Biorthogonal Wilson Bases

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ABSTRACT

Wilson bases consist of products of trigonometric functions with window functions which have good time-frequency localization, so that the basis functions themselves are well localized in time and frequency. Therefore, Wilson bases are well suited for time-frequency analysis. Daubechies, Jaffard and Journe have given conditions on the window function for which the resulting Wilson basis is orthonormal. In particular, they constructed an example where the basis functions have exponential decay in the time and the frequency domain. Here, we investigate biorthogonal Wilson bases with arbitrary shape. Necessary and sufficient conditions for the Riesz stability of these bases are given. Furthermore, we determine exact Riesz bounds and the dual bases.

Keywords: Wilson bases, folding, Zak transform, biorthogonal bases, time frequency analysis

1. INTRODUCTION

Gabor frames \( \{ g(\cdot - an)e^{2\pi ibm} : m, n \in \mathbb{Z}, a, b > 0 \} \), have found wide applications in signal processing, in particular in time-frequency analysis (cf. Ref. 1). However, by the Balian-Low theorem one knows that a Gabor frame for \( L^2(\mathbb{R}) \) which is a Riesz basis necessarily has bad localization properties in time or frequency. Therefore, Wilson\(^2,3\) suggested to consider functions which are localized around the positive and negative frequency of the same order. This idea was used by Daubechies, Jaffard, and Journe\(^4\) to construct orthonormal “Wilson bases” which consist of functions \( \psi_j \) given by

\[
\psi_j(x) := \begin{cases} 
\sqrt{2}w(x - \frac{j}{2}), & \text{if } k = 0 \text{ and } j \text{ is even}, \\
2w(x - \frac{j}{2}) \cos(2k\pi x), & \text{if } k \in \mathbb{N} \text{ and } j \text{ is even}, \\
2w(x - \frac{j}{2}) \sin(2(k + 1)\pi x), & \text{if } k \in \mathbb{N}_0 \text{ and } j \text{ is odd},
\end{cases}
\]

with a smooth, well localized window function \( w \). For such bases the disadvantage described in the Balian-Low theorem is completely removed.

Independently from the result of Daubechies, Jaffard, and Journe, orthonormal local trigonometric bases consisting of the functions \( w_j \cos((k + \frac{1}{2})\pi(x - j)), j \in \mathbb{Z}, k \in \mathbb{N}_0 \), were introduced by Malvar\(^5\) in a so called two-overlapping setting. Here, two-overlapping means that the window functions are assumed to be compactly supported and only immediately neighboring windows are allowed to have overlapping supports. There exist many generalizations of the Malvar bases (see e.g. Ref. 6,7). To obtain more freedom for the choice of window functions biorthogonal bases were investigated in Ref. 8–13. Such Malvar bases were applied in speech processing\(^14\) or in image compression\(^15,16\) to reduce blocking effects.

A drawback of Malvar’s construction is the restriction on the support of the window functions. Therefore, we consider here the Wilson bases of Daubechies, Jaffard, and Journe,\(^4\) where the window \( w \) can have arbitrary support. But the restriction on orthonormal bases allows only a small class of window functions. To obtain more freedom usually orthonormality is replaced by Riesz stability. In this way one still obtains bases which allow a stable decomposition of functions from \( L^2(\mathbb{R}) \) but one has more freedom to include other desirable features. Therefore, Coifman and Meyer\(^17\) investigated Wilson bases, where the Gaussian \( w = e^{-c(\cdot - \frac{1}{4})^2}, \text{Re}(c) > 0 \), is used as the window function. The main result is that these Gaussian bases are Riesz bases of \( L^2(\mathbb{R}) \), and an explicit expression for the corresponding dual basis is given. An investigation of biorthogonal Wilson bases with an arbitrary symmetric

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window function, i.e. \( w(x) = w(\frac{1}{2} - x) \), was given by Chui and Shi. In particular, they proved a condition for Riesz stability, and they also gave an explicit formulation of the dual basis.

In this paper we want to study a more general approach for the construction of biorthogonal Wilson bases. In particular, we will investigate window functions of arbitrary shape which do not need to be symmetric. Furthermore, we consider window functions with different shapes for even and odd indices \( j \) of \( \psi^j_k \). This is motivated by the fact that we have different trigonometric functions for even and odd \( j \). As a main tool for our purposes we introduce a folding operator which maps the Wilson basis onto an orthonormal basis with a simple structure. In particular, this folding operator leads to an explicit formula for the Riesz bounds of a Wilson basis. Such results are new even for the bases of Chui and Shi as well as for the Gaussian bases of Coifman and Meyer.

The paper is organized as follows. In Sect. 2 we introduce the folding operator using the Zak transform. Section 3 contains our main results on the Riesz bounds and the dual bases. Furthermore, we give necessary and sufficient conditions for orthonormality. In Sect. 4 we apply our results to the Gaussian bases of Coifman and Meyer. In particular, we are able to compute the Riesz bounds for each positive \( \zeta \) and we give a short proof for the result of Coifman and Meyer on the dual bases.

2. FOLDING OPERATORS FOR WILSON BASES

In the sequel, we want to investigate Wilson bases with window functions of arbitrary shape. For this, we consider the trigonometric functions

\[
d^j_k(x) := \begin{cases} 
\epsilon_k \cos(2k\pi x), & \text{if } j \text{ is even}, \\
2 \sin(2(k+1)\pi x), & \text{if } j \text{ is odd},
\end{cases} \quad k \in \mathbb{N}_0, \ j \in \mathbb{Z},
\]

with

\[
\epsilon_k := \begin{cases} 
\sqrt{2}, & \text{if } k = 0, \\
2, & \text{otherwise}.
\end{cases}
\]

