Signal restoration for linear systems with weighted inputs.
Singular value analysis for two cases of low-pass filtering

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Abstract. The a priori knowledge that the input of a linear system is weighted by some kind of profile function can be exploited for restoring the output signal by means of the singular value technique. We determine the analytic expression of the singular system for two typical cases in which the system behaves like a low-pass filter and the profile function has the same form as the impulse response of the filter. The analysis leads us to predict the performance of the restoration process in the presence of noise. It is shown that under suitable conditions the resolution of the restored signal can be twice that of the un-restored output.

1. Introduction

In a recent series of papers [1-4] the application of singular value analysis to optical problems has been presented. In particular, for images produced by optical systems acting on the object Fourier spectrum as low-pass filters the possibility of restoration has been studied in the case of uncertain localisation or non-uniform illumination of the object [4]. Although the problem was stated there in optical terms it can occur in different areas of signal processing. Therefore, we shall state the problem without reference to any particular physical device. A few examples will be given later. We shall refer to one-dimensional cases, although the basic concepts apply to two- and three-dimensional cases as well.

Let us consider a linear, shift-invariant system characterised by an impulse response \( H(x) \). When an input function \( f(x) \) is fed to the system, the output function, say \( g(x) \), is given by the convolution

\[
 g(x) = \int_{-\infty}^{\infty} H(x-y) f(y) \, dy \quad (-\infty < x < \infty). \tag{1.1}
\]

In many cases of physical interest the system behaves like a low-pass filter in the sense that its transfer function, namely the Fourier transform of \( H \), is significantly different from zero only for low frequencies. Cases described by equation (1.1) are treated by ordinary Fourier analysis.

Sometimes the input function is weighted by some kind of entrance window whose effect is represented by a profile function \( P(x) \). The output, say \( g(x) \), is then given by

\[
 g(x) = \int_{-\infty}^{\infty} H(x-y) P(y) f(y) \, dy \quad (-\infty < x < \infty). \tag{1.2}
\]
The exact meaning of \( P(x) \) varies from one case to another. For example, if \( f(x) \) is the transmission function of a coherently illuminated transparency, \( P(x) \) can be the field distribution produced across the transparency by the illuminating beam. In particular, for a lowest-order mode laser beam, \( P(x) \) would be a gaussian function, whereas for a diffraction-limited focused beam produced by a uniformly filled lens it would be a sinc function (see equation (1.8)). Alternatively, we could know that \( f(x) \) vanishes outside some finite domain. In that case \( P(x) \) is the characteristic function of the domain (equaling one inside the domain and zero outside). As another example, \( P(x) \) can represent a first approximation to the localisation of a finite support signal.

It will be noted that in all the cases mentioned above \( P(x) \) supposes a priori knowledge about the input. Nevertheless, from the mathematical point of view the profile function \( P(x) \) pertains to the system. The overall impulse response becomes \( H(x-y)P(y) \) and it is no longer shift invariant.

Several examples of the occurrence of equation (1.2) are familiar from the literature on signal processing. To quote one of the most celebrated, we recall the relationship

\[
g(x) = \int_{-\infty}^{\infty} \frac{\sin(2\pi \nu M (x-y))}{\pi (x-y)} f(y) \, dy \quad (-\infty < x < \infty),
\]

(1.3)

giving the output of a system whose transfer function is one for any frequency \( \nu \) such that \( \nu \leq \nu_M \) and is zero otherwise, when the input \( f(x) \) is known to vanish for \( |x| > a \). In other words, \( g(x) \) gives a band-limited version of the (band-unlimited) input \( f(x) \). The possibility of recovering \( f(x) \) by exploiting the prior knowledge of its finite support has been the subject of countless papers and it is still an active area of research. Basically, this is the problem of extrapolating a band-limited function.

Whatever the form of \( P(x) \), the problem is that of restoring the signal. One looks for a correction procedure to be applied to \( g(x) \) in order to produce a function whose distance (in some suitable sense) from \( f(x) \) is smaller than that between \( g(x) \) and \( f(x) \). The singular value analysis furnishes such a procedure. Actually, it applies under more general conditions than stated above (e.g. when \( H \) is not shift invariant or when the output is not available on the whole \( x \) axis) [4].

In this paper we shall evaluate analytically the singular systems (singular values and singular functions) for two cases of low-pass filtering systems. We shall first (§ 3) refer to a function \( H(x) \) of the form

\[
H(x) = \frac{\sin(2\pi \nu M x)}{\pi x},
\]

(1.4)

i.e. of the form occurring in equation (1.3). The filter described by equation (1.4) can be called a sharp low-pass filter because of the sudden transition of the transfer function at \( \nu = \pm \nu_M \). The same form will be assumed for the profile function. As mentioned before this is a case arising in diffraction-limited optics. It has already been studied in reference [4] where it was shown that the use of such a profile function can increase the resolution of a diffraction-limited microscope by a factor of two. The singular system was determined numerically. Our analysis leads to the exact structure of the restored signal which can be represented in the noiseless case through an equivalent shift-variant impulse response. The influence of the zeros of the profile function will be made clear and a simple rule will be found for evaluating that part of the input signal which can be recovered from the output.

The analytical expressions we shall find for the singular functions are particularly simple and suggest certain analogies between singular function expansions and sampling expansions. The number of degrees of freedom of the restored signal in the presence of noise will be estimated and its physical meaning will be discussed.
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The second case to be treated (§ 4) deals with a gaussian function $H$,

$$H(x) = \exp\left(-\frac{x^2}{2\gamma^2}\right)/\sqrt{2\pi\gamma},$$

when the profile function is also gaussian. The form (1.5) for $H$ gives a transfer function which decreases smoothly from one to very small values. The corresponding system will be termed 'smooth low-pass'. It will be shown that in the noiseless case a perfect restoration of the signal would be possible. The singular functions turn out to be the familiar Hermite-Gauss functions, whose well known properties give the restoration process a clear meaning. An estimate of the number of degrees of freedom for noisy cases will be obtained. Their difference in meaning from the previous case will be discussed.

It is thought that, to a certain extent, the two cases discussed here can be assumed to be representative of other systems, sharing with them either a sharp or a smooth drop of the transfer function as well as similar behaviour of the Fourier transform of $P(x)$. Apart from their immediate applicability to particular physical systems they are made attractive by the existence of analytically simple singular systems allowing easy investigations on several aspects of the restoration process.

A brief summary of the concepts employed in our analysis is given in § 2. It will also be shown there that the phase of $P(x)$ is immaterial as far as the evaluation of the singular system is concerned.

