Abstract. Let $G$ be a locally compact Abelian group. We denote by $L^{1}_{w}(G)$ the Beurling algebra on a locally compact Abelian group $G$, [15] and by $C^{0}_{w-1}(G)$ the space of all function $h$ such that $\frac{h}{w} \in C_{0}(G)$. It is known that $C^{0}_{w-1}(G)$ is a Banach space with the norm

$$\|f\|_{\infty,w} = \sup_{x \in G} \left| \frac{f(x)}{w(x)} \right|.$$ 

In [1], Cigler introduced a space $S_{w}(G)$ on $G$ which is a kind of generalization of the Segal algebras. In an earlier paper [10] we studied some properties and multipliers of this space. The present paper is a sequel to my paper [10]. In this paper first we discussed some properties of the space $C^{0}_{w-1}(G)$. Later we prove that every dense essential Banach ideal in $L^{1}_{w}(G)$ is an $S_{w}(G)$-space. We also discussed multipliers of some $L^{1}_{w}(G)$—convolution Banach modules.

**Key words**: Multipliers, Banach module, Banach ideal, Factorization, weighted $L^{p}(G)$—spaces

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0.1 Introduction.

Throughout this paper $G$ will denote a locally compact Abelian group (non-compact and non discrete) with dual group $\hat{G}$ having Haar measure $dx$ and $d\hat{x}$ respectively. $L^{p}(G), (1 \leq p \leq \infty)$ denote the usual Lebesgue spaces. Also $C_{c}(G)$ and $C_{0}(G)$ will be the space of all continuous, complex valued functions on $G$ with compact support, and the algebra of continuous complex valued functions on $G$ that vanish at infinity respectively. $M(G)$ will denote the space of bounded Radon measure on $G$.

We set for $1 \leq p \leq \infty$

$$L^{p}_{w}(G) = \{ f : fw \in L^{p}(G) \},$$

(1)

where $w$ is the Beurling’s weight function on $G$, i.e. $w$ is a continuous function satisfying $w \geq 1$ and $w(x + y) \leq w(x) \cdot w(y)$ for all $x, y \in G$. It is known that $L^{p}_{w}(G)$ is a Banach space under the norm $\|f\|_{p,w} = \|fw\|_{p}$. For $p = 1$, $L^{1}_{w}(G)$ is a Banach algebra under convolution, called Beurling algebra. We say that $w_{1} \leq w_{2}$ if and only if there exists a constant $C > 0$ such that $w_{1}(x) \leq C \cdot w_{2}(x)$ for all $x \in G$ [15].

Let $A$ be a Banach algebra and $V, W$ be Banach $A$-modules. Then a module homomorphism is a bounded linear operator $T$ from $V$ into $W$ which
commutes with the module multiplication i.e. \( T(a.b) = a.T(b) \) for all \( a \in A \) and \( b \in V \). We denote by \( \text{Hom}_A(V,W) \) the space of module homomorphism from \( V \) to \( W \). The elements of \( \text{Hom}_A(V,W) \) are traditionally called multipliers from \( V \) to \( W \). Sometimes this space is also denoted by \( M(V,W) \), [16]. Furthermore we denote the projective tensor product of \( V \) and \( W \) as Banach space by \( V \otimes_p W \).

Let \( K \) be the closed linear subspace of \( V \otimes \gamma W \) spanned by elements of the form \((f.g) \otimes h - g \otimes (f.h), f \in L^1_w(G), g \in V \) and \( h \in W \). Then the \( L^1_w \)-module tensor product \( V \otimes L^1_w W \) is the quotient Banach space \( V \otimes \gamma W / K \), [16]. It is known that any \( t \in V \otimes L^1_w W \) can be written in the form

\[
t = \sum_{k=1}^{\infty} g_k \otimes h_k, \quad g_k \in V, h_k \in W, \quad \text{where} \quad \sum_{k=1}^{\infty} ||g_k|| ||h_k|| < \infty.
\]

(2)

It is known that the space of multipliers of \( L^1_w(G) \) is isomorphic to the Banach space \((M(w), ||.||_w), [8], where

\[
M(w) = \left\{ \mu : \mu \in M(G), \int_G wd|\mu| < \infty \right\}
\]