It is well known, that for each \( j \in \mathbb{Z} \) the functions \( d^j_k, k \in \mathbb{N}_0 \), form an orthonormal basis of \( L^2([\frac{1}{2}, \frac{3}{2}]) \). Furthermore, we introduce window functions \( w_j \in L^2(\mathbb{R}) \) which are integer translations of \( w_0 \) resp. \( w_{-1} \) for even resp. odd \( j \in \mathbb{Z} \). Namely, we have

\[
w_{j+s}(x) = w_j(x-j), \ j \in \mathbb{Z}, \ s \in \{-1,0\}.
\]

**Definition 1.** The Wilson system for the window functions \( w_0, w_{-1} \in L^2(\mathbb{R}) \) is given by \( \{ \psi^j_k : j \in \mathbb{Z}, k \in \mathbb{N}_0 \} \) with

\[
\psi^j_k(x) := w_j(x) d^j_k(x) = \begin{cases} 
w_0(x - \frac{j}{2}) \epsilon_k \cos(2k\pi x), & \text{if } j \text{ is even}, \\
w_{-1}(x - \frac{j+1}{2}) 2 \sin(2(k+1)\pi x), & \text{if } j \text{ is odd}.
\end{cases}
\]

In particular, if the Wilson system \( \{ \psi^j_k \} \) is a basis we call it a Wilson basis.

To investigate the basis properties of Wilson systems we will introduce a folding operator. For this we need the Zak transform.

**Definition 2.** The Zak transform of \( f \in L^2(\mathbb{R}) \) is defined by

\[
Zf(x, \xi) := \sum_{n \in \mathbb{Z}} f(x + j) e^{2\pi i n \xi}, \ x, \xi \in \mathbb{R}.
\]

For a fixed \( x \in \mathbb{R} \) the Zak transform can be interpreted as a Fourier series with coefficients \( f(x + k) \). This view results in some features which turn out to be very useful for our purposes.

**Lemma 3.**

(i) The Zak transform is quasi-periodic in the sense that

\[
Zf(x + n, \xi + m) = e^{-2\pi i n \xi} Zf(x, \xi), \ x, \xi \in \mathbb{R}; \ n, m \in \mathbb{Z}.
\]

Thus, \( Zf \) is completely determined on \( \mathbb{R}^2 \) by its values on the square \( Q := [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}] \).
(ii) The Zak transform is an unitary map, i.e., \( Z : L^2(\mathbb{R}) \to L^2(\mathbb{Q}) \) is bijective and
\[
\|f\|_{L^2(\mathbb{R})} = \|Zf\|_{L^2(\mathbb{Q})}.
\]

(iii) For \( f, g \in L^2(\mathbb{R}) \) it holds that
\[
\langle f, g \rangle = \iint_Q Zf(x, \xi) \overline{Zg(x, \xi)} \, dx \, d\xi.
\]

(iv) Every function \( f \in L^2(\mathbb{R}) \) can be determined from \( Zf \) by
\[
f(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} Zf(x, \xi) \, d\xi.
\]

(v) Let the Fourier transform be given by \( \hat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} \, dx \). For \( f \in L^2(\mathbb{R}) \) it holds that
\[
Zf(x, \xi) = e^{-2\pi i x \xi} Z\hat{f}(-\xi, x).
\]

For the proof we refer the reader to Ref. 19, where these statements are shown.

**Theorem 4.** Let \( w_0 \) and \( w_{-1} \) satisfy the inequality
\[
|w_s(x)| \leq C(1 + |x|)^{-1-\epsilon}, \quad s \in \{-1, 0\},
\]
for some positive constant \( C \) and some \( \epsilon > 0 \). Furthermore, let the matrix
\[
M(x, \xi) := \begin{pmatrix}
Zw_0(x, \xi) & Zw_0(-x, \xi) \\
-Zw_{-1}(x, \xi) & Zw_{-1}(-x, \xi)
\end{pmatrix}
\]
be given. Then, a bounded operator \( T_w : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) is defined by
\[
\begin{pmatrix}
ZT_wf(x, \xi) \\
ZT_wf(-x, \xi)
\end{pmatrix} = M(x, \xi) \begin{pmatrix}
Zf(x, \xi) \\
Zf(-x, \xi)
\end{pmatrix}, \quad (x, \xi) \in Q^+ := (0, \frac{1}{2}) \times [-\frac{1}{2}, \frac{1}{2}).
\]

This operator satisfies the equality
\[
\langle f, \psi_k \rangle = \int_{\frac{1}{2}}^{\frac{1}{2}} T_wf(x) \, d\psi_k(x) \, dx.
\]

**Proof.** From (7) it follows that \( w_0, w_{-1} \in L^2(\mathbb{R}) \) and we obtain
\[
|Zw_s(x, \xi)| \leq \sum_{n \in \mathbb{Z}} |w_s(x + n)| \leq C \sum_{n \in \mathbb{Z}} (1 + |x + n|)^{-1-\epsilon} \leq 2C \sum_{n=2}^{\infty} n^{-1-\epsilon} < \infty.
\]

Hence, \( Zw_0 \) and \( Zw_{-1} \) are contained in \( L^\infty(\mathbb{Q}) \). Now, one obtains immediately that \( ZT_wf \in L^2(\mathbb{Q}) \) if \( Zf \in L^2(\mathbb{Q}) \). By the unitarity of the Zak transform (3) it follows that \( T_w \) is a bounded linear mapping from \( L^2(\mathbb{R}) \) into \( L^2(\mathbb{R}) \).

