On Finite Sections of Band-dominated Operators

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Abstract. In an earlier paper we showed that the sequence of the finite sections \( P_n A P_n \) of a band-dominated operator \( A \) on \( l^p(\mathbb{Z}^+) \) is stable if and only if the operator \( A \) is invertible, every limit operator of the sequence \( (P_n A P_n) \) is invertible, and if the norms of the inverses of the limit operators are uniformly bounded. The purpose of this short note is to show that the uniform boundedness condition is redundant.

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1. Introduction

Let \( 1 < p < \infty \). We will work on the Banach space \( l^p(\mathbb{Z}^+) \) of all sequences \( (x_n)_{n=0}^{\infty} \) of complex numbers with \( \sum |x_n|^p < \infty \). We provide this space with its standard basis which consists of all sequences \( e_i := (0, \ldots, 0, 1, 0, \ldots) \) with the 1 standing at the \( i \)th position. Every bounded linear operator on \( l^p(\mathbb{Z}^+) \) admits a matrix representation \( (a_{ij})_{i,j \in \mathbb{Z}^+} \) with respect to the standard basis. We call an operator \( A \in L(l^p(\mathbb{Z}^+)) \) a band operator if the associated matrix is a band matrix, i.e., if there is a \( k \) such that \( a_{ij} = 0 \) whenever \( |i - j| \geq k \). The operator \( A \) is said to be band-dominated if it is the norm limit of a sequence of band operators.

Let \( n \in \mathbb{N} \). The \( n \)th finite section of an operator \( A \in L(l^p(\mathbb{Z}^+)) \) with matrix representation \( (a_{ij})_{i,j \in \mathbb{Z}^+} \) is the \( n \times n \)-matrix \( (a_{ij})_{i,j=0}^{n-1} \). We identify this matrix with the operator \( P_n A P_n \) where \( P_n \) is the projection

\[
P_n : l^p(\mathbb{Z}^+) \to l^p(\mathbb{Z}^+), \quad (x_0, x_1, \ldots) \mapsto (x_0, \ldots, x_{n-1}, 0, 0, \ldots).
\]

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The sequence \((P_n A P_n)\) of the finite sections of \(A\) is said to be stable if there is an \(n_0\) such that the operators \(P_n A P_n : \text{im} P_n \to \text{im} P_n\) are invertible for every \(n \geq n_0\) and if the norms of their inverses are uniformly bounded.

There is an intimate relation between stability of sequences and Fredholmness of operators. For, we associate to the sequence \(A = (P_n A P_n)\) the block diagonal operator
\[ Op(A) := \text{diag} (P_1 A P_1, P_2 A P_2, P_3 A P_3, \ldots) \tag{1.1} \]
considered as acting on \(l^p(Z^+) = \text{im} P_1 \oplus \text{im} P_2 \oplus \text{im} P_3 \oplus \cdots\). It is an easy exercise to show that the sequence \(A\) is stable if and only if the operator \(Op(A)\) is a Fredholm operator on \(l^p(Z^+)\), i.e., if its kernel and its cokernel have finite dimension. In general, the equivalence between stability and Fredholmness seems to be of less use. But if we start with the sequence \(A = (P_n A P_n)\) of a band-dominated operator \(A\), then we end up with a band-dominated operator \(Op(A)\) on \(l^p(Z^+)\) again. But for band-dominated operators on \(l^p(Z^+)\), there is a general Fredholm criterion which expresses the Fredholm property of a band-dominated operator in terms of its limit operators. To state this result, we need a few notations.

It will be convenient to work on the Banach space \(l^p(Z)\) of the two-sided infinite sequences. The space \(l^p(Z^+)\) can be considered as a closed subspace of \(l^p(Z)\) in a natural way. We let \(P\) denote the projection
\[ P : l^p(Z) \to l^p(Z), \quad (x_n) \mapsto (\ldots, 0, 0, x_0, x_1, x_2, \ldots) \]
and write \(Q\) for the complementary projection \(I - P\). Usually we will identify an operator \(A\) on \(l^p(Z^+)\) with the operator \(P A P\) acting on \(l^p(Z)\).

For every \(m \in \mathbb{Z}\), we introduce the shift operator
\[ U_m : l^p(Z) \to l^p(Z), \quad (x_n) \mapsto (x_{n-m}). \]
Let further \(\mathcal{H}\) stand for the set of all sequences \(h : \mathbb{N} \to \mathbb{N}\) which tend to infinity. An operator \(A_h \in L(l^p(Z))\) is called a limit operator of \(A \in L(l^p(Z^+))\) with respect to the sequence \(h \in \mathcal{H}\) if \(U_{-h(n)} P A P U_{h(n)}\) tends \(*\)-strongly to \(A_h\) as \(n \to \infty\). Here, \(*\)-strong convergence means strong convergence of the sequence itself and of its adjoint sequence.