A few words about notation. The following functions will be used

$$\text{rect}(x) = \begin{cases} 1 & |x| \leq \frac{1}{2} \\ 0 & |x| > \frac{1}{2} \end{cases},$$

$$\Lambda(x) = \begin{cases} 1-|x| & |x| \leq 1 \\ 0 & |x| > 1 \end{cases},$$

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}.$$ (1.8)

The symbol $L^2_{\mathbb{C}}$ is an abbreviation for the space of functions square integrable in $(-\infty, \infty)$. The Fourier transform (FT) of any $(L^2_{\mathbb{C}})$ function $f(x)$ will be denoted either by $\mathcal{F}\{f\}(\nu)$ or by $\mathcal{F}\{f\}(\nu)$. The symmetrical form will be adopted:

$$\mathcal{F}\{f\}(\nu) = \mathcal{F}\{f\}(\nu) = \int_{-\infty}^{\infty} f(x) \exp(-2\pi i\nu x) \, dx \quad (-\infty < \nu < \infty).$$

(1.9)

The space of (band-limited) functions whose FT vanishes for $|\nu| > \sigma$ will be denoted by $B_{\sigma}$. The symbol $B^\perp_{\sigma}$ stands for the orthogonal complement of $B_{\sigma}$ (i.e. for the space of functions whose FT vanishes in $[-\sigma, \sigma]$). For an operator $A$, the symbol $\mathcal{Z}_A$ denotes the null space (the space of functions $f$ such that $Af = 0$) and $\mathcal{Z}_A^\perp$ its orthogonal complement.

2. The singular value analysis

In this section we recall briefly the main concepts involved in the singular value analysis of the input–output relation. We shall limit ourselves to the case of the systems discussed in § 1. For a detailed exposition of this subject in the general case the reader is referred to the papers [1–6].

Let us define an operator $A$ from $L^2_{\mathbb{C}}$ to $L^2_{\mathbb{C}}$ as

$$\langle Af\rangle(x) = \int_{-\infty}^{\infty} H(x-y)P(y)f(y) \, dy \quad (-\infty < x < \infty).$$

(2.1)
where \( f \in L^1_{\infty} \). We shall assume that the profile function satisfies the condition
\[
|P(y)| \leq 1 \quad (-\infty < y < \infty),
\]
and that both \( H \) and \( P \) have finite \( L^2 \) norm:
\[
\|H\| = \left( \int_{-\infty}^{\infty} |H(t)|^2 \, dt \right)^{1/2} < +\infty \quad (2.3)
\]
\[
\|P\| = \left( \int_{-\infty}^{\infty} |P(t)|^2 \, dt \right)^{1/2} < +\infty. \quad (2.4)
\]
In this case, it is easily seen that the operator \( A \) is of the Hilbert–Schmidt class. In fact, we have
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |H(x-y)P(y)|^2 \, dx \, dy = (\|H\| \|P\|)^2 < +\infty. \quad (2.5)
\]
Then equation (2.1) defines a compact operator in \( L^2_{\infty} \). The adjoint operator, say \( A^* \), is defined by the relation
\[
(A^* g)(y) = P^*(y) \int_{-\infty}^{\infty} H^*(x-y)g(x) \, dx \quad (-\infty < y < \infty) \quad (2.6)
\]
where \( g \in L^2_{\infty} \) and the asterisk indicates complex conjugate. Let us now consider the operators \( A^* A \) and \( AA^* \), whose action can be described by the relations
\[
(A^* A f)(y) = \int_{-\infty}^{\infty} S(y, y')f(y') \, dy' \quad (-\infty < y < \infty) \quad (2.7)
\]
\[
(AA^* g)(x) = \int_{-\infty}^{\infty} U(x, x')g(x') \, dx' \quad (-\infty < x < \infty) \quad (2.8)
\]
where
\[
S(y, y') = P^*(y)P(y') \int_{-\infty}^{\infty} H^*(x-y)H(x-y') \, dx \quad (-\infty < y, y' < \infty) \quad (2.9)
\]
\[
U(x, x') = \int_{-\infty}^{\infty} H(x-y)P(y)^2H^*(x'-y) \, dy \quad (-\infty < x, x' < \infty). \quad (2.10)
\]
Both \( A^* A \) and \( AA^* \) are non-negative, compact operators. As a consequence, each of them has a discrete set of eigenfunctions corresponding to a common set of non-negative eigenvalues. Omitting the possible eigenfunctions corresponding to the null eigenvalue (i.e., belonging either to \( Z_A \) or to \( Z_{A^*} \)) we denote the eigenfunctions of \( A^* A \) and \( AA^* \) by \( u_k(x) \) and \( v_k(x) \), respectively, and the corresponding eigenvalues arranged in decreasing order by \( \alpha_k^2 \) (\( \alpha_k \) real positive):
\[
A^* A u_k = \alpha_k^2 u_k \quad (k = 0, 1, \ldots). \quad (2.11)
\]
The functions \( u_k \) and \( v_k \) are the singular functions of the operator \( A \). The (positive) numbers \( \alpha_k \) are the singular values. In addition to equation (2.11) the so called singular
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system \{u_k, v_k, \alpha_k\}_{k=0}^\infty satisfies the set of coupled equations

\begin{align*}
A u_k &= \alpha_k v_k \\
A^\dagger v_k &= \alpha_k u_k
\end{align*}

(k = 0, 1, ...).

(2.12)

The singular functions \(u_k(v_k)\) form an orthonormal basis in \(L^2(Z_A)\).

A simple remark can be made about the role played by the phase distribution of the profile function \(P(y)\). To this end, we change \(P(y)\) by an arbitrary phase factor:

\[ \tilde{P}(y) = \exp(i\psi(y))P(y) \]

where \(\psi(y)\) is real. It is immediately seen that the kernels \(S\) and \(U\) in equations (2.9) and (2.10) are replaced by

\begin{align*}
\tilde{S}(y, y') &= \exp[-i(\psi(y) - \psi(y'))]S(y, y') \\
\tilde{U}(x, x') &= U(x, x').
\end{align*}

(2.13)

Accordingly, the new singular system is given by

\begin{align*}
\tilde{u}_k(x) &= \exp(-i\psi(x))u_k(x) \\
\tilde{v}_k(x) &= v_k(x) \\
\tilde{\alpha}_k &= \alpha_k
\end{align*}

(2.14)

(2.15)

(2.16)

This means that it is not substantially restrictive to think that \(P(y)\) is purely real or even non-negative if needed.