(3)

and

\[
||\mu||_w = \int_G wd|\mu| < \infty.
\]

(4)

0.2 General Results.

Let \( G \) be a locally compact Abelian group. For any Beurling’s weight function \( w \) we denote by \( C^0_{w^{-1}}(G) \) the space of all function \( h \) such that \( \frac{h}{w} \in C_0(G) \). The space \( C^0_{w^{-1}}(G) \) is a Banach space with the norm

\[
||f||_{\infty,w} = \sup_{x \in G} \left|\frac{f(x)}{w(x)}\right|, \quad f \in C^0_{w^{-1}}(G).
\]

(5)

It is known that the dual space of \( \left( C^0_{w^{-1}}(G), ||.||_{\infty,w} \right) \) is \( M(w) \).

In [1], Cigler has given a kind of generalization of Segal algebra as follows:

Let \( S_w = S_w(G) \) be a subalgebra in \( L^1_w(G) \) satisfying the following conditions:

S1) \( S_w \) is dense in \( L^1_w(G) \).

S2) \( S_w \) is a Banach algebra under some norm \( ||.||_{S_w} \) and invariant under translations.

S3) \( ||L_a f||_{S_w} \leq w(a)||f||_{S_w} \) for all \( a \in G \) and for each \( f \in S_w \).

S4) Given any \( f \in S_w \), and \( \varepsilon > 0 \) there is a neighbourhood \( U \) of the unit element \( e \) of \( G \) such that \( ||L_y f - f||_{S_w} < \varepsilon \) for all \( y \in U \).

S5) \( ||f||_{1,w} \leq ||f||_{S_w} \) for all \( f \in S_w \).

If \( S(G) \) be a Segal algebra on \( G \) it is known that \( S(G) * C_0(G) \neq C_0(G) \) for some Segal algebras ( [13], Theorem 5). It is also discussed the necessary condition for \( S(G) * C_0(G) = C_0(G) \) in [11], [12] and [13]. Since every Segal algebra is a \( S_w(G) \)-space then the equality \( C^0_{w^{-1}}(G) = S_w(G) * C^0_{w^{-1}}(G) \) is not true in general.
Lemma 1 Let $K > 1$ is a bound for a bounded approximate identity in $L^1_w(G)$ and $(N, ||\cdot||_N)$ be an essential Banach convolution ideal in $L^1_w(G)$. If $T$ is a multiplier for $L^1_w(G)$ with the operator norm $\|T\|$ and $f \in N$ then 

$$\|Tf\|_N \leq K \|T\| \|f\|_N.$$ 

Proof. Let $f \in N$ and $\varepsilon > 0$ be given. Since $N$ is essential Banach ideal then by the Factorization Theorem (see Theorem 16.5 in [4]) there exists $g \in L^1_w(G)$ and $h \in N$ such that $f = g \ast h$, $\|g\|_{1,w} \leq K$ and 

$$\|h - f\|_N = \|h - g \ast h\|_N < \varepsilon. \quad (6)$$ 

Hence from (6) we have 

$$\|Tf\|_N = \|T(g \ast h)\|_N = \|h \ast T(g)\|_N \leq \|Tg\|_{1,w} \|h\|_N \leq K \|T\| \|h\|_N \leq K \|T\| (\|f\|_N + \varepsilon). \quad (7)$$

Since this inequality remains true for all $\varepsilon > 0$ the proof is completed.

It is proved in Proposition 2.2 in [10] that $S^1_w(G)$ is an essential Banach ideal in $L^1_w(G).$ Now we will prove the converse of this Proposition 2.2 in the following.

Theorem 2 Let $G$ be a locally compact Abelian group. If $N$ is a dense essential Banach convolution ideal in $L^1_w(G)$ then there exists a Beurling’s weight function $\omega$ on $G$ such that $w$ is equivalent to $\omega$ and $N$ is an $S^1_w$-space.