Notice that every operator can possess at most one limit operator with respect to a given sequence \(h \in \mathcal{H}\). The set \(\sigma_{op}(A)\) of all limit operators of a given operator \(A\) is the operator spectrum of \(A\).

Notice further that every sequence \(h : \mathbb{N} \to \mathbb{N}\) which tends to infinity has a strongly monotonically increasing subsequence, \(g\) say, and that the existence of the limit operator \(A_g\) implies the existence of \(A_h\) and the identity \(A_h = A_g\). Thus, it is sufficient to consider limit operators with respect to strongly monotonically increasing sequences.

It is not hard to see that every limit operator of a Fredholm operator is invertible. A basic result of [2] claims that the operator spectrum of a band-dominated operator is rich enough in order to guarantee the reverse implications. Here is a
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summary of the results from [2] needed in what follows. A comprehensive treatment of this topic is in [4]; see also the references mentioned there.

**Theorem 1.1.** Let \( A \in L(l^p(\mathbb{Z}^+)) \) be a band-dominated operator. Then
(a) every sequence \( h \in \mathcal{H} \) possesses a subsequence \( g \) such that the limit operator \( A_g \) exists.
(b) the operator \( A \) is Fredholm if and only if each of its limit operators is invertible and if the norms of their inverses are uniformly bounded.

An elegant proof which also works for band-dominated operators on other discrete groups than \( \mathbb{Z} \) is due to Roe [6].

Thus, and by the above mentioned equivalence between stability of the sequence \( A \) and Fredholmness of the associated operator \( \text{Op}(A) \), one will get a stability criterion for \( A \) by computing all limit operators of \( \text{Op}(A) \). This computation has been carried out in [2, 3, 5], see also Chapter 6 in [4]. Here is the result.

**Theorem 1.2.** Let \( A \in L(l^p(\mathbb{Z}^+)) \) be a band-dominated operator. Then the finite sections sequence \( (P_n A P_n)_{n \geq 1} \) is stable if and only if the operator \( P A P + Q \) and all operators \( Q A h Q + P \) with \( A \in \sigma_{\text{op}}(A) \) are invertible on \( l^p(\mathbb{Z}) \), and if the norms of their inverses are uniformly bounded.

The goal of this note is to show that the uniform boundedness condition in Theorem 1.2 can be removed.

2. Main result

For our goal, we will need a subsequence version of Theorem 1.2. We choose and fix a strongly monotonically increasing sequence \( \eta : \mathbb{N} \to \mathbb{N} \). Further, we write \( \mathcal{H}_\eta \) for the set of all (infinite) subsequences of \( \eta \) and \( \sigma_{\text{op}, \eta}(A) \) for the collection of all limit operators of \( A \) with respect to subsequences of \( \eta \). Then we have the following version of Theorem 1.2.

**Theorem 2.1.** Let \( A \in L(l^p(\mathbb{Z}^+)) \) be a band-dominated operator and \( \eta : \mathbb{N} \to \mathbb{N} \) a strongly monotonically increasing sequence. Then the sequence \( (P_{\eta(n)} A P_{\eta(n)})_{n \geq 1} \) is stable if and only if the operator \( P A P + Q \) and all operators \( Q A h Q + P \) with \( A \in \sigma_{\text{op}, \eta}(A) \) are invertible on \( l^p(\mathbb{Z}) \), and if the norms of their inverses are uniformly bounded.

Thus, instead of all limit operators of \( A \) with respect to monotonically increasing sequences \( h \), only those with respect to subsequences of \( \eta \) are involved. The following proof of Theorem 2.1 is an adaptation of the proof of Theorem 1.2 given in [5].
Proof. Let $A \in L(l^p(Z^+))$ be a band-dominated operator, set $A_\eta := (P_{\eta(n)}AP_{\eta(n)})$, and associate with the sequence $A_\eta$ the block diagonal operator

$$
\text{Op}(A_\eta) := \text{diag}(P_{\eta(1)}AP_{\eta(1)}, P_{\eta(2)}AP_{\eta(2)}, P_{\eta(3)}AP_{\eta(3)}, \ldots)
$$

$$
= \sum_{n=1}^{\infty} U_{\mu(n)}P_{\eta(n)}AP_{\eta(n)}U_{-\mu(n)}
$$

acting on $l^p(Z^+)$ where $\mu(1) := 0$ and $\mu(n) := \eta(1) + \cdots + \eta(n-1)$ for $n \geq 2$, and where the series converges in the strong operator topology. It is still true that $\text{Op}(A_\eta)$ is a band-dominated operator on $l^p(Z^+)$ and that the sequence $A_\eta$ is stable if and only if the operator $\text{Op}(A_\eta)$ is Fredholm.