The input–output relation of a linear system described by the operator \(A\)

\[ o(x) = (A0)(x) \]

where \(i(x)\) and \(o(x)\) are the input and the output, respectively, can be analysed by means of the singular system of \(A\) along the following lines. Suppose \(i(x)\) is expanded in a series of \(u_k(x)\)

\[ i(x) = \sum_{k=0}^{\infty} i_k u_k(x) + r(x) \]

(2.18)

where

\[ i_k = \int_{-\infty}^{\infty} i(x)u_k^*(x) \, dx \quad (k = 0, 1, ...) \]

(2.19)

and where \(r(x)\) is the (possibly vanishing) projection of \(i(x)\) onto the null space of \(A\). On inserting equation (2.18) into (2.17) one obtains

\[ o(x) = \sum_{k=0}^{\infty} \alpha_k i_k \tilde{v}_k(x) \]

(2.20)

where equation (2.12) has been taken into account. As the system of the \(v_k\) is complete with respect to the output functions the expansion can be made

\[ o(x) = \sum_{k=0}^{\infty} o_k v_k(x) \]

(2.21)

where

\[ o_k = \int_{-\infty}^{\infty} o(x)\tilde{v}_k^*(x) \, dx \quad (k = 0, 1, ...) \]

(2.22)
A comparison between equations (2.20) and (2.21) shows that the unknown coefficients of the projection of \( f(x) \) onto \( Z_1 \), namely the coefficients \( i_k \), could be recovered as follows:
\[
i_k = \alpha_k / \alpha_k \quad (k = 0, 1, \ldots).
\]

(2.23)

According to a general feature of compact operators, \( \alpha_k \to 0 \) when \( k \to +\infty \). This prevents the use of equation (2.23) for arbitrarily large values of \( k \) because of noise and round-off errors. This difficulty is due to the ill-posed nature of the problem of inverting equation (2.17). A possible way out is to use equation (2.23) up to a maximum value of \( k \), say \( M \), which can be thought of as the 'number of degrees of freedom' of the output. The value of \( M \) depends, of course, on the signal-to-noise ratio. More explicitly, \( M \) is the maximum value of \( k \) such that
\[
\alpha_M \geq \sigma / \epsilon
\]

(2.24)

where \( \sigma^2 \) and \( \epsilon^2 \) are the power spectra of the noise and the signal, respectively (assumed to be white noise processes).

3. Sharp low-pass systems

3.1. General remarks

In this section we deal with an operator of the form (2.1) when the function \( H \) has the form (1.4). Written down explicitly, the operator \( A \) is
\[
(Af)(x) = 2\nu_M \int_{-\infty}^{\infty} \text{sinc}[2\nu_M(x-y)]P(y)f(y) \, dy \quad (-\infty < x < \infty).
\]

(3.1)

Provided that \( P(y) \) satisfies both conditions (2.2) and (2.4) the analysis of the previous section can be applied. In fact the sinc function satisfies condition (2.3).

This kind of operator has already been studied in reference [4] where several properties of the pertaining singular system have been established. We shall quote some of these properties in the following. Before specialising the form of \( P(y) \) we add some considerations about \( Z_A \), the null space of \( A \).

If a function \( q(x) \) has to belong to \( Z_A \), i.e., if
\[
(Aq)(x) = 0 \quad (-\infty < x < \infty)
\]

(3.2)

then the \( \text{FT} \) of \( Aq \), namely
\[
\mathcal{F}(Aq)(\nu) = \text{rect} \left( \frac{\nu}{2\nu_M} \right) \int_{-\infty}^{\infty} \tilde{P}(\xi)\tilde{q}(\nu - \xi) \, d\xi
\]

(3.3)

must vanish too:
\[
\mathcal{F}(Aq)(\nu) = 0 \quad (-\infty < \nu < \infty).
\]

(3.4)

Suppose \( \tilde{P} \) has a bounded support. Because \( P(y) \) can be thought of as real, so that \( \overline{\tilde{P}(-\nu)} = \tilde{P}^*(\nu) \), the support of \( \tilde{P} \) can be assumed to be symmetrical with respect to the origin. Let us limit ourselves to the case in which the support of \( \tilde{P} \) is a single interval \([-\nu_1, \nu_1]\) and let
\[
\nu = \nu_M + \nu_1.
\]

(3.5)

As noticed in reference [4], equation (3.4) is certainly satisfied by any function \( q \in B_{\nu}^\nu \). It remains to be seen whether a function belonging to \( B_{\nu} \) can satisfy equation (3.4). We shall
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now prove that it can whenever the function $P(y)$ has any zeros. To this end, let us consider the following function of $B_{\beta}$:

$$q(x) = 2\nu \text{sinc}[2\nu(x - x_0)],$$

(3.6)

whose FT is

$$\tilde{q}(\nu) = \exp(-2\pi i\nu x_0) \text{rect}(\nu/2\nu).$$

(3.7)

On inserting from equation (3.7) into equation (3.3) we obtain

$$\mathcal{F}\{Aq\}(\nu) = \text{rect}\left(\frac{\nu}{2\nu_M}\right) \exp(-2\pi i\nu x_0) \int_{-\nu_M}^{\nu+\nu_M} \tilde{P}(\xi) \exp(2\pi i\xi x_0) \, d\xi.$$  

(3.8)

For any $\nu$ within $[-\nu_M, \nu_M]$ the integration domain in equation (3.8) contains the support of $\tilde{P}$; therefore equation (3.8) can be written as

$$\mathcal{F}\{Aq\}(\nu) = \text{rect}(\nu/2\nu_M) \exp(-2\pi i\nu x_0) P(x_0).$$

(3.9)

Therefore, the function $q(x)$ defined by equation (3.6) belongs to both $B_{\beta}$ and $Z_A$ if $P(x_0) = 0$. In such a case the singular functions do not form a basis in $B_{\beta}$. This does not prove, of course, that the opposite is true when $P(y)$ has no real zeros.

As we shall see in a moment, much more can be said when $P(y)$ is a sine function.

3.2. Sine profile

Let us assume that $P(y)$ has the form of a sinc function. We shall limit ourselves to the practically important case $\nu_1 = \nu_M$, i.e.

$$P(y) = \text{sinc}(2\nu_M y).$$

(3.10)

The function (3.10) satisfies both conditions (2.2) and (2.4). Note that equation (3.5) becomes

$$\tilde{P}(\xi) = 2\nu_M.$$

(3.11)

We want to find $Z_A$. According to the previous discussion it is sufficient to find the intersection between $B_{\beta}$ and $Z_A$. On inserting the FT of equation (3.10) into equation (3.3) we obtain

$$\mathcal{F}\{Aq\}(\nu) = \text{rect}\left(\frac{\nu}{2\nu_M}\right) \int_{-\nu_M}^{\nu+\nu_M} \tilde{q}(\xi) \, d\xi,$$

(3.12)

so that $q$ belongs to $Z_A$ if

$$\int_{-\nu_M}^{\nu+\nu_M} \tilde{q}(\xi) \, d\xi = 0 \quad |\nu| \leq \nu_M.$$  

(3.13)

This is possible if and only if $\tilde{q}(\nu)$ behaves like a periodic function with period $2\nu_M$ and mean value zero within the whole interval $[-\nu, \nu]$, i.e. $[-2\nu_M, 2\nu_M]$ (see equation (3.11)).

