COMMUTING TOEPLITZ OPERATORS
WITH HARMONIC SYMBOLS

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This paper shows that on the Bergman space, two Toeplitz operators with harmonic symbols commute only in the obvious cases. The main tool is a characterization of harmonic functions by a conformally invariant mean value property.

We begin by recalling some standard notation and definitions. Let $dA$ denote the usual area measure on the open unit disk $D$ in the complex plane $\mathbb{C}$. The complex space $L^2(D, dA)$ is a Hilbert space with the inner product
\[ \langle f, g \rangle = \int_D f \overline{g} \, dA. \]

The Bergman space $L_a^2$ is the set of those functions in $L^2(D, dA)$ that are analytic on $D$ (the $a$ in $L_a^2$ stands for "analytic"). The Bergman space $L_a^2$ is a closed subspace of $L^2(D, dA)$, and so there is an orthogonal projection $P$ from $L^2(D, dA)$ onto $L_a^2$. For $\varphi \in L^\infty(D, dA)$, the Toeplitz operator with symbol $\varphi$, denoted $T_\varphi$, is the operator from $L_a^2$ to $L_a^2$ defined by $T_\varphi f = P(\varphi f)$. By a harmonic function we mean a complex-valued function on $D$ whose Laplacian is identically 0.

The following theorem is the main result of this paper.

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THEOREM 1: Suppose that \( \varphi \) and \( \psi \) are bounded harmonic functions on \( D \). Then

\[
T_\varphi T_\psi = T_\psi T_\varphi
\]

if and only if

1. \( \varphi \) and \( \psi \) are both analytic on \( D \),
2. \( \varphi \) and \( \psi \) are both analytic on \( D \),
3. there exist constants \( a, b \in \mathbb{C} \), not both 0, such that \( a\varphi + b\psi \) is constant on \( D \).

As we will see, the “if” direction of the above theorem is trivial, but the proof of the “only if” direction requires a converse to an invariant form of the mean value property.

The statement of Theorem 1 is similar to the analogous result proved by Arlen Brown and Paul Halmos ([3], Theorem 9) for Toeplitz operators with symbol in \( L^\infty(\partial D) \) acting on the Hardy space \( H^2(\partial D) \). Brown and Halmos proved their result by examining the matrix (with respect to the usual orthonormal basis for the Hardy space \( H^2(\partial D) \)) of products of Hardy space Toeplitz operators. On the Bergman space \( L^2_a(D) \), Toeplitz operators do not have nice matrices, and the techniques used by Brown and Halmos do not seem to work in this context. Thus function theory techniques rather than matrix manipulations, play a large role in our proof.

A special case of Theorem 1 was proved by Sheldon Axler and Pamela Gorkin ([2], Theorem 7), using function theory techniques quite different from those that we use here. Dechao Zheng ([5], Theorem 5) also proved a special case of Theorem 1; our proof makes use of some of his ideas.

Functions in \( L^\infty(\partial D) \) correspond, via the Poisson integral, to bounded harmonic functions on \( D \), so perhaps the restriction in Theorem 1 to consideration only of Toeplitz operators with harmonic symbols (as opposed to Toeplitz operators with symbol in \( L^\infty(D, dA) \)) is natural. More importantly, Theorem 1 does not hold if “harmonic” is replaced by “measurable”. For example, Paul Bourdon has pointed out to us that if \( \varphi \) and \( \psi \) are any two radial functions in \( L^1(D, dA) \), then \( T_\varphi T_\psi = T_\psi T_\varphi \) (a function is called radial if its value at \( z \) depends only on \( |z| \)). Thus the following open problem may be hard: Find conditions on functions \( \varphi \) and \( \psi \) in \( L^\infty(D, dA) \) that are necessary and sufficient for \( T_\varphi \psi \) to commute with \( T_\psi \).

The next section of this paper discusses an invariant mean value property that will be used in the proof of Theorem 1. The final section of the paper contains the proof of Theorem 1, along with a corollary that describes the normal Toeplitz operators with harmonic symbol.

THE INVARIANT MEAN VALUE PROPERTY

A continuous function on the disk \( D \) is harmonic if and only if it has the mean value property. In this section we characterize harmonic functions in terms of an invariant mean value property.

Let \( \text{Aut}(D) \) denote the set of analytic, one-to-one maps of \( D \) onto \( D \) ("\( \text{Aut} \)" stands for "Automorphism"). A function \( h \) on \( D \) is in \( \text{Aut}(D) \) if and only if there exist \( \alpha \in \partial D \) and \( \beta \in D \) such that

\[
h(z) = \alpha \frac{\beta - z}{1 - \beta \bar{z}}
\]

for all \( z \in D \).