Let $h \in \mathcal{H}$ be a sequence which tends to infinity and for which the limit operator $\text{Op}(A_\eta)_h$ exists. We call numbers of the form $\eta(1) + \eta(2) + \cdots + \eta(n)$ $\eta$-triangular and distinguish between two cases: Either there is a subsequence $g$ of $h$ such that the distance from $g(n)$ to the set of all $\eta$-triangular numbers tends to infinity as $n \to \infty$, or there are a $k \in \mathbb{Z}$ and a subsequence $g$ of $h$ such that $g(n) + k$ is $\eta$-triangular for all $n$. The figure illustrates the shifted operator $U_{-g(n)}\text{Op}(A_\eta)U_{g(n)}$ in the neighborhood of its 00-entry (marked by 0).

In the first case, we let $\Delta_n$ denote the largest $\eta$-triangular number which is less than $g(n)$. Then $l(n) := g(n) - \Delta_n$ defines a sequence $l$ which tends to infinity, and the limit operator $\text{Op}(A_\eta)_h = \text{Op}(A_\eta)_g$ of $\text{Op}(A_\eta)$ coincides with the limit operator $A_l$ of $A$.

Let now $g$ be a subsequence of $h$ such that each $g(n) + k$ is $\eta$-triangular for some integer $k$. Then the sequence $l$ defined by $l(n) := g(n) + k$ tends to infinity, the limit operator of $\text{Op}(A_\eta)$ with respect to the sequence $l$ exists, and

$$
\text{Op}(A_\eta)_l = U_{-k}\text{Op}(A_\eta)_gU_k.
$$
Let \( d(n) \) be the (uniquely determined) positive integer such that 
\[
l(n) = \eta(1) + \eta(2) + \cdots + \eta(d(n)).
\]
The sequence \( d \) is strongly monotonically increasing. Thus, the sequence \( \eta \circ d \) is a subsequence of \( \eta \) and tends to infinity. Without loss we can assume that the limit operator of \( A \) with respect to the sequence \( \eta \circ d \) exists (otherwise we pass to a suitable subsequence of \( d \) and, hence, of \( l \) and \( g \)). Then 
\[
\text{Op}(A)_{\eta} = \text{Op}(A)_{\eta} + U_k \text{Op}(A)_{1} U_{-k} + U_k(Q A_{\eta \circ d} Q + P A P) U_{-k}.
\]
Thus, each limit operator of \( \text{Op}(A)_{\eta} \) is either a limit operator of \( A \) or of the form 
\[
U_k(Q A_{\eta \circ d} Q + P A P) U_{-k} \quad \text{with } k \in \mathbb{Z} \text{ and } A_{\eta \circ d} \in \sigma_{op, \eta}(A).
\] (2.1)
Next we are going to show that, conversely, each limit operator of \( A \) and each operator of the form (2.1) appears as a limit operator of \( \text{Op}(A)_{\eta} \). Let \( A_l \) be a limit operator of \( A \) with respect to a sequence \( l \in \mathcal{H} \). Choose a strongly monotonically increasing sequence \( d : \mathbb{N} \to \mathbb{N} \) such that \( \eta(d(n) + 1) - l(n) \to \infty \) and set 
\[
h(n) := (\eta(1) + \eta(2) + \cdots + \eta(d(n))) + l(n).
\]
Then \( h \in \mathcal{H} \), the limit operator \( \text{Op}(A)_{\eta} h \) exists, and it is equal to \( A_l \).

Let now \( d : \mathbb{N} \to \mathbb{N} \) be a strongly monotonically increasing sequence such that the limit operator \( A_{\eta \circ d} \) of \( A \) exists, and let \( k \in \mathbb{Z} \). Consider 
\[
h(n) := (\eta(1) + \eta(2) + \cdots + \eta(d(n))) + k.
\]
Again, \( h \in \mathcal{H} \), the limit operator \( \text{Op}(A)_{\eta} h \) exists, and now this limit operator is equal to \( U_{-k}(Q A_{\eta \circ d} Q + P A P) U_k \). Thus, 
\[
\sigma_{op}(\text{Op}(A)_{\eta}) = \sigma_{op}(A) \cup \{U_{-k}(Q A_{k} Q + P A P) U_k : k \in \mathbb{Z}, A_h \in \sigma_{op, \eta}(A)\}.
\]
This equality shows that the conditions of the theorem are necessary. They are also sufficient since the invertibility of \( A \) implies those of all limit operators of \( A \), and if both \( A \) and \( Q A_{k} Q + P \) are invertible then the operator \( U_{-k}(Q A_{k} Q + P A P) U_k \) is invertible for every integer \( k \).