In conclusion, $Z_A$ contains $B_{\beta}$ as well as those functions of $B_{\beta}$ whose FT are periodic functions (with period $2\nu_M$ and vanishing mean value) truncated to $[-\nu, \nu]$.

Given any function $f(x) \in L_2$, we can uniquely decompose it into two parts

$$f(x) = f^{(0)}(x) + f^{(\nu)}(x)$$

(3.14)
where \( f^{(0)} \in \mathbb{Z}_A \) and \( f^{(0)} \in \mathbb{Z}_A^d \). In physical terms, this is to say that a typical input function splits into a first component, \( f^{(0)}(x) \), giving no output and a second component, \( f^{(t)}(x) \), which is transmitted to the output (hence the superscript 't') although in a modified form. We shall show how such a decomposition can be made. As \( B^t \) is entirely contained in \( \mathbb{Z}_A \) we can limit ourselves to a typical function \( f \in B_p \). Within the interval \([-2\nu_M, 2\nu_M]\) we expand \( f^{(t)}(\nu) \) into a Fourier series of the form

\[
\tilde{f}^{(t)}(\nu) = \text{rect} \left( \frac{\nu}{4\nu_M} \right) \sum_{n=0}^{\infty} \varphi_n \exp \left( -2\pi i n \frac{\nu}{4\nu_M} \right) \tag{3.15}
\]

where

\[
\varphi_n = \frac{1}{4\nu_M} \int_{-2\nu_M}^{2\nu_M} f^{(t)}(\nu) \exp \left( 2\pi i n \frac{\nu}{4\nu_M} \right) d\nu \equiv \frac{1}{4\nu_M} \int \left( \frac{n}{4\nu_M} \right) \tag{3.16}
\]

In equation (3.15) each exponential term with even \( n \neq 0 \) is a periodic function with period \( 2\nu_M \) and mean value zero. As such, it belongs to \( \mathbb{Z}_A \). Accordingly, we can write

\[
\mathcal{F}\{f^{(0)}(\nu)\} = \text{rect} \left( \frac{\nu}{4\nu_M} \right) \sum_{n=-\infty}^{\infty} \varphi_{2k} \exp \left( -2\pi i k \frac{\nu}{2\nu_M} \right) \tag{3.17}
\]

where the prime on the sum means that the term \( k=0 \) is excluded. The remaining terms of the series (3.15) give the FT of \( f^{(t)}(x) \)

\[
\mathcal{F}\{f^{(t)}(\nu)\} = \text{rect} \left( \frac{\nu}{4\nu_M} \right) \left[ \varphi_0 + \sum_{k=-\infty}^{\infty} \varphi_{2k+1} \exp \left( -2\pi i \left( \frac{(2k+1)\nu}{2\nu_M} \right) \right) \right] \tag{3.18}
\]

Equations (3.17) and (3.18) can be immediately inverted to give

\[
f^{(0)}(x) = \sum_{k=-\infty}^{\infty} f \left( \frac{k}{2\nu_M} \right) \text{sinc} \left[ 4\nu_M \left( x - \frac{k}{2\nu_M} \right) \right] \tag{3.19}
\]

\[
f^{(t)}(x) = f^{(0)}(x) \text{sinc}(4\nu_M x) + \sum_{k=-\infty}^{\infty} f \left( \frac{2k+1}{4\nu_M} \right) \text{sinc} \left[ 4\nu_M \left( x - \frac{2k+1}{4\nu_M} \right) \right] \tag{3.20}
\]

where use has been made of equation (3.16). It will be noticed that \( f^{(0)}(x) \) is given by a series of functions of the form (3.6). Both equations (3.19) and (3.20) look like sampling expansions. As a matter of fact, when added to one another they furnish a sampling expansion of the band-limited function \( f(x) \) (with sampling rate \( 1/4\nu_M \)). Thus, we have a very simple rule for evaluating \( f^{(0)}(x) \). Take the sampling expansion of \( f(x) \) and delete all the samples centred at even, non-zero multiples of \( 1/4\nu_M \).

An alternative way to find \( f^{(0)}(x) \) without the use of series is obtained by noting that equation (3.18) can also be written as

\[
\mathcal{F}\{f^{(0)}(\nu)\} = \text{rect}(\nu/4\nu_M) \left[ \varphi_0 + \frac{1}{2} \left[ \tilde{f}(\nu-2\nu_M) + \tilde{f}(\nu) - \tilde{f}(\nu+2\nu_M) \right) \right] \tag{3.21}
\]

It is easily seen in fact that the combination within the square brackets destroys the even-numbered Fourier components (including the zeroth term) and doubles the odd ones over the whole interval \([-2\nu_M, 2\nu_M]\).

3.3. The structure of the recovered signal in the noiseless case

As we saw in § 2, in the ideal, noiseless case the singular value technique leads to the recovery of the projection of the input signal \( f(x) \) onto \( \mathbb{Z}_A^d \). Such a projection can be
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denoted by $f^0(x)$ according to the notation introduced in § 3.2 and we have (see equation (2.18))

$$
 f^0(x) = \sum_{k=0}^{\infty} t_k u_k(x)
$$

(3.22)

where the coefficients $t_k$ are given by equation (2.19). We know from equations (3.18)–(3.21) how to evaluate $f^0(x)$. However, we can gain a further insight into the performance of the recovery procedure by a different approach. Using equations (3.22) and (2.19) the ideal recovered signal can be written as

$$
 f^0(x) = \int_{-\infty}^{\infty} W(x, y) \delta(y) \, dy
$$

(3.23)

where

$$
 W(x, y) = \sum_{k=0}^{\infty} t_k u_k(x) u_k^*(y).
$$

(3.24)

Equation (3.23) shows that the recovered signal is obtained by a superposition integral where $W(x, y)$ plays the role of an overall impulse response (corresponding to the joint effect of the operator $A$ and the recovery procedure). Then the whole process is characterised by $W(x, y)$, regardless of the particular form of $\delta(x)$ we choose.