A function \( u \in C(D) \) is said to have the invariant mean value property if

\[
\int_0^{2\pi} u(h(re^{i\theta})) \frac{d\theta}{2\pi} = u(h(0))
\]

for every \( h \in \text{Aut}(D) \) and every \( r \in [0, 1) \). Here "invariant" refers to conformal invariance, meaning invariance under composition with elements of \( \text{Aut}(D) \). If \( u \) is harmonic on \( D \), then \( u \) has the invariant mean value property. The converse is also true (see [4], Corollary 2 to Theorem 4.2.4): if a function \( u \in C(D) \) has the invariant mean value property, then \( u \) is harmonic on \( D \).

The invariant mean value property concerns averages over circles with respect to arc length measure. Because we are dealing with the Bergman space \( L^2_a(D) \), we need an invariant condition stated in terms of an area average over \( D \). Thus we say that a function \( u \in C(D) \cap L^1(D, dA) \) has the area version of the invariant mean value property if

\[
\int_D u \cdot h \, dA = u(h(0))
\]

for every \( h \in \text{Aut}(D) \). If \( u \) is in \( C(D) \cap L^1(D, dA) \), then so is \( u \cdot h \) for every \( h \in \text{Aut}(D) \), so the left-hand side of the above equation makes sense. Note that the area version of the invariant mean value property deals with integrals over all of \( D \), as opposed to integrals over \( rD \) for \( r \in (0, 1) \).

If \( u \) is harmonic on \( D \) and in \( L^1(D, dA) \), then so is \( u \cdot h \) for every \( h \in \text{Aut}(D) \). Thus, by the mean value property, harmonic functions have the area version of the invariant mean value property. Whether or not the converse is true is an open question. In other words, if \( u \in C(D) \cap L^1(D, dA) \) has the area version of the invariant mean value property, must \( u \) be harmonic? This question has an affirmative answer if we replace the hypothesis that \( u \) is in \( C(D) \cap L^1(D, dA) \) with the stronger hypothesis that \( u \) is in \( C(D) \); see [4], Proposition 13.4.4 or [1], Proposition 10.2.
We need to consider functions that are not necessarily continuous on the closed disk, so the result mentioned in the last sentence will not suffice. However, our functions do have the property that their radializations (defined below) are continuous on the closed disk, and we will prove that this property, along with the area version of the invariant mean value property, is enough to imply harmonicity; see Lemma 2.

If \( u \in C(D) \), then the radialization of \( u \), denoted \( \mathcal{R}(u) \), is the function on \( D \) defined by
\[
\mathcal{R}(u)(w) = \int_0^{2\pi} u(we^{i\theta}) \frac{d\theta}{2\pi}
\]
In the following lemma, which will be a key tool in our proof of Theorem 1, the statement \( \mathcal{R}(u) \in C(D) \) means \( \mathcal{R}(u) \) can be extended to a continuous complex-valued function on \( D \).

**Lemma 2:** Suppose that \( u \in C(D) \cap L^1(D, dA) \). Then \( u \) is harmonic on \( D \) if and only if
\[
\int_D u \, dA = u(h(0))
\]
and
\[
\mathcal{R}(u) \in C(D) \quad \text{for every } h \in \text{Aut}(D).
\]

**Proof:** We first prove the easy direction (the other direction will be the one that we need in the proof of Theorem 1). Suppose that \( u \) is harmonic on \( D \). Let \( h \in \text{Aut}(D) \). As we discussed earlier, \( uh \) is harmonic and so (3) holds. The mean value property implies that \( \mathcal{R}(uh) \) is a constant function on \( D \), with value \( u(h(0)) \), so (4) also holds.

To prove the other direction, suppose that (3) and (4) hold. Let \( h \in \text{Aut}(D) \), and let
\[
v = \mathcal{R}(uh).
\]
By (4), \( v \in C(D) \).

