathbf{u}_T(q,t,t') = \int \mathbf{T}(q',q) L^{(a)}(q',q) dq'dq'.
\]

Eq. (25) is a constraint on \( \mathbf{u}_T \) that corresponds to property (24) \( \mathbf{T} \) satisfies property (25) if and only if \( \mathbf{u}_T \) satisfies constraint (25). Corresponding constraints can also be formulated for \( \mathbf{v}_T, \mathbf{w}_T, \).

**Example 5.** We consider the moment property

\[
\mathbf{m}^{(a)}(t_0) = \int T(t,\omega) \mathbf{u}_T(\mathbf{e}(q)) dq \delta(t-t_0).
\]

Inserting the impulse response of \( \mathbf{m}^{(a)} \) (see example 1) into (25), we obtain the corresponding constraint on \( \mathbf{u}_T \),

\[
\delta(t-t_0) = \int \mathbf{T}(q,\omega) \mathbf{u}_T^{(a)}(q,\omega) dq \delta(t-t_0).
\]

This constraint is not met if we insert for \( \mathbf{u}_T \) the spectrogram's impulse response

\[
\mathbf{u}_T^{(a)}(q,\omega) = \frac{1}{2\pi} T_{q}(t-t',\omega) \tilde{e}^{-i\omega t}.
\]

We have thus shown that the spectrogram does not satisfy the moment property (26).

2. **Compatibility properties**: we desire that, to given signal transformations \( \mathbf{x}, \mathbf{y} \), the BSR responds by a given transformation \( L^{(xy)} \):

\[
T_{x,y} = L^{(xy)} T_{f,g}.
\]

The corresponding constraint on \( \mathbf{u}_T \) is found by combining the results of the two previous sections, i.e., equating (19), (23):

\[
\mathbf{u}_T = L^{(xy)} \mathbf{u}_T.
\]

Again, constraints can be formulated also for \( \mathbf{v}_T, \mathbf{w}_T, \).

**Example 6.** A BSR is compatible with time shifts if a time shift of signals causes a corresponding time shift of the BSR. Combining the results of examples 3 and 4, we obtain a constraint on \( \mathbf{u}_T \): \( \mathbf{u}_T(t,\omega,t',\omega') = \mathbf{u}_T(t-t_0,\omega) \) \( \forall t_0 \).

**Derivation of the Cohen class.** A constraint for the shift invariance property (i.e., compatibility with both time and frequency shifts) is best formulated using \( \mathbf{v}_T \). It reads

\[
\mathbf{v}_T(t_0,t,\omega,\omega') = \mathbf{v}_T(t-t_0,\omega,\omega') \forall t_0, \omega_0.
\]

Hence \( \mathbf{v}_T \) must be a convolution-type kernel for which we write

\[
\mathbf{v}_T(t_0,\omega,\omega') = \mathbf{v}_T(t-t_0,\omega,\omega') \forall t_0, \omega_0.
\]

and the normal form (13) is a convolution of Wigner distribution,

\[
T = \mathbf{v}_T \mathbf{W} = \mathbf{v}_T \mathbf{W}.
\]

But this defines the class of BSRs introduced by Cohen[1], and we have thus shown the Cohen class to be the class of shift invariant BSRs (comp. [2], pp. 30-34).

**INVERSION AND CONVERSION OF BSRs**

We shall call a BSR \( T \) reversible or regular if the family of signals \( f_t = C_f, g_t = (\gamma/c)^t g \) (C being an unknown complex constant) can be recovered from \( T_{f,g} \). We use this restricted notion of inversion since it is easily seen from the normal form (11) that \( f \) and \( g \) themselves can never be recovered from a BSR. - Recovery of \( \tilde{f}, \tilde{g} \), now, is equivalent to recovery of \( q \) from \( T=U_q \). A BSR \( T \) is therefore reversible if and only if its operator \( \mathbf{u}_T \) is reversible, i.e., there exists an inverse operator \( \mathbf{u}_T^\dagger \) such that

\[
q = \mathbf{u}_T^\dagger T, \quad q(t,\omega) = \int T(\mathbf{q}) \mathbf{u}_T^\dagger(t,\omega) d\omega dt.
\]

For a regular BSR, also the inverses \( \mathbf{v}_T, \mathbf{w}_T, \mathbf{v}_T^\dagger, \mathbf{w}_T^\dagger \) will exist, so that \( \Theta = \mathbf{w}_T^\dagger T, \quad \mathbf{w}_T^\dagger, \mathbf{v}_T, \mathbf{v}_T^\dagger \) are mutually related by one- and two-dimensional Fourier transformation.