Proof. Let $(e_{\alpha}), \alpha \in I$ be an approximate identity in $L^1_w(G)$ bounded by $K > 1.$ Define a function $\omega(x) = Kw(x).$ It is easy to show that $\omega$ is another Beurling weight on $G$ and, $\omega$ and $w$ are equivalent. Therefore, $L^1_w(G) = L^1_w(G).$ Now, if we write $\delta_{a}$ for the point mass at $a \in G,$ we have $L_a f = \delta_{a} \ast f$ for $f \in N.$ Also thinking $\delta_{a}$ as multiplier of $L^1_w(G)$ and $\|\delta_{a}\|$ the operator norm of $\delta_{a},$ we have 

$$\|\delta_{a} \ast f\|_{1,w} = \|L_a f\|_{1,w} \leq w(a) \|f\|_{1,w} \quad (8)$$

and 

$$\|\delta_{a}\| = \sup \left\{ \frac{\|\delta_{a} \ast f\|_{1,w}}{\|f\|_{1,w}} : f \neq 0, f \in L^1_w(G) \right\} \quad (9)$$

$$= \sup \left\{ \frac{\|L_a f\|_{1,w}}{\|f\|_{1,w}} : f \neq 0, f \in L^1_w(G) \right\}$$

$$\leq \sup \left\{ \frac{w(a) \|f\|_{1,w}}{\|f\|_{1,w}} : f \neq 0, f \in L^1_w(G) \right\} = w(a).$$

Now, we show that $\|f\|_{1,w} \leq \|f\|_N.$ Let’s define $|f| = \|f\|_{1,w} + \|f\|_N$ on $N.$ One can prove that $(N, |\cdot|)$ is a Banach space. Hence, by the Closed Graph Mapping Theorem one can find $C > 0$ such that $\|f\|_{1,w} \leq C \|f\|_N$ for all $f \in N.$ Since the
norms $\| \cdot \|_N = C \| f \|_N$ and $\| f \|_N$ are equivalent, then we can use the norm $| \cdot |_N$ instead of $\| f \|_N$. Again, since the approximate identity $(e_\alpha)_\alpha \in I$ of $L^1_w(G)$ is bounded by $K > 1$ then by formula (9) and Lemma 1, we write

$$|L_\alpha f|_N = |\delta_\alpha * f|_N \leq K. \| \delta_\alpha \| \cdot |f|_N \leq Kw(\alpha) \cdot |f|_N = \omega(\alpha) \cdot |f|_N. \quad (10)$$

We have shown that $N$ satisfies S3. This also implies that $N$ is invariant under translation. Also, by the definition of Banach ideal $N$ satisfies (S5). Let $f, g \in N$. Since $N$ is an $L^1_w(G)$–Banach convolution ideal then we have

$$\| f * g \|_N \leq \| f \|_N \cdot \| g \|_{1,w} \leq \| f \|_N \cdot \| g \|_N.$$

It is easy to show the other conditions of to be Banach convolution algebra. Now, let $f \in N$ and $y \in G$ be given. Since $N$ is an essential Banach ideal then there exists $g \in L^1_w(G)$ and $h \in N$ such that $f = g * h$. Hence we write

$$\| L_y f - f \|_N = \| L_y (g * h) - g * h \|_N = \| (L_y g) * h - g * h \|_N \quad (11)$$

$$\leq \| (L_y g - g) * h \|_N \leq \| L_y g - g \|_{1,w} \| h \|_N.$$

It is known by Lemma 1.6 in [7] that the mapping $y \rightarrow L_y f$ of $G$ into $L^1_w(G)$ is continuous. Then given any $\varepsilon > 0$ there exists a neighbourhood $U$ of unit in $G$ such that

$$\| L_y g - g \|_{1,w} \leq \frac{\varepsilon}{\| h \|_N} \quad (12)$$

for all $y \in U$. By using (11) and (12) we obtain

$$\| L_y f - f \|_N \leq \| L_y g - g \|_{1,w} \| h \|_N < \frac{\varepsilon}{\| h \|_N} \cdot \| h \|_N = \varepsilon.$$

Thus $N$ satisfies (S4). This completes the proof.

The proof of the following Corollary 3 is clear by Theorem 2.

**Corollary 3** Since the weight function $\omega$ is equivalent to the weight function $w$, then we have $S_{w}(G) = S_{w}(G)$. That means $N$ is an $S_{w}$–space.