Using the 1-periodicity of \( d^k_j \) we obtain
\[
Z \psi_{k+j+s}^k(x, \xi) = \sum_{n \in \mathbb{Z}} w_s(x + n - j) e^{2\pi i n \xi} d^k_j(x) = Zw_s(x, \xi) e^{2\pi i j \xi} d^k_j(x), \quad j \in \mathbb{Z}, \ k \in \mathbb{N}_0, \ s \in \{-1, 0\}.
\]
In particular, we have for \( w_0 = \chi_{[0, \frac{1}{2})} \) that

\[
Z \left( \chi_{[j, j + \frac{1}{2})} d_0 \right)(x, \xi) = \begin{cases} 
\varepsilon_k \cos(2k\pi x), & \text{if } x \in [0, \frac{1}{2}), \\
0, & \text{if } x \in (-\frac{1}{2}, 0),
\end{cases}
\]

and for \( w_{-1} = \chi_{(-\frac{1}{2}, 0]} \) that

\[
Z \left( \chi_{[j-\frac{1}{2}, j)} d_{-1} \right)(x, \xi) = \begin{cases} 
0, & \text{if } x \in [0, \frac{1}{2}), \\
e^{2\pi i j \xi} 2 \sin(2(k + 1)\pi x), & \text{if } x \in (-\frac{1}{2}, 0),
\end{cases}
\]

where \( \chi_I \) denotes the characteristic function of an interval \( I \). Thus, it follows by (4) that

\[
\langle f, \psi_{j+s} \rangle = \int_Q Z f(x, \xi) \overline{Z \psi_j(x, \xi)} e^{2\pi i j \xi} d_\xi(x) \, dx \, d\xi.
\]

Since \( \cos(2k\pi \cdot) \) is an even function, this implies for \( s = 0 \) that

\[
\langle f, \psi_{2j} \rangle = \int_Q Z f(x, \xi) \overline{Z \psi_j(x, \xi)} e^{-2\pi i j \xi} \cos(2k\pi x) \, dx \, d\xi
\]

Analogously, one shows for \( s = -1 \)

\[
\langle f, \psi_{2j-1} \rangle = \int_{\frac{j}{2}}^{1} T_w f(x) d_{2j-1}(x) \, dx
\]

using that \( \sin(2k\pi \cdot) \) is an odd function. 0

We denote the operator \( T_w \) as folding operator since it has the same property (9) as the folding operators introduced by Wickerhauser\(^{20,21}\) or as similar folding operators in numerous other papers (cf. Ref. 8,9,11-13,22,23). Furthermore, the dual operator \( T_w^* \) is denoted as unfolding operator. By simple calculations one obtains that \( T_w^* \) is given by

\[
\left( \begin{array}{c}
Z T_w f(x, \xi) \\
Z \overline{T_w f(x, \xi)}
\end{array} \right) = M^H(r, \xi) \left( \begin{array}{c}
Z f(x, \xi) \\
Z \overline{f(x, \xi)}
\end{array} \right), \quad (r, \xi) \in \Omega^+.
\]

with \( M^H = M^T \). From equality (9) one concludes immediately

\[
T_w^* \left( \chi_{[j, j+\frac{1}{2})} d_j \right) = \psi_{j},
\]

The folding operator can also be described without using the Zak transform.

**Corollary 5.** Let the window functions \( w_0 \) and \( w_{-1} \) be given as in Theorem 4. For \( f \in L^2(\mathbb{R}) \) and \( w_{2j+1} = w_j(-2j) \) the folding operator \( T_w \) satisfies

\[
T_w f(x) = \sum_{r \in \mathbb{Z}} \left( w_j(2r + x)f(2r + x) + (-1)^j w_j(2r - x)f(2r - x) \right), \quad x \in \left[ \frac{1}{2}, \frac{j+1}{2} \right), \quad j \in \mathbb{Z}.
\]

\[\text{Proof.}\] Since the Zak transform is defined as a Fourier series we have

\[
Z f(x, \xi) \overline{Z w_j(x, \xi)} = \sum_{r \in \mathbb{Z}} \left( \sum_{j \in \mathbb{Z}} f(x + j) w_j(x + r) \right) e^{2\pi i (j-r) \xi}
\]

\[
= \sum_{r \in \mathbb{Z}} \left( \sum_{j \in \mathbb{Z}} f(x + j + r) w_j(x + r) \right) e^{2\pi i j \xi}.
\]

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From (8) it follows now for \( x \in [0, \frac{1}{2}) \) by comparison of coefficients
\[
T_w f(2(x + j)) = \sum_{r \in \mathbb{Z}} \left( f(x + r + j)w_0(x + r) + f(-x + r + j)w_0(-x + r) \right).
\]
By the substitution \( x \rightarrow x - j, j \in \mathbb{Z} \), we obtain (14) for \( x \in [j, j + \frac{1}{2}) \). Analogously, one shows (14) for \( x \in [j - \frac{1}{2}, j) \). Therefore, the assertion is proved for all \( x \in \mathbb{R} \).

Corollary 5 states that one obtains \( T_w f \) in \([\frac{3}{4}, \frac{1}{4} + \frac{1}{2})\) by “folding” the function \( w_j f \) repeatedly at the points \( \frac{j}{2} \) and adding or subtracting the folded parts. Similarly to the classical folding operator of Wickerhauser we have for smooth window functions and smooth \( f \) that \( T_w f \) is smooth in each interval \([\frac{3}{4}, \frac{1}{4} + \frac{1}{2})\). Furthermore, \( T_w f \) has smooth even resp. odd extensions over the interval boundaries for even resp. odd \( j \in \mathbb{Z} \).

Remark 6. It is also possible to construct Wilson bases using the antiperiodic functions \( c_j(x) := \cos((2k + 1)\pi(x - \frac{1}{2})) \) instead of the periodic functions \( d_j \) (see e.g. Ref. 17, 18). A folding operator which satisfies (9) with these functions \( c_j \) is given by
\[
\begin{pmatrix}
Z_2 f(x, \xi) \\
Z_2 f(-x, \xi)
\end{pmatrix}
= M(x, \xi + \frac{1}{2})
\begin{pmatrix}
Z_1 f(x, \xi) \\
Z_1 f(-x, \xi)
\end{pmatrix}, \quad (x, \xi) \in \mathbb{R}^+. 
\]
Since we obtain similar statements for these bases we will consider only the functions \( d_j \) in the sequel.