We want to show now that $W(x, y)$ can be given a closed form. More precisely, we maintain that

$$
 W(x, y) = w^0(x, y)
$$

(3.25)

where

$$
 w(x, y) = 2\pi \frac{\text{sinc}[2\pi(x-y)]}{2\pi(x-y)}
$$

(3.26)

In equation (3.25), we mean by $w^0(x, y)$ the projection onto $Z^*_1$ of $w(x, y)$ when considered as a function of $x$ for fixed $y$. To prove the assertion, we note that $w^0(x, y)$ can be expanded into a series of $u_k$ functions,

$$
 w^0(x, y) = \sum_{k=0}^{\infty} \tau_k(y) u_k(x)
$$

(3.27)

where the expansion coefficients $\tau_k$ (depending on the value of the parameter $y$) are given by

$$
 \tau_k(y) = \int_{-\infty}^{\infty} w(x, y) u_k^*(x) \, dx \quad (k = 0, 1, \ldots).
$$

(3.28)

On inserting from equation (3.26) into equation (3.28) we obtain

$$
 \tau_k(y) = 2\pi \int_{-\infty}^{\infty} \text{sinc}[2\pi(x-y)] u_k^*(x) \, dx \quad (k = 0, 1, \ldots).
$$

(3.29)

The projection operator occurring in equation (3.29) leaves $u_k^*$ unchanged because the functions $u_k$ belong to $B_\infty$. Therefore

$$
 \tau_k(y) = u_k^*(y) \quad (k = 0, 1, \ldots).
$$

(3.30)

Replacing $\tau_k$ by $u_k^*$ in equation (3.27) we see that equations (3.24) and (3.27) are identical. Thus equation (3.25) is proved.
It remains to give $w^{(0)}(x, y)$ a closed form. This can be done with the aid of equation (3.21). On inserting into it the $\mathcal{F}$ (for fixed $y$) of equation (3.26) we obtain

$$\mathcal{F}[w^{(0)}(\nu) = \text{rect} \left( \frac{\nu}{4\nu_M} \right) \left( \text{sinc}(4\nu_M y) - \frac{1}{2} \text{rect} \left( \frac{\nu - \nu_M}{2\nu_M} \right) \exp[-2\pi i(\nu - 2\nu_M) y] \right)$$

$$+ \frac{1}{2} \text{rect} \left( \frac{\nu}{4\nu_M} \right) \exp(-2\pi i y) - \frac{1}{2} \text{rect} \left( \frac{\nu + \nu_M}{2\nu_M} \right) \exp[-2\pi i(\nu + 2\nu_M) y] \right),$$

(3.31)

Figure 1. The ideal overall impulse response $W(x, y)$ of the sharp low-pass system after restoration for some values of $y$: (a) $y=0$; (b) $y=0.1$; (c) $y=0.4$; (d) $y=0.6$; (e) $y=0.9$; (f) $y=1.1$. 
Signal restoration: singular value analysis

The inverse FT of equation (3.31) is easily evaluated and gives the result (see equation (3.25))

\[ W(x, y) = 4\nu_M (\text{sinc}(4\nu_M x) \text{sinc}(4\nu_M y) + \text{sinc}(2\nu_M(x-y)) \sin(2\pi\nu_M x) \sin(2\pi\nu_M y)) \]  

(3.32)

Equation (3.32) gives the impulse response of the whole process in the noiseless case. Such an impulse response is not shift invariant. In particular it is seen that

\[ W(x, 0) = 4\nu_M \text{sinc}(4\nu_M x) \]  

(3.33)

\[ W(x, 2n/4\nu_M) = 0 \]  

\(n = \pm 1, \pm 2, \ldots\)  

(3.34)

\[ W(x, (2n+1)/4\nu_M) = 4\nu_M \text{sinc}(4\nu_M x - 2n - 1) \]  

\(n = \pm 1, \pm 2, \ldots\)  

(3.35)

It is further seen that the 'central value' of the impulse response, i.e. \(W(x, x)\), has the form

\[ W(x, x) = 4\nu_M (\text{sinc}^2(4\nu_M x) + \sin^2(2\pi\nu_M x)). \]  

(3.36)

A few curves showing the structure of \(W(x, y)\) for fixed \(y\) are given in figure 1 where a value \(\nu_M = 0.5\) is assumed. The present impulse response should be compared with that of the original sharp low-pass filtering system when no profile function is used, i.e. to the shift-invariant impulse response \(2\nu_M \text{sinc}(2\nu_M(x-y))\). Such a comparison shows that the use of the profile (3.10) joined to the recovery procedure gives rise to an increase of resolution by a factor of two. This conclusion was also outlined in reference [4] although the impulse response was evaluated for \(y = 0\) only (see equation (3.33)). Equation (3.34) reveals that the signal values at \(y = k/2\nu_M\) \((k = \pm 1, \pm 2, \ldots)\) cannot be recovered. This can be physically ascribed to the fact that the profile function cancels the signal at those points. However, this does not impair the performance of practical devices using the above scheme, such as the scanning microscope [7], where the signal is sequentially moved across the profile.

3.4. Analytic evaluation of the singular system

In order to apply the singular value analysis to the actual recovery procedure the singular system must, of course, be evaluated. This can be done by numerical methods as was done in reference [4] for the case of the profile (3.10). For this case the singular system can also be evaluated analytically, as we shall see presently.

Starting from equations (2.7)–(2.12) it can be shown [4] that for sharp low-pass systems the RFS of the singular functions satisfy the integral equation

\[ \int_{-\nu_M}^{\nu_M} \tilde{v}_k(\xi) \tilde{Q}(\nu - \xi) \, d\xi = \alpha_k^2 \tilde{v}_k(\nu) \quad (|\nu| \leq \nu_M; k = 0, 1, \ldots) \]  

(3.37)

where \(\tilde{Q}(\nu) = \mathcal{F}\{|P|\} (\nu)\). For the profile (3.10) equation (3.37) then becomes

\[ \frac{1}{2\nu_M} \int_{-\nu_M}^{\nu_M} \tilde{v}_k(\xi) \Lambda \left( \frac{\nu - \xi}{2\nu_M} \right) \, d\xi = \alpha_k^2 \tilde{v}_k(\nu) \quad (|\nu| \leq \nu_M; k = 0, 1, \ldots). \]  

(3.38)

Taking the second derivative of both sides of equation (3.38), we easily obtain

\[ \frac{1}{2\nu_M^2} \tilde{v}_k(\nu) = \alpha_k^2 \frac{d^2}{d\nu^2} (\tilde{v}_k(\nu)) \quad (|\nu| \leq \nu_M; k = 0, 1, \ldots). \]  

(3.39)

According to equation (3.39) each function \(\tilde{v}_k\) must be a harmonic function, i.e. it must have the form

\[ \tilde{v}_k(\nu) = N_k \cos(\beta_k \nu + \theta_k) \]  

(3.40)
for constant $N, \beta_k$ and $\theta_k$. Direct substitution of equation (3.40) into equation (3.38) shows that the allowed values of $\theta_k$ can be reduced to 0, $\frac{1}{4}\pi$. In other words, the eigenfunctions of equation (3.38) are either even or odd. The possibility of degenerate eigenvalues is then excluded [8]. This will be apparent in the following. The allowed values of $\beta_k$ are obtained from equation (3.38) by inserting into it either a sinusoidal or a cosinusoidal function. It is then found that the values of $\beta_k$ corresponding to even eigenfunctions are the solutions of the transcendental equation

$$\tan(\beta_{2n} \nu_M) = 1/\beta_{2n} \nu_M \quad (n = 0, 1, \ldots)$$

(3.41)

whereas for the odd eigenfunctions we find

$$\beta_{2n+1} = (2n + 1)\pi/2\nu_M \quad (n = 0, 1, \ldots).$$

(3.42)