**Proposition 4** If $w$ is symmetric then $C^0_{w^{-1}}(G)$ is an essential Banach module over $L^1_w(G)$.

**Proof.** Let $f \in C^0_{w^{-1}}(G)$ and $g \in L^1_w(G)$. Since $\frac{f}{w} (Lt g) w \in L^1_w(G)$ then $f * g$ is well defined. Also since $w$ is symmetric we write

$$\left| \frac{f * g}{w} (x) \right| = \left| \int_G f(t) g(x-t) \frac{dt}{w(x)} \right| \leq \int_G \left| f(t) \right| \frac{w(t) g(x-t)}{w(x)} \frac{dt}{w(x)} \leq \| f \|_{\infty,w} \cdot \left| g \right|_{1,w}, \quad (13)$$

where $x - t = u$. Thus we have

$$\| f * g \|_{\infty,w} \leq \| f \|_{\infty,w} \| g \|_{1,w}. \quad (14)$$
Hence \( C^0_{w^{-1}}(G) \) is a \( L^1_w(G) \)-Banach module.

Now we show that it is a \( L^1_w(G) \)-essential Banach module. It is known that \( L^1_w(G) \) has a bounded approximate identity \( (e_\alpha) \), \( \alpha \in I \) with compactly supported [18]. Assume that \( \|e_\alpha\|_{1,w} \leq M \) for all \( \alpha \in I \) for some \( M > 0 \). Also it is easy to see that \( C_0(G) \) is dense in \( C^0_{w^{-1}}(G) \). Let \( h \in C^0_{w^{-1}}(G) \) and \( \varepsilon > 0 \) be given. There exists \( g \in C_0(G) \) such that

\[
\|f - g\|_{\infty,w} \leq \frac{\varepsilon}{3M} \tag{15}
\]

Also since \( L^1_w(G) \) is dense in \( L^1(G) \) then it is easy to show that \( (e_\alpha) \) is also a bounded approximate identity in \( L^1(G) \). It is also known that \( C_0(G) \) is essential Banach module over \( L^1(G) \), [10]. Then \( e_\alpha * g \in C_0(G) \subset C^0_{w^{-1}}(G) \) for all \( \alpha \in I \) and there exists \( \alpha_0 \in I \) such that

\[
\|e_\alpha * g - g\|_{\infty} \leq \frac{\varepsilon}{3M} \tag{16}
\]

for all \( \alpha > \alpha_0 \) by 15.4 Corollary in [4]. Hence from (14) we write

\[
\|e_\alpha * h - e_\alpha * g\|_{\infty,w} \leq \|e_\alpha\|_{1,w} \cdot \|h - g\|_{\infty,w} \leq M \cdot \|h - g\|_{\infty,w}. \tag{17}
\]

Also from (15), (16) and (17) we have

\[
\|e_\alpha \ast h - h\|_{\infty,w} \leq \|e_\alpha \ast h - e_\alpha \ast g\|_{\infty,w} + \|e_\alpha \ast g - g\|_{\infty,w} + \|h - g\|_{\infty,w} \\
\leq M \cdot \|h - g\|_{\infty,w} + \frac{\varepsilon}{3M} + \frac{\varepsilon}{3M} < \varepsilon
\]

for all \( \alpha > \alpha_0 \). Since \( e_\alpha \ast g \in L^1_w(G) \ast C^0_{w^{-1}}(G) \) then our assertion is proved by Corollary 15.3 in [14].

**Definition 5** Let \( V \) be a \( L^1_w(G) \)-convolution Banach module. We write \( S_w(G) \otimes V \) for the space of all \( t \in V \) for which there are sequences \( (g_k) \subset S_w(G) \), \( (h_k) \subset V \) with \( t = \sum_{k=1}^{\infty} g_k \ast h_k \) and \( \sum_{k=1}^{\infty} \|g_k\|_{S_w} \|h_k\|_V < \infty \).

It follows immediately from Theorem 6 of [17] that \( S_w(G) \otimes V \) is a Banach space with the norm

\[
\|t\| = \inf \left\{ \sum_{k=1}^{\infty} \|g_k\|_{S_w} \|h_k\|_V : t = \sum_{k=1}^{\infty} g_k \ast h_k \right\} \tag{18}
\]

and \( \|t\|_V \leq \|t\| \).