3. BIORTHOGONAL BASES AND RIESZ BOUNDS

With the folding operator \( T_w \) introduced in (8) we can investigate the basis properties of Wilson systems. In particular, we are interested in the question for which windows \( w_0 \) and \( w_{-1} \) the functions \( \psi_j \) constitute a Riesz basis.

Definition 7. A sequence \( \{f_n : n \in \mathbb{Z}\} \) is called a Riesz basis if it is complete in \( L^2(\mathbb{R}) \), and there exist constants \( A, B > 0 \) such that
\[
A \sum_{n \in \mathbb{Z}} |c_n|^2 \leq \left\| \sum_{n \in \mathbb{Z}} c_n f_n \right\|_{L^2(\mathbb{R})}^2 \leq B \sum_{n \in \mathbb{Z}} |c_n|^2
\]
for arbitrary coefficients \( c_n \in \mathbb{C} \). The constants \( A \) and \( B \) are called Riesz bounds. Furthermore, the Riesz bounds \( A_0 \) and \( B_0 \) are called the best possible Riesz bounds if the inequality (15) does not hold for any \( A > A_0 \) or any \( B < B_0 \).

For our further investigations we will need the Euclidean norm \( |x| := \sqrt{|x_1|^2 + \ldots + |x_n|^2} \) of a vector \( x \in \mathbb{R}^n \) and the spectral norm \( \|A\|_2 := \sup_{\|x\|_2=1} |Ax| = \sqrt{\rho(A^* A)} \) of a matrix \( A \).

Theorem 8. Let the functions \( \psi_{2j+1}^k(x) := w_j(x - 2j)d_{2j+1}^k(x), j \in \mathbb{Z}, x \in \{-1, 0\}, k \in \mathbb{N}_0, \) be given, where the window functions \( w_0 \) and \( w_{-1} \) satisfy the decay property (7). Furthermore, we define
\[
A_0 := \text{ess inf}_{(x, \xi) \in \mathbb{Q}^+} \|M^{-1}(x, \xi)\|_{L^2(\mathbb{R})}^2 = \text{ess inf}_{(x, \xi) \in \mathbb{Q}^+} \frac{\Delta(x, \xi)^2}{2} - \sqrt{\frac{\Delta(x, \xi)^2}{4} - |\det M(x, \xi)|^2}, 
\]
\[
B_0 := \text{ess sup}_{(x, \xi) \in \mathbb{Q}^+} \|M(x, \xi)\|_{L^2(\mathbb{R})}^2 = \text{ess sup}_{(x, \xi) \in \mathbb{Q}^+} \frac{\Delta(x, \xi)^2}{2} + \sqrt{\frac{\Delta(x, \xi)^2}{4} - |\det M(x, \xi)|^2}
\]
with
\[
\Delta(x, \xi) := |Z w_0(x, \xi)|^2 + |Z w_0(-x, \xi)|^2 + |Z w_{-1}(x, \xi)|^2 + |Z w_{-1}(-x, \xi)|^2.
\]
The Wilson system \( \{\psi_j^k\} \) is a Riesz basis of \( L^2(\mathbb{R}) \) with the Riesz bounds \( A \) and \( B \) if \( 0 < A \leq A_0 \leq B_0 \leq B < \infty \).
Proof. Assume $0 < A_0 \leq B_0 < \infty$. Let the folding operator $T_w$ be defined by (8). For $f \in L^2(\mathbb{R})$ we conclude from unitarity of the Zak transform

$$
\|T_w f\|_{L^2(\mathbb{R})}^2 = \|ZT_w f\|_{L^2(Q)}^2 = \int_{Q^+} \left| \left( ZT_w f(x,\xi) \right) \right|^2 dx d\xi
$$

$$
= \int_{Q^+} \left| M(x,\xi) \left( \frac{Zf(x,\xi)}{Zf(-x,\xi)} \right) \right|^2 dx d\xi
$$

$$
\leq \int_{Q^+} \|M(x,\xi)\|^2 \left( |Zf(x,\xi)|^2 + |Zf(-x,\xi)|^2 \right) dx d\xi
$$

(18)

$$
\leq B_0 \int_{Q^+} |Zf(x,\xi)|^2 dx d\xi = B_0 \|f\|_{L^2(\mathbb{R})}^2 \tag{19}
$$

Hence, it is proved that

$$
B_0 \geq \frac{\|T_w f\|_{L^2(\mathbb{R})}^2}{\|f\|_{L^2(\mathbb{R})}^2}, \quad f \neq 0.
$$

Now, we will show that $B_0$ is the smallest upper bound in the inequality above. Let for all $(x,\xi) \in Q^+$ the vector $(Zf(x,\xi), Zf(-x,\xi))^T$ be the largest eigenvector of $M^H(x,\xi)M(x,\xi)$ corresponding to the maximal eigenvalue $\rho(M^H(x,\xi)M(x,\xi))$. Then, equality is attained in (18). Now, the norm $g(x,\xi) = \sqrt{|Zf(x,\xi)|^2 + |Zf(-x,\xi)|^2}$ of this eigenvector can be an arbitrary non-negative function of $L^2(Q^+)$. Hence, $B_0$ is the smallest possible constant in the Hölder inequality

$$
\int_{Q^+} \rho \left( M^H(x,\xi)M(x,\xi) \right) \left| g(x,\xi) \right|^2 dx d\xi \leq B_0 \int_{Q^+} \left| g(x,\xi) \right|^2 dx d\xi.
$$

i.e., in (19). Therefore, it is shown that

$$
B_0 = \sup_{f \neq 0} \frac{\|T_w f\|_{L^2(\mathbb{R})}^2}{\|f\|_{L^2(\mathbb{R})}^2} = \|T_w\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})}^2 \tag{20}
$$

Since $A_0$ is positive we know that the matrix $M(x,\xi)$ is invertible a.e. on $Q^+$ and $\|M^{-1}(x,\xi)\|_2^2 \leq A_0^{-1}$. Then, it follows immediately that $T_w$ is bijective with

$$
\begin{pmatrix}
ZT_w^{-1}f(x,\xi) \\
ZT_w^{-1}f(-x,\xi)
\end{pmatrix}
= M^{-1}(x,\xi)
\begin{pmatrix}
Zf(x,\xi) \\
Zf(-x,\xi)
\end{pmatrix}, \quad (x,\xi) \in Q^+.
$$

(21)

Analogously to (20) one shows now $\|T_w^{-1}\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} = A_0^{-1}$.