**Corollary 2.2.** Let \( A \in L(l^{p}(\mathbb{Z}^{+})) \) be a band-dominated operator, and let \( \eta : \mathbb{N} \to \mathbb{N} \) be a strongly monotonically increasing sequence for which the limit operator \( A_{\eta} \) exists. Then the sequence \( (P_{\eta(n)} A P_{\eta(n)})_{n \geq 1} \) is stable if and only if the operators \( P A P + Q \) and \( Q A_{\eta} Q + P \) are invertible on \( l^{p}(\mathbb{Z}) \).

Indeed, under the conditions of the corollary, the set \( \sigma_{op, \eta}(A) \) is a singleton. Here is the announced main result of the present paper.
Theorem 2.3. Let $A \in L(l^p(\mathbb{Z}^+))$ be a band-dominated operator and $\eta : \mathbb{N} \rightarrow \mathbb{N}$ a strongly monotonically increasing sequence. Then the sequence $(P_{\eta(n)}AP_{\eta(n)})_{n \geq 1}$ is stable if and only if the operator $PA P + Q$ and all operators $QA_h Q + P$ with $A_h \in \sigma_{op, \eta}(A)$ are invertible on $l^p(\mathbb{Z})$.

Proof. The necessity of invertibility of the mentioned operators follows from Theorem 2.1. Conversely, let $PA P + Q$ and all operators $QA_h Q + P$ with $A_h \in \sigma_{op, \eta}(A)$ be invertible on $l^p(\mathbb{Z})$. Contrary to what we want to show, assume that the sequence $A^\eta = (P_{\eta(n)}AP_{\eta(n)})$ fails to be stable. Then there is a strongly monotonically increasing sequence $d : \mathbb{N} \rightarrow \mathbb{N}$ such that
\[
\| (P_{\eta(d(n))}AP_{\eta(d(n))})^{-1} \| \geq n \quad \text{for all } n \in \mathbb{N}
\]
where we agree upon writing $\| A_n^{-1} \| = \infty$ if the matrix $A_n$ fails to be invertible. Thus, no subsequence of the sequence $A^\eta_{odd}$ is stable.

Let $g$ be a subsequence of $\eta \circ d$ for which the limit operator $A_g$ exists. (The existence of a sequence $g$ with these properties follows from Theorem 1.1 (a).) Then $A_g \in \sigma_{op, \eta}(A)$, and the operators $PA P + Q$ and $QA_g Q + P$ are invertible by hypothesis. Corollary 2.2 implies that the subsequence $A_g$ of $A^\eta_{odd}$ is stable. Contradiction. $\square$

There is also a version of Theorem 2.3 for band-dominated operators on $l^p(\mathbb{Z})$ which we will briefly sketch. For $n \in \mathbb{N}$, consider the projections
\[R_n : l^p(\mathbb{Z}) \rightarrow l^p(\mathbb{Z}), \quad (R_n x)(m) := \begin{cases} x(m) & \text{if } -n \leq m < n \\ 0 & \text{otherwise.} \end{cases}\]
The finite sections sequence for an operator $A$ on $l^p(\mathbb{Z})$ is the sequence $(R_n AR_n)$ where the $R_n AR_n$ are viewed of as operators on $\text{im} R_n$, provided with the norm induced by the norm on $L(l^p(\mathbb{Z}))$. The stability of the sequence $(R_n AR_n)$ as well as the notion of a band-dominated operator on $l^p(\mathbb{Z})$ are defined as above, with obvious modifications.

Let again $\eta : \mathbb{N} \rightarrow \mathbb{N}$ be a strongly monotonically increasing sequence. In $\sigma_{+, \eta}(A)$ and $\sigma_{-, \eta}(A)$, we collect all limit operators of $A$ with respect to subsequences of $\eta$ and of $-\eta$, tending to $+\infty$ and $-\infty$, respectively. The following can be proved in the same vein as Theorem 2.3.

Theorem 2.4. Let $A \in L(l^p(\mathbb{Z}))$ be a band-dominated operator. Then the finite sections sequence $(R_n AR_n)_{n \geq 1}$ is stable if and only if the operator $A$, all operators $QA_h Q + P$ with $A_h \in \sigma_{+}(A)$ and all operators $PA_h P + Q$ with $A_h \in \sigma_{-}(A)$ are invertible on $l^p(\mathbb{Z})$. 

We would like to mention that the stability of the finite sections sequence for band-dominated operators on $l^\infty$ can be studied as well. This involves some technical subtleties (when working with adjoint sequences, for instance), but it is easier with respect to the concern of the present paper: Indeed, for $p = \infty$, the uniform boundedness condition in Theorem 1.1 (b) is already redundant. For much more on this topic, we refer to the recent textbook [1].

It remains an open question whether the uniform boundedness condition in Theorem 1.1 (b) is redundant for $p \in (1, \infty)$ or at least for $p = 2$.

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