It is known that if \( V \) be a \( L^1_w(G) \) convolution Banach module then

\[
S_w(G) \otimes_{L^1_w} V \cong S_w(G) \otimes V \tag{19}
\]

the isomorphism being isometric, (Proposition 2.6 of [10]).

Since the dual of \( C^0_{w^{-1}}(G) \) is \( M(w) \), the proof of the following Theorem 6 and Corollary 7 is easy by Theorem 2.7 in [10].
Theorem 6 Let \( w \) be a symmetric weight function on a locally compact abelian group \( G \). Then following are equivalent:

1) \( S_w(G) \otimes C^0_{w^{-1}}(G) = C^0_{w^{-1}}(G) \).
2) \( \text{Hom}_{L^1_w}(S_w(G), M(w)) \cong M(w) \).

Corollary 7 Let \( w \) be a symmetric weight function on a locally compact abelian group \( G \). If

\[
\text{Hom}_{L^1_w}(S_w(G), M(w)) = \text{Hom}_{L^1_w}(S_w(G), S_w(G))
\]

then following are equivalent:

1) \( S_w(G) \otimes C^0_{w^{-1}}(G) = C^0_{w^{-1}}(G) \).
2) \( \text{Hom}_{L^1_w}(S_w(G), S_w(G)) \cong M(w) \).

Proposition 8 Let \( N \) be an \( L^1_w(G) \)-convolution module and suppose that

1) \( S_w(G) \ast N \subseteq C^0_{w^{-1}}(G) \).
2) There exists a constant \( B > 0 \) such that

\[
\|f \ast h\|_{\infty,w} \leq B \|f\|_{S_w} \|g\|_{N}
\]

for every \( f \in S_w(G), g \in N \). If \( S_w(G) \ast N = C^0_{w^{-1}}(G) \) then \( \text{Hom}_{L^1_w}(S_w(G), N^*) \cong M(w) \).

Proof. From the assumptions we write

\[
C^0_{w^{-1}}(G) = S_w(G) \ast N \subseteq S_w(G) \otimes N.
\]

Conversely take any \( u = \sum_{k=1}^{\infty} g_k \ast h_k \in S_w(G) \otimes N \). From the assumption (1) and (2) we have

\[
\|u\|_{\infty,w} \leq B \sum_{k=1}^{\infty} \|f\|_{S_w} \|g\|_{N} < \infty.
\]

Hence given \( \varepsilon > 0 \) there exists a natural number \( k_0 \) such that

\[
\left\| \sum_{k=k_0+1}^{\infty} f_k \ast g_k \right\|_{\infty,w} \leq \frac{\varepsilon}{2}.
\]

Also since \( f_i \ast g_i \in C^0_{w^{-1}}(G), (i = 1, 2, ..., k_0) \), then there exist compacts \( K_i, (i = 1, 2, ..., k_0) \) such that

\[
\left| f_i \ast g_i \right|_{w(x)} \leq \frac{\varepsilon}{2k_0}.
\]
for all $x \notin K_i, (i = 1, 2, \ldots, k_0).$ Now put $K = \bigcup_{i=1}^{\infty} K_i.$ Thus from (21) we write

$$
\left\| \frac{u(x)}{w(x)} \right\| \leq \sum_{k=1}^{k_0} \left( \frac{f_k * g_k}{w(x)} \right) + \sum_{k=k_0+1}^{\infty} \left( \frac{f_k * g_k}{w(x)} \right) \right) \quad (25)
$$

Thus

$$u \in C_0^{w-1}(G) \text{ and we have } S_w(G) \otimes N \subset C_0^{w-1}(G). \quad (26)$$

From (19) and (24) we obtain

$$S_w(G) \otimes N = C_0^{w-1}(G). \quad (27)$$

Also the unit function $i$ is continuous from $S_w(G) \otimes N$ onto $C_0^{w-1}(G)$ by (20) and (25). Then $i$ is a topological isomorphism by Closed Graph Mapping Theorem. Hence by Theorem 1.4 in Rieffel [16] and (17) we have

$$H_{om_{L_b}}(S_w(G), N^*) \cong (S_w(G) \otimes L_b^\prime N)^* \cong (S_w(G) \otimes N)^* \equiv (C_0^{w-1}(G))^* = M(w). \quad (28)$$

References


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