Since $\{d_j^k : k \in \mathbb{N}_0\}$ is an orthonormal basis of $L^2(\mathbb{Z}, \mathbb{Z}^+)$ the functions $X_{j,0}(x,\xi) = \langle f, d_j^k \rangle_{L^2(\mathbb{Z}, \mathbb{Z}^+)} d_j^k$, $j \in \mathbb{Z}$, $k \in \mathbb{N}_0$, constitute an orthonormal basis of $L^2(\mathbb{R})$. Thus, if $f$ is in $L^2(\mathbb{R})$ the function $T_w^{-1}f \in L^2(\mathbb{R})$ has the expansion

$$
T_w^{-1}f = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}_0} c_{n,k} X_{j,0}(x,\xi) d_j^k \tag{22}
$$

with $c_{n,k} := \langle T_w f, X_{j,0}(\xi,\xi) \rangle_{L^2(\mathbb{R})}^2$. Note that $T_w^{-1} := (T_w^*)^{-1}$ has the same norm as $T_w^{-1}$. Because $T_w$ is also a bounded, linear operator equality (22) implies

$$
f = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}_0} c_{n,k} T_w^* X_{j,0}(\xi,\xi) d_j^k.
$$

Now, from (13) we conclude that

$$
f = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}_0} c_{n,k} \varphi_j^k.
$$
In particular, we see that \( \{ \psi_j^d \} \) is complete.

Furthermore, from (22) it follows by the orthonormality of the functions \( \chi_{\{d, \frac{\pi}{2d} \}^d} \) that
\[
\| T_w^{-1} f \|_{L^2(\mathbb{R})}^2 = \| T_w^{-1} f \|_{L^2(\mathbb{R})}^2 = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}_0} |c_{n,k}|^2.
\]
Applying the boundedness of the folding operator \( T_w \) and its inverse \( T_w^{-1} \) we obtain the Riesz inequality (15) with the best possible Riesz bounds \( A_0 \) and \( B_0 \). Hence, it is shown that \( \{ \psi_j^d \} \) is a Riesz basis with Riesz bounds \( A \) and \( B \) if \( 0 < A \leq A_0 \) and \( B_0 \leq B < \infty \).

Conversely, we assume now that \( \{ \psi_j^d \} \) is a Riesz basis with Riesz bounds \( A \) and \( B \). Using the orthonormality of \( \{ \chi_{\{d, \frac{\pi}{2d} \}^d} \} \) we conclude from the Riesz inequality and the Parseval identity
\[
A \left\| \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}_0} c_{n,k} \chi_{\{d, \frac{\pi}{2d} \}^d} \right\|_{L^2(\mathbb{R})}^2 \leq B \left\| \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}_0} c_{n,k} \psi_j^d \right\|_{L^2(\mathbb{R})}^2 \leq B \left\| \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}_0} c_{n,k} \chi_{\{d, \frac{\pi}{2d} \}^d} \right\|_{L^2(\mathbb{R})}^2.
\]
Then, by (13) it follows immediately that \( A \leq \| T_w^{-1} \|_{L^2(\mathbb{R})}^2 = A_0 \) and \( B \geq \| T_w^{-1} \|_{L^2(\mathbb{R})}^2 = B_0 \).

Finally, the identities in (16) and (17) are obtained by computing the eigenvalues of \( M(x, \xi) M(x, \xi) \). \( \square \)

Remark 9. If we have only one real-valued, symmetric window \( w \), i.e. \( w_0(x) = w_{-1}(x + \frac{1}{2}) = w(\frac{1}{2} - x) = w'(-x) \), then for the Zak transform of \( w \) it holds that \( Z_{w_{-1}}(-x, \xi) = Z_w(\frac{1}{2} - x, \xi) = Z_w(x, \xi) \). Hence,
\[
\det M(x, \xi) = |Z_{w}(x, \xi)|^2 + |Z_{w}(-x, \xi)|^2 = \frac{\Delta(x, \xi)}{2}
\]
and we have the Riesz bounds
\[
A_0 = \text{ess inf}_{(x, \xi) \in Q} \left( \det M(x, \xi) \right) \quad \text{and} \quad B_0 = \text{ess sup}_{(x, \xi) \in Q} \left( \det M(x, \xi) \right).
\]

Now, we want to investigate for which window functions we have an orthonormal Wilson basis.

Theorem 10. Let the functions \( \psi_j^d \) satisfy the assumptions of Theorem 8. The system \( \{ \psi_j^d \} \) is an orthonormal basis if and only if
\[
w_{-1}(x) = \pm w_0(-x), \tag{23}
\]
\[
\sum_{r \in \mathbb{Z}} \left( w_0(j + r + x) \overline{w_0(r + x)} + w_0(j + r - x) \overline{w_0(r - x)} \right) = \delta_{j,0}, \quad j \in \mathbb{Z}. \tag{24}
\]