The following theorem shows that regularity is a fundamental property of BSRs:
From a BSR $T_2$, any other BSR $T_3$ can be recovered,
$$T_3 = L_{t_2} T_1,$$
if and only if $T_2$ is regular.

Any bilinear time-frequency signal representation and any bilinear signal parameter can thus be recovered from a regular BSR by a linear transformation. The conversion operator $L_{t_2}$ is constructed using the inverse of e.g. $U_{t_2}$:
$$T_2 = u_{t_2} q = u_{t_2} (u_{t_2}^* T_1) = (u_{t_2} u_{t_2}^*) T_1$$
so that
$$L_{t_2} = u_{t_2} u_{t_2}^*$$

Other forms are $L_{t_2} = U_{t_2} U_{t_2}^* = V_{t_2} V_{t_2}^*$.

Example 3. The elementary BSRs $q, Q, W, A$ are all regular. Hence any BSR can be derived from them which is expressed by the normal forms (11)-(14). Finding the inverse operators of the elementary BSRs is rather trivial, merely involving Fourier transformations.

Example 4. Rihaczek distribution
$$R_{t,t}(t_2) = \frac{1}{2\pi} \tilde{F}(\omega) \tilde{g}(\omega)^{-1/2} \delta(t + t_2)$$
is a regular BSR whose inverse impulse response,
$$u_t^* (t_2; t_2) = \delta(t + t_2) e^{-j\omega t},$$
is easily found by inverse Fourier transformation. Let us see how the moment $m_{t_2}^{(t_2)}$ is derived from $R$. Inserting into (33) $u_{t_2}$ for $u_{t_2}^*$ and $u_{t_2}$ for $u_{t_2}$ (see (16)) we obtain the conversion kernel
$$L_{t_2} (\sigma, \tau) = (t - t_2)^n.$$
Hence we have (compare with (17))
$$m_{t_2}^{(t_2)} = \int R(t_2; -t - t_2) dt_2.$$

Regular BSRs are furthermore distinguished in that signal transformations are always equivalent to BSR transformations; i.e. we always have
$$T_{X_t, Y_t} = L_{X_t}^{(X_t)} T_{Y_t}$$
with some $L_{X_t}^{(X_t)}$ which follows from (29):
$$L_{X_t}^{(X_t)} = u_{\epsilon t} u_{t_2}^*.$$

Note that, in contrast to a compatibility property, $L_{X_t}^{(X_t)}$ is not necessarily a "desirable" transformation.

Example 5. For Wigner distribution $W$ and ambiguity function $A$, (35) yields
$$L_{X_t}^{(X_t)} (\omega, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(t, \tau; \omega) \delta^{(\omega - \omega')} dt \omega' \delta^{(\tau - \tau')} d\tau \tau'.$$

A BSR which is not regular could be called singular. A simple indirect proof shows that no regular BSR can be derived from a singular BSR. Singularity of a BSR $T$ also means that there exist two non-trivially different sets of signals $(t_1, \beta_1)$ and $(t_2, \beta_2)$ whose BSRs are identical,
$$T_{t_1, \beta_1} = T_{t_2, \beta_2}.$$
By "trivial difference" we mean
$$t_2 = C t_1, \beta_2 = (\chi/C) \beta_1, \chi \neq 1.$$

Example 6. Any spectrogram with finite-time-support window $q$ (i.e., any practical spectrogram) is singular. Proof: it can be shown that the spectrogram's double Fourier transform, denoted by $\hat{S}$, is given by
$$\hat{S}(t_2, \omega) = A(\omega t_2) A_2(\omega t_2).$$
If $|\omega| > 0$, $|t_2| > T$, then $A_2(\omega t_2) = 0$, $|t_2| > 2T$. $\hat{S}$ is thus a clipped version of ambiguity function which is easily shown to be singular. But if $\hat{S}$ is singular, then $S$ is singular, too.

CONCLUSION

The theory of bilinear signal representations (BSRs) presented in this paper characterizes BSRs by linear operators or, equivalently, kernel functions. It is based on the fact that BSRs are, if at all, connected by linear transformations, and that this linear relationship is the underlying principle of virtually all BSR properties. Thus the theory provides the tools for a systematic discussion of BSRs; e.g., for finding constraints to given BSR properties. A central concept in this theoretical framework is regularity: from a regular BSR, all other BSRs can be derived, and also the signals themselves can be recovered up to a constant factor. It was shown that, while Wigner distribution, Rihaczek distribution and ambiguity function are all regular, this is not the case for practical spectrograms.

References