As shown by equation (3.42) the odd-numbered values of $\beta_k$ are equally spaced. This is not the case for the even-numbered ones. However, a simple inspection of the transcendental equation (3.41) shows that the roots $\beta_{2n} \nu_M$ tend to

$$\beta_{2n} \nu_M \longrightarrow n\pi$$

(3.43)

so that the $\beta_{2n}$ become approximately equally spaced for large values of $n$. The first few roots of equation (3.41) are given in table 1. It is seen that from the fourth root onwards the actual values agree with the asymptotic expression (3.43) to better than one per cent.

The singular values $\alpha_k$ are related to the $\beta_k$ by (see, e.g., equation (3.39))

$$\alpha_k = 1/\beta_k \nu_M \sqrt{2} \quad (k = 0, 1, \ldots)$$

(3.44)

so that for odd $k$ equations (3.42) and (3.44) give

$$\alpha_{2n+1} = \sqrt{2}/\pi(2n + 1) \quad (n = 0, 1, \ldots).$$

(3.45)

For even $k$ the numerical values of the roots of equation (3.41) must be inserted into equation (3.44). Taking into account the asymptotic expression (3.43) we see that

$$\alpha_{2n} \longrightarrow \sqrt{2}/\pi(2n).$$

(3.46)

We can now write the normalised form of the eigenfunctions of equation (3.38). For odd indices the normalising factor is independent of the index and we have

$$\tilde{v}_{2n+1}(\nu) = \frac{1}{\sqrt{\nu_M}} \sin \left( (2n + 1) \frac{\pi \nu}{2\nu_M} \right) \text{rect} \left( \frac{\nu}{2\nu_M} \right) \quad (n = 0, 1, \ldots).$$

(3.47)

*Table 1. Roots of the equation $\tan(\beta_{2n} \nu_M) = 1/\beta_{2n} \nu_M$.*

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\beta_{2n} \nu_M$</th>
</tr>
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<td>0</td>
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<tr>
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<td>2</td>
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<td>18.902</td>
</tr>
<tr>
<td>7</td>
<td>22.036</td>
</tr>
</tbody>
</table>
Signal restoration: singular value analysis

For even indices we write

\[ \tilde{v}_{2n}(\nu) = N_{2n} \cos(\beta_{2n} \nu) \text{rect}(\nu/2\nu_M) \quad (n = 0, 1, \ldots) \]  

(3.48)

where \( N_{2n} \) turns out to be

\[ N_{2n} = \left( \frac{1 + (\beta_{2n} \nu_M)^2}{\nu_M [2 + (\beta_{2n} \nu_M)^2]} \right)^{1/2} \quad (n = 0, 1, \ldots). \]  

(3.49)

In order to find the singular functions \( v_k(x) \) we have simply to perform an inverse FT on equations (3.47) and (3.48). This gives

\[ v_{2n+1}(x) = \sqrt{\nu_M} \left( \text{sinc} \left[ 2\nu_M \left( x - \frac{2n + 1}{4\nu_M} \right) \right] 
- \text{sinc} \left[ 2\nu_M \left( x + \frac{2n + 1}{4\nu_M} \right) \right] \right) \quad (n = 0, 1, \ldots) \]  

(3.50)

\[ v_{2n}(x) = \nu_M N_{2n} \left( \text{sinc} \left[ 2\nu_M \left( x - \frac{\beta_{2n}}{2\pi} \right) \right] 
+ \text{sinc} \left[ 2\nu_M \left( x + \frac{\beta_{2n}}{2\pi} \right) \right] \right) \quad (n = 0, 1, \ldots). \]  

(3.51)

In equation (3.50) we omitted an inessential factor \(-i\). The other set of characteristic functions \( \{u_k\} \) is obtained by application of \( A^\dagger \) to \( \{v_k\} \) (see equation (2.12)). Taking the band-limited nature of the \( \{v_k\} \) into account, it is easily seen [4] that \( A^\dagger u_k(x) = P(x) v_k(x) \). Therefore, we have

\[ u_k(x) = (1/\alpha_k) \text{sinc}(2\nu_M x) v_k(x) \quad (k = 0, 1, \ldots) \]  

(3.52)

where the \( \{v_k\} \) are given by equations (3.50) and (3.51). We add that by simple trigonometric identities it can be shown that the odd-numbered functions \( u_k \) have the form

\[ u_{2n+1}(x) = \sqrt{2\nu_M} \left( \text{sinc} \left[ 4\nu_M \left( x - \frac{2n + 1}{4\nu_M} \right) \right] 
- \text{sinc} \left[ 4\nu_M \left( x + \frac{2n + 1}{4\nu_M} \right) \right] \right) \quad (n = 0, 1, \ldots). \]  

(3.53)

Therefore, disregarding a factor \( \sqrt{2} \) (consistent with the normalisation condition), \( u_{2n+1} \) differs from \( v_{2n+1} \) only in the width of the sinc functions.

As far as the even functions are concerned we observe that equations (3.43), (3.46), (3.49), (3.51) and (3.52) lead to the asymptotic expression

\[ u_{2n}(x) \xrightarrow{n \to \infty} 2n\sqrt{2\nu_M} \frac{\sin^2(2\pi \nu_M x)}{\pi([2\nu_M x]^2 - \pi^2)}. \]  

(3.54)
This kind of function is mostly concentrated near \( x = \pm n/2vM \). Furthermore, it is non-positive inside the interval \([-n/2vM, n/2vM]\) and non-negative outside.

We finally note that equation (3.37) could also be solved along the present lines in the case \( P(x) = \text{sinc}(2\nu_1 x) \) with \( \nu_1 > \nu_M \).

### 3.5. Degrees of freedom

In practical applications the recovery procedure involves a finite number of singular functions (§ 2). This number, i.e. the number of degrees of freedom, is determined by equation (2.24). In the present case the singular values are independent of \( \nu_M \) as shown by equations (3.41), (3.42) and (3.44). This may be surprising at first because in similar problems the \( \{\alpha_k\} \) do depend on \( \nu_M \). However, the reason for such independence is easily seen. In fact, due to the condition \( \nu_1 = \nu_M \) an increase of \( \nu_M \) entails a decrease of the width of \( P(x) \) so that the product of the bandwidth \( (2\nu_1) \) and the width of \( P(x) \) does not change with \( \nu_M \). An alternative way of expressing the same result is to note that the sum of the squares of the singular values is given by

\[
\sum_{k=0}^{\infty} \alpha_k^2 = \text{Tr}(A^\dagger A) = \text{Tr}(AA^\dagger) = 1,
\]

regardless of \( \nu_M \). Equation (3.55) is evaluated, for example, by tracing the kernel of equation (3.38).