Proof. Assume that \( \psi_j^d \) is an orthonormal basis. By Parseval identity we know that \( A_0 = B_0 = 1 \) and hence \( M^H(x, \xi) M(x, \xi) = I \). Computing the entries of \( M^H(x, \xi) M(x, \xi) \) this means
\[
|Z_{w_0}(x, \xi)|^2 + |Z_{w_{-1}}(x, \xi)|^2 = 1, \tag{25}
\]
\[
Z_{w_0}(x, \xi) \overline{Z_{w_0}(-x, \xi)} = Z_{w_{-1}}(x, \xi) \overline{Z_{w_{-1}}(-x, \xi)} \tag{26}
\]
for \( (x, \xi) \in Q \). Applying (25) one obtains
\[
|Z_{w_{-1}}(x, \xi)|^2 |Z_{w_{-1}}(-x, \xi)|^2 = 1 - |Z_{w_0}(x, \xi)|^2 - |Z_{w_0}(-x, \xi)|^2 + |Z_{w_0}(x, \xi)|^2 |Z_{w_0}(-x, \xi)|^2.
\]
Together with (26) this implies
\[
|Z_{w_0}(x, \xi)|^2 + |Z_{w_0}(-x, \xi)|^2 = 1. \tag{27}
\]
From (25) it follows furthermore that \( |Z_{w_0}(x, \xi)| = |Z_{w_{-1}}(-x, \xi)| \). This results in two possibilities.
1. For $Zw_0(x, \xi) = \pm Zw_{-1}(-x, \xi)$ it follows by comparison of coefficients that $w_0(x + j) = \pm w_{-1}(-y + 2j), y \in \mathbb{R}, j \in \mathbb{Z}$. By the substitution $y = x + j$ we obtain $w_0(y) = \pm w_{-1}(-y + 2j), y \in \mathbb{R}, j \in \mathbb{Z}$. But this implies that $w_0$ and $w_{-1}$ are 2-periodic, which is a contradiction to $w_0 \in L^2(\mathbb{R})$ and $w_0 \neq 0$.

2. If $Zw_0(x, \xi) = \pm Zw_{-1}(-x, \xi)$ then $w_0(x + j) = \pm w_{-1}(-x - j), x \in \mathbb{R}, j \in \mathbb{Z}$. Substituting $y = -(x + j)$ this leads to (23).

Furthermore, it follows from (27) that

$$\sum_{j \in \mathbb{Z}} \sum_{r \in \mathbb{Z}} \left( w_0(j + r + x) \bar{w}_0(r + x) + w_0(j + r - x) \bar{w}_0(r - x) \right) e^{2\pi i \xi} = 1. \quad (x, \xi) \in \mathbb{Q}.$$

By comparison of coefficients, equality (24) follows immediately.

On the other hand, if (23) and (24) are valid one shows analogously that $M_{\mathbb{F}}(x, \xi) M(x, \xi) = I$. Hence, $A_0 = B_0 = 1$ and thus $\{ \psi_j^k \}$ is an orthonormal basis.

Remark 11. If we have only one non-negative window $w = w_0 = w_{-1}$, then equality (23) implies that $w(x) = w(\frac{1}{2} - x)$. Therefore, (27) can be written as $|Zw(x, \xi)|^2 + |Zw(x + \frac{1}{2}, \xi)|^2 = 1$. Applying (6), we obtain for the Fourier transform of $w$ that $|\hat{Z}w(x, \xi)|^2 + |\hat{Z}w(x + \frac{1}{2}, \xi)|^2 = 1$. By comparison of coefficients this equality reads as

$$\sum_{r \in \mathbb{Z}} \hat{w}(2j + r + \xi) \hat{w}(r + x) = \delta_{j, 0}, \quad j \in \mathbb{Z},$$

which corresponds to the result of Daubechies, Jaffard, and Journé on orthonormal Wilson bases.

If $\{ \psi_j^k \}$ is a Riesz basis with Riesz bounds $A$ and $B$, then there exists a dual basis $\{ \tilde{\psi}_j^k \}$ which is a Riesz basis with Riesz bounds $B^{-1}$ and $A^{-1}$. This dual basis is uniquely determined by the biorthogonality condition $(\psi_j^k, \tilde{\psi}_l^m) = \delta_{j, -\delta_{k, l}},$ where $\delta_{k, l}$ is the Kronecker symbol. To describe the dual basis we introduce the dual window functions $\tilde{w}_0$ and $\tilde{w}_{-1}$ which are defined by

$$\tilde{w}_0(x) := \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{Zw_{-1}(-x, \xi)}{|\xi|} d\xi \quad \text{and} \quad \tilde{w}_{-1}(x) := \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{Zw_0(-x, \xi)}{|\xi|} d\xi. \quad (28)$$

It turns out that the dual basis is a Wilson basis, too.

Theorem 12. Let the functions $\psi_j^k$ satisfy the assumptions of Theorem 8. Then, the dual basis is constituted by the functions

$$\tilde{\psi}_j^k(x) := \tilde{w}_j(x) d_j^k(x) = \left\{ \begin{array}{ll} w_0(x - \frac{1}{2}) \cos(2k\pi x), & \text{if } j \text{ is even}, \\ w_{-1}(x - \frac{1}{2}) 2 \sin(2(k - 1)\pi x), & \text{if } j \text{ is odd}, \end{array} \right. \quad j \in \mathbb{Z}, k \in \mathbb{N}_0.$$

Proof. Since the assumptions of Theorem 8 are satisfied, $\{ \psi_j^k \}$ is a Riesz basis and the folding operator $T_w$ and its inverse $T_w^{-1}$ are bounded. By the biorthogonality condition and property (9) of the folding operator we have

$$\delta_{j, r} \delta_{k, l} = (\tilde{\psi}_r^l, \psi_j^k) = (T_w \tilde{\psi}_r^l, \chi_{\frac{1}{2}, \frac{1}{2}}) d_j^k). \quad (29)$$