An estimate of the number \( M \) of degrees of freedom is obtained through equation (2.24), taking into account the law of decrease of the singular values expressed by equations (3.45) and (3.46). It is then found that

\[
M = \frac{\sqrt{2} E}{\pi \hat{e}}.
\]

A comment about the meaning of \( M \) is in order. For problems where the signal support is finite any increase of the number of degrees of freedom gives rise to a corresponding increase of resolution in the recovered signal. Presently, however, even in the case \( M \rightarrow \infty \) the resolution does not exceed a limiting value (twice that of the low-pass filter), as we know from the results of § 3.3. That was remarked in reference [4] where it was also shown by numerical evaluation that the impulse response centred at the origin \( W(x,0) \) (see equation (3.33)) is well approximated by summing a relatively small number of \( \alpha_k \). What advantage is then to be expected from any increase in \( M \)? An inspection of the expression for the singular functions (equations (3.50)-(3.53)) suggests that by progressively increasing \( M \) we shall obtain a good approximation of \( W(x,y) \) over a progressively increasing range of values of \( x \) and \( y \). In other words, the larger \( M \) the wider the interval of the \( x \) axis, say \( \Delta x \), over which the signal is recovered. This feature together with the presence of sinc functions in the expression for the singular functions reveals an analogy between the truncated series of \( \{\alpha_k\} \) and a truncated sampling expansion [9]. We can take as an estimate of \( \Delta x \) the distance between the centres of the two outer lobes characterising the \( \{\alpha_k\} \). It then follows from equations (3.53) and (3.54) that \( \Delta x = M/2\nu_M \). Let us recall that, according to the rule found at the end of § 3.2, the transmittable part of a signal is determined by the set of its samples at \( 0, \pm 1/4\nu_M, \pm 3/4\nu_M, \ldots \). Except for the sample at the origin, the other samples are spaced \( 1/2\nu_M \) from one another. We see then that \( M \) is (approximately) equal to the number of such samples falling within \( \Delta x \). This is consistent with the analogy to truncated sampling expansions noted above.
4. Smooth low-pass systems

4.1. General remarks

We assume here that the function $H(x)$ is given by equation (1.5), i.e. that the operator $A$ is of the form

$$A(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-y)^2}{2\gamma^2}\right] P(y) f(y) \, dy \quad (-\infty < x < \infty) \quad (4.1)$$

and the function $P(y)$ is subjected to conditions (2.2) and (2.4). As far as the null space $Z_A$ is concerned, if a function $q(x)$ has to satisfy the equation

$$(AQ)(x) = 0 \quad (-\infty < x < \infty) \quad (4.2)$$

or in terms of FT

$$\exp(-2\pi^2\nu^2y^2) \int_{-\infty}^{\infty} P(\xi) \overline{q}(\nu - \xi) \, d\xi = 0 \quad (-\infty < \nu < \infty) \quad (4.3)$$

then the convolution between $P$ and $\overline{q}$ must vanish everywhere or, equivalently,

$$P(y)q(y) = 0 \quad (-\infty < y < \infty). \quad (4.4)$$

If $P(y)$ is analytic so that it cannot vanish identically in any finite interval, equation (4.4) cannot be satisfied by any function $q \not\equiv 0$, unless we extend the class of functions considered so far, for example by including distributions. In particular, this holds true whenever $P(\nu)$ has finite support. A familiar example is $P(x) = \text{sinc}(2\nu_1 x)$. In that case equation (4.3) could be satisfied by a strictly periodic function $\overline{q}(\nu)$ with period $2\nu_1$ and mean value zero and this would give a series of $\delta$ functions for $q(x)$. The only possibility of satisfying equation (4.4) is then when both $P(y)$ and $q(y)$ have finite supports, the two supports being disjoint.

We note again that the kernel $S(y,y')$ of equation (2.9) has the simple form

$$S(y,y') = \frac{1}{2\sqrt{\pi}} \frac{P^*(y)P(y') \exp\left[-\frac{(y-y')^2}{4\gamma^2}\right]}{\gamma^2} \quad (-\infty < y, y' < \infty). \quad (4.5)$$

4.2. Gaussian profile

Let us assume the following form of $P(x)$:

$$P(x) = \exp(-x^2/2\gamma^2). \quad (4.6)$$

This form of $P$ clearly meets conditions (2.2) and (2.4) and, according to the previous remarks, the operator $A$ is now injective. In order to determine the singular system the integral equations with the kernels (2.9) and (2.10) are to be solved. The kernel $S$ is obtained immediately by inserting equation (4.6) into equation (4.5). The evaluation of $U$ through equations (2.10), (1.5) and (4.6) is also straightforward. The resulting integral equations for the singular functions $u_k$ and $v_k$ are

$$\frac{1}{2\gamma\sqrt{\pi}} \int_{-\infty}^{\infty} u_k(y') \exp\left[-\frac{y^2+y'^2 + (y-y')^2}{2\gamma^2}\right] \, dy' = \alpha_k^2 u_k(y) \quad (k = 0, 1, \ldots; -\infty < y < \infty) \quad (4.7)$$

$$\frac{1}{2\gamma\sqrt{\pi}} \frac{d}{d\gamma^2} \int_{-\infty}^{\infty} v_k(x') \exp\left[-\frac{\rho^2}{\rho^2 + \gamma^2} \left(\frac{x^2 + x'^2}{2\rho^2} + \frac{(x-x')^2}{4\gamma^2}\right)\right] \, dx' = \alpha_k^2 v_k(x) \quad (k = 0, 1, \ldots; -\infty < x < \infty). \quad (4.8)$$
It is seen that equation (4.8) transforms into equation (4.7) by a change of variables. This means that the singular functions $v_k$ are scaled versions of the $u_k$. More precisely, we have

$$v_k(x) = F_k u_k(x \sqrt{\rho^2 + \gamma^2}) \quad (k = 0, 1, \ldots)$$  \hspace{1cm} (4.9)

where $F_k$ is a proportionality factor to allow for normalisation. Equation (4.7) can be solved analytically \cite{10, 11} and it turns out that the solutions are the well known Hermite–Gauss functions. Their normalised expression is

$$u_k(x) = \left(\frac{2c}{\pi}\right)^{1/4} \frac{1}{\sqrt{2^k k!}} H_k(x \sqrt{2c}) \exp(-cx^2) \quad (k = 0, 1, \ldots)$$  \hspace{1cm} (4.10)

where $H_k$ is the $k$th Hermite polynomial. Here $c$ has the value

$$c = \frac{1}{2\rho^2} \left(1 + \frac{\rho^2}{\gamma^2}\right)^{1/2}$$  \hspace{1cm} (4.11)

The pertaining singular values are

$$\alpha_k = \alpha_0 \eta^k \quad (k = 0, 1, \ldots)$$  \hspace{1cm} (4.12)

where

$$\alpha_0 = \left[\frac{\rho}{\sqrt{\rho^2 + \gamma^2}}\right]^{1/2}$$  \hspace{1cm} (4.13)

$$\eta = \alpha_0^2.$$  \hspace{1cm} (4.14)

As expected from the injectivity of the operator $A$, the singular functions form a complete set in $L^2$. Therefore, in the ideal, noiseless case a perfect restoration of the signal would be possible. This is to be contrasted with the case treated in § 3.3.