Since $\{ \chi_{\frac{1}{2}, \frac{1}{2}} d_j^k \}$ is an orthonormal basis of $L^2(\mathbb{R})$ equality (29) is equivalent to $T_w \tilde{\psi}_r^l = \chi_{\frac{1}{2}, \frac{1}{2}} d_j^k$. Hence, the dual basis functions are given by $\tilde{\psi}_j^k = T_w^{-1} \chi_{\frac{1}{2}, \frac{1}{2}} d_j^k$. By Cramer's rule it follows that

$$M^{-1}(x, \xi) = \frac{1}{\det M(x, \xi)} \begin{pmatrix} \frac{Zw_{-1}(-x, \xi)}{Zw_0(-x, \xi)} & -\frac{Zw_0(-x, \xi)}{Zw_0(x, \xi)} \\ \frac{Zw_{-1}(x, \xi)}{Zw_0(x, \xi)} & -\frac{Zw_0(x, \xi)}{Zw_0(-x, \xi)} \end{pmatrix}. \quad 417$$
Using the assertions on the inverse folding operator (21) and the Zak transform of \( \chi_{(j,j+\frac{1}{2})} \) in (11) we obtain now

\[
Z\tilde{\psi}^k_j(x, \xi) = \frac{Zw_{-1}(-x, \xi)}{\det M(x, \xi)} e^{2\pi ij\xi} \epsilon_k \cos(2k\pi x).
\]

From the quasi-periodicity (2) of the Zak transform we conclude

\[
Z\tilde{\psi}^k_j(x + j, \xi) = \frac{Zw_{-1}(-x, \xi)}{\det M(x, \xi)} \epsilon_k \cos(2k\pi x).
\]

Now, the application of the inverse Zak transform (5) leads to

\[
\psi^k_j(x + j) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{Zw_{-1}(-x, \xi)}{\det M(x, \xi)} d\xi \epsilon_k \cos(2k\pi x)
\]

and thus \( \tilde{\psi}^k_j(x) = \tilde{w}_0(x - j) \epsilon_k \cos(2k\pi x). \) Analogously one shows \( \tilde{\psi}^k_{j-1}(x) = \tilde{w}_{-1}(x - j) 2\sin(2(k + 1)\pi x). \)

4. Gaussian Bases of Coifman and Meyer – An Example with Good Time-Frequency Localization

Coifman and Meyer\(^7\) investigated Wilson bases with the Gaussian

\[ w_0(x) = w_{-1}(x - \frac{1}{2}) = w(x) := e^{-\xi(x-\frac{1}{2})^2}, \quad \xi > 0, \]

as window function. For this they took advantage of a special property of the Gaussian. In particular, they proved that the Gaussian bases are Riesz bases and they determined the corresponding dual bases. Using our approach we can derive the results of Coifman and Meyer in a simple way. As a new result we gain the Riesz bounds for the Gaussian bases.

Obviously, \( w_\frac{1}{2}(x) = e^{-(\frac{1}{4} - \frac{1}{2})^2} = e^{-\xi(\frac{1}{4} - \frac{1}{2})^2} = w(x) \) and we can use Remark 9. First, we determine \( \det M(x, \xi) \). Using the identity \((j + t)^2 + (\ell + t)^2 = \frac{1}{2}(2t + (j + \ell))^2 + \frac{1}{2}(j - \ell)^2 \) we obtain

\[
Zw(x, \xi)Zw(x, \xi) = \sum_{j \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}} e^{-\frac{1}{2}(j-x-\frac{1}{2})^2 + (\ell + x-\frac{1}{2})^2} e^{2\pi ij\xi} \epsilon_k
\]

With the substitution \( k = j + \ell \) this yields

\[
Zw(x, \xi)Zw(x, \xi) = \sum_{k \in \mathbb{Z}} e^{-\frac{1}{2}(2x-\frac{1}{2}+k)^2} \sum_{j \in \mathbb{Z}} e^{-\frac{1}{2}(2j-k)^2} e^{2\pi ij\xi} \epsilon_k.
\]

Analogously the substitution \( k = 1 - j - \ell \) leads to

\[
Zw(-x, \xi)Zw(-x, \xi) = \sum_{j \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}} e^{-\frac{1}{2}(2x-\frac{1}{2}+j+\ell)^2} e^{-\frac{1}{2}(j-\ell)^2} e^{2\pi ij\xi} \epsilon_k
\]

Summing up, we obtain by using Remark 9 that

\[
\det M(x, \xi) = |Zw(x, \xi)|^2 + |Zw(-x, \xi)|^2 = \left( \sum_{k \in \mathbb{Z}} e^{-\frac{1}{2}(2x-\frac{1}{2}+k)^2} \right) \left( \sum_{j \in \mathbb{Z}} e^{-\frac{1}{2}j^2} e^{2\pi ij\xi} \right).
\]

Thus, for the Gaussian bases the determinant of \( M(x, \xi) \) is a tensor product of two 1-periodic functions. Furthermore, by the Poisson summation formula we obtain

\[
\det M(x, \xi) = \sqrt{\frac{\pi}{2\xi}} \left( \sum_{k \in \mathbb{Z}} e^{-\frac{1}{2}(2x-\frac{1}{2}+k)^2} \right) \left( \sum_{j \in \mathbb{Z}} e^{-2\xi^2(j+\ell)^2} \right).
\]
Applying Theorem 8 we can now determine the Riesz bounds of the Gaussian bases.

**Theorem 13.** Let \( \zeta > 0 \) be given. The functions 
\[
\psi_f(x) := e^{-\zeta (x - \frac{1}{2} - j)^2} d_f(x)
\]
constitute a Riesz basis of \( L^2(\mathbb{R}) \) with the best possible Riesz bounds
\[
A_0 = \left( \sum_{k \in \mathbb{Z}} e^{-\frac{\zeta}{\sqrt{2}} (k+1/2)^2} \right) \left( \sum_{j \in \mathbb{Z}} (-1)^j e^{-\frac{\zeta}{\sqrt{2}} j^2} \right) > 0 \quad \text{and} \quad B_0 = \left( \sum_{k \in \mathbb{Z}} e^{-\frac{\zeta}{\sqrt{2}} k^2} \right)^2 < \infty.
\] (31)

**Proof.** Since \( w(x) = w(\frac{1}{2} - x) \) we can use Remark 9, i.e.,
\[
A_0 = \operatorname{ess inf}_{(x, \xi) \in Q^*} (\det M(x, \xi)) \quad \text{and} \quad B_0 = \operatorname{ess sup}_{(x, \xi) \in Q^*} (\det M(x, \xi)).
\]

We define
\[
g_\zeta(x) := \sum_{k \in \mathbb{Z}} e^{-\frac{\zeta}{\sqrt{2}} (x + k)^2}
\]
such that \( \det M(x, \xi) = \sqrt{2\pi} g_\zeta(2x - \frac{1}{2}) g_{2\zeta}(\xi) \). To prove the theorem we have to find lower and upper bounds for the 1-periodic function \( g_\zeta \).