### 4.3. Degrees of freedom

An estimate of the number $M$ of degrees of freedom in the presence of noise can be obtained from equations (2.24) and (4.12). The number $M$ is determined by the condition

$$\alpha_0^{2M+1} \approx E/e$$  \hspace{1cm} (4.15)

or taking equation (4.13) into account

$$M = -\frac{\ln(E/e)}{\ln((\rho/\gamma)/(1 + [1 + (\rho/\gamma)^2]^{1/2}))} - \frac{1}{2}.$$  \hspace{1cm} (4.16)

The dependence of $M$ on the signal-to-noise ratio $E/e$ is of logarithmic nature. That should be compared with the case of the sharp low-pass filter with sinc profile (see equation (3.56)), where a proportionality between $M$ and $E/e$ is exhibited. The logarithmic dependence is obviously less favourable than the linear dependence. Nevertheless, the comparison between the two cases is not completely significant unless we clarify the meaning of the degrees of freedom for the present case. As we shall see in a moment, the meaning is not the same as we found in § 3.5. There the number of degrees of freedom was linearly related to the extension of the interval over which the signal could be recovered. This is because the resolution of the restored signal cannot exceed twice the resolution of the original low-pass filter, whatever the value of $M$ (even if $M \to \infty$). Now, the situation is different, because for $M \to \infty$ an infinite resolution would be attained. In order to understand what kind of restored signal is obtained with a finite value of $M$ we focus our...
Signal restoration: singular value analysis

attention on the function

\[ W_M(x, y) = \sum_{k=0}^{M-1} u_k(x)u_k^*(y) \]  \hspace{1cm} (4.17)

where the functions \( u_k \) are given by equation (4.10). It is easily realised that \( W_M(x, y) \) plays the role of an overall impulse response of the process (see the analogous discussion at the beginning of § 3.3). It is well known that the \( k \)th Hermite–Gauss function appearing in \( u_k(x) \) becomes negligibly small when \( |x|\sqrt{2\pi} \) exceeds a value roughly equal to \( \sqrt{2k} \). As a consequence, the interval, say \( \Delta x \), over which the signal can be recovered is roughly equal to

\[ \Delta x \approx 2\sqrt{M/c}. \]  \hspace{1cm} (4.18)

At the same time, an increase of \( M \) gives rise to a reduction of the width of the central lobe of \( W_M \) when considered as a function of \( x \) for fixed \( y \). To give an idea of the behaviour of \( W_M(x, y) \), a few curves relating to different values of \( M \) and \( y \) are drawn in figure 2.

Figure 2. The truncated impulse response \( W_M(x, y) \) of the smooth low-pass system after restoration for some values of \( M \) and \( y \): (a) \( M = 32, y = 0 \); (b) \( M = 32, y = 4.0 \); (c) \( M = 64, y = 0 \); (d) \( M = 64, y = 6.0 \); (e) \( M = 128, y = 0 \); (f) \( M = 128, y = 8.0 \).
assuming $c = 0.5$. Although $W_M(x, y)$ is not shift invariant, the width of the central peak does not change very much across $\Delta x$. As a rule of thumb, such a width, say $\delta x$, is found to be

$$\delta x \approx 2\sqrt{cM}.$$  \hfill (4.19)

As a conclusion, an increase of $M$ has two simultaneous effects. It gives rise to an expansion of $\Delta x$ and to a reduction of $\delta x$. Assuming $\delta x$ to be a measure of the resolution, we find from equations (4.18) and (4.19) that the number of resolved elements in $\Delta x$ is

$$\Delta x / \delta x \approx M.$$  \hfill (4.20)

If the ratio $\rho/\gamma$ exceeds one the approximation

$$- \left( \ln \left( \frac{\rho/\gamma}{1 + (1 + (\rho/\gamma)^2)^{1/2}} \right) \right)^{-1} \approx \frac{\rho}{\gamma}$$  \hfill (4.21)

can be used in equation (4.16) (within a few per cent). Using equations (4.16), (4.20) and (4.21) we obtain

$$\Delta x / \delta x \approx M \approx (\rho/\gamma) \ln(E/e).$$  \hfill (4.22)

It will be noted that $\rho/\gamma$ is the ratio between the width of the gaussian profile and the width pertaining to the impulse response of the original gaussian smooth low-pass filter. In other words, $\rho/\gamma$ can be taken as a measure of the number of degrees of freedom of the unrestored output signal. The number of degrees of freedom of the restored signal can be significantly higher than $\rho/\gamma$ if the signal-to-noise ratio is reasonably large. To complete the

Figure 3. The truncated impulse response $W(x, y)$ for some values of $y$: (a) $y = 0$; (b) $y = 0.4$; (c) $y = 0.8$; (d) $y = 1.2$. The gaussian curve in (a) is the original impulse response.
comparison with the case of § 3.3, we add something about the case \( \rho = y \). Remember in fact that for the sharp low-pass system the hypothesis \( \nu_M = \nu_1 \) was made.

Letting \( \rho = y \) in equation (4.13) we find \( \alpha_0 = \left[ \frac{1}{1 + \sqrt{2}} \right]^{1/2} \). Assuming that \( \varepsilon/s \) is not too large, say 10, we obtain \( M = 3 \) from equation (4.15). Does a restored signal with only three degrees of freedom show any significant improvement with respect to the unrestored signal? To answer this question, it is sufficient to compare the original impulse response of the smooth low-pass filter, namely equation (1.5), with \( W_3(x, y) \). This can be done with the aid of figure 3 where \( c = 0.5 \) (i.e. \( y = \sqrt{2} \)) is assumed. It is seen that the width of the main lobe of \( W_3 \) is about one half of the width of the gaussian response over the whole interval in which the recovery is effective. Therefore the recovered signal will exhibit a resolution twice that of the unrestored one.

Acknowledgment

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References