From the Poisson summation formula it follows that
\[
g_\zeta(x) = \sqrt{\frac{\pi}{\zeta}} \sum_{k \in \mathbb{Z}} e^{-\frac{\pi^2 k^2}{\zeta}} e^{2\pi ikx} = \sqrt{\frac{\pi}{\zeta}} \vartheta \left( 2\pi x, e^{-\frac{\zeta}{\sqrt{2}}} \right)
\]
with the Theta function
\[
\vartheta(x, q) := \sum_{k \in \mathbb{Z}} q^{k^2} e^{ikx}, \quad x \in \mathbb{R}, \ q \in \mathbb{C}, \ |q| < 1.
\]
The Theta function enjoys the infinite product formula (see Ref. 24, Sect. 21.3)
\[
\vartheta(x, q) = \prod_{l=0}^{\infty} \left( 1 - q^{2l} \right) \left( 1 + q^{2l+1} \cos(x) + q^{2l+2} \right).
\]

Hence, the function \( g_\zeta(x) = \sqrt{\frac{\pi}{\zeta}} \vartheta \left( 2\pi x, e^{-\frac{\zeta}{\sqrt{2}}} \right) \) is decreasing for \( x \in [0, \frac{1}{2}] \) (cf. Ref. 25, Proposition 2.3). Since \( g_\zeta \) is even and 1-periodic, this implies \( \max_{x \in \mathbb{R}} g_\zeta(x) = g_\zeta(0) \) and \( \min_{x \in \mathbb{R}} g_\zeta(x) = g_\zeta(\frac{1}{2}) \). Therefore, the best possible Riesz bounds are
\[
A_0 = \sqrt{\frac{2\pi}{\zeta}} g_\zeta(\frac{1}{2}) g_{2\zeta}(\frac{1}{2}) \quad \text{and} \quad B_0 = \sqrt{\frac{2\pi}{\zeta}} g_\zeta(0) g_{2\zeta}(0) = g_{\frac{3}{2}}(0),
\]
i.e., the identities in (31) are proved. \( \square \)

In Fig. 1 the Riesz bounds in dependence of \( \zeta \) are shown. If \( \zeta \) is close to \( 2\pi \) then the Gaussian basis is stable, while for very large or very small \( \zeta \) the Riesz stability becomes worse. This can also be seen from the behavior of the dual window functions (cf. Fig. 2 and 3).

Furthermore, the factorization of \( \det M(x, \xi) \) leads to a simple representation of the dual window \( \tilde{\omega} \), which was also found by Coifman and Meyer.\(^{17}\) Applying our results we can present a short proof of their statement.

**Theorem 14.** For the window function \( w(x) := e^{-\zeta (x - \frac{1}{2})^2} \) the dual window is given by
\[
\tilde{\omega}(x) = \frac{\sum_{k \in \mathbb{Z}} g_{2\zeta}(x + k)^2}{\sum_{k \in \mathbb{Z}} e^{-\frac{\zeta}{\sqrt{2}} (2x - \frac{1}{2} + k)^2}},
\]

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Figure 1. Upper and lower Riesz bound for Gaussian bases \( \{ e^{x(1-j-\frac{1}{2})^2} d_j^2 \} \) in dependence of \( \zeta \).

Figure 2. Dual window (solid line) for Gaussian bell (dashed) for \( \zeta = 2\pi \).

Figure 3. Dual window (solid line) for Gaussian bell (dashed) for \( \zeta = 1 \) (left) and \( \zeta = 20 \) (right).

where

\[
\gamma_\zeta := \sqrt{\frac{\zeta}{2\pi}} \int_{-1}^{1} \frac{e^{2\pi i \xi}}{\sum_{j \in \mathbb{Z}} e^{-2\pi^2 (\xi+j)^2}} d\xi.
\]

Proof. By Fourier series expansion of \( 1/(\det M(x, \xi)) \) one obtains

\[
\frac{1}{\det M(x, \xi)} = \frac{\sum_{\xi \in \mathbb{Z}} \gamma_\zeta e^{-2\pi i \xi}}{\sum_{k \in \mathbb{Z}} e^{-\xi^2 (2\pi-\frac{1}{2}+k)^2}}.
\]

Now, the application of (28) leads to

\[
\tilde{w}(x) = \frac{1}{\det M(x, \xi)} \int_{-\frac{1}{2}}^{\frac{1}{2}} Z w(x, \xi) \, d\xi = \frac{1}{\sum_{k \in \mathbb{Z}} e^{-\xi^2 (2\pi-\frac{1}{2}+k)^2}} \sum_{j, \xi} \left( w(x+j) \gamma_\zeta \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i (j-\xi) \xi} \, d\xi \right)
\]

\[
= \frac{\sum_{\xi \in \mathbb{Z}} \gamma_\zeta e^{-\zeta^2 (\xi-\frac{1}{2}+\xi)^2}}{\sum_{k \in \mathbb{Z}} e^{-\xi^2 (2\pi-\frac{1}{2}+k)^2}}.
\]

Note, that \( 1/(\sum_{j \in \mathbb{Z}} e^{-2\pi^2 (\xi+j)^2}) \) is analytic for each \( x \in \mathbb{R} \) and therefore the Fourier coefficients \( \gamma_\zeta \) have exponential decay. Hence, all sums in the proof are convergent. \( \square \)
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