We want to show that \( v \) has the area version of the invariant mean value property. To do this, fix \( g \in \text{Aut}(D) \). Then
\[
\int_D v \, dA = \int_D \mathcal{R}(uh)(g(w)) \, dA(w) = \int_D \int_0^{2\pi} u(h(g(w)e^{i\theta})) \frac{d\theta}{2\pi} \, dA(w).
\]

To check that interchanging the order of integration in the last integral is valid, for each \( \theta \in [0, 2\pi] \) define \( f_\theta \in \text{Aut}(D) \) by
\[
f_\theta(w) = h(g(w)e^{i\theta}).
\]

The inverse (under composition) \( f_\theta^{-1} \) of \( f_\theta \) is also an analytic automorphism of \( D \), so there exist \( \alpha \in \partial D \) and \( \beta \in D \) such that
\[
f_\theta^{-1}(z) = \frac{\beta - z}{1 - \beta \bar{z}} \quad \text{for all } z \in D.
\]
Thus
\[
|f_\theta^{-1}(z)| \leq \frac{1 + |\beta|}{1 - |\beta|} \quad \text{for all } z \in D.
\]
Note that \( \beta = f_\theta(0) = h(g(0)e^{i\theta}) \); we are thinking of \( h \) and \( g \) as fixed, so the above inequality shows there is a constant \( K \) (depending only on \( h \) and \( g \)) such that
\[
|f_\theta^{-1}(z)| \leq K \quad \text{for all } z \in D \text{ and all } \theta \in [0, 2\pi].
\]
Now
\[
\int_0^{2\pi} \int_D |u(h(g(w)e^{i\theta}))| \, dA(w) \frac{d\theta}{2\pi} = \int_D \int_0^{2\pi} u(z) |f_\theta^{-1}(z)|^2 \, dA(z) \frac{d\theta}{2\pi}
\]
\[
\leq K^2 \int_D u(z) \, dA(z)
\]
\[
< \infty.
\]
Thus we can apply Fubini's Theorem to (5), getting
\[
\int_D v \, dA = \int_0^{2\pi} \int_D u(h(g(w)e^{i\theta})) \, dA(w) \frac{d\theta}{2\pi} = \int_0^{2\pi} \int_D u(f_\theta(0)) \, dA(w) \frac{d\theta}{2\pi} = \int_0^{2\pi} u(h(g(0)e^{i\theta})) \, d\theta = \mathcal{R}(uh)(g(0))
\]
\[
= v(g(0)).
\]
Thus \( v \) is a continuous function on \( \overline{D} \) that has the area version of the invariant mean value property. Hence (see [4], Proposition 13.4.4 and Remark 4.1.4 or [1], Proposition 10.2, combined with a change of variables) \( v \) is harmonic on \( D \). Because \( v \) is also a radial function, the mean value property implies that \( v \) is a constant function on \( D \), with value \( v(0) \).

Recall that \( v = \mathcal{R}(uh) \), so
for every \( r \in [0, 1) \) and for each \( h \in \text{Aut}(D) \). In other words, \( u \) has the invariant mean value property (the usual version, not the area version). Thus (see [4], Corollary 2 to Theorem 4.2.4 and Remark 4.1.4) \( u \) is harmonic on \( D \). ■

As mentioned earlier, it is unknown whether Lemma 2 remains true if (4) is deleted. We believe that the following proposition, which reduces this question to a tempting integral equation, is the best way to attack this problem. Patrick Ahern and Walter Rudin also independently proved Lemma 2 and Proposition 6 (with similar proofs) at about the same time we did.

**Proposition 6:** Suppose that the constant functions are the only functions \( v \in C([0, 1]) \cap L^1([0, 1]) \) such that
\[
V(t) = (1 - t)^2 \int_0^1 \frac{1 + ts}{(1 - ts)^3} V(s) \, ds \quad \text{for every } t \in [0, 1).
\]
Then every function in \( C(D) \cap L^1(D, dA) \) having the area version of the invariant mean value property is harmonic.

**Proof:** First suppose that \( v \) is a radial function in \( C(D) \cap L^1(D, dA) \) having the area version of the invariant mean value property. We will show that \( v \) is constant on \( D \). For \( \alpha \in [0, 1) \), let \( h_\alpha \in \text{Aut}(D) \) be defined by
\[
h_\alpha(z) = \frac{\alpha - z}{1 - \alpha z}.
\]

Note that \( h_\alpha \) is its own inverse under composition. For each \( \alpha \in [0, 1) \) we have
\[
v(\alpha) = \int_D v(h_\alpha)(z) \, dA(z) = \int_D v(w) \, |h_\alpha'(w)|^2 \, dA(w)
= (1 - \alpha^2)^2 \int_0^1 v(r) \, \int_0^1 \frac{1 + 2\alpha^2r^2}{(1 - \alpha^2r^2)^3} \, ds \, dr
= (1 - \alpha^2)^2 \int_0^1 \frac{1 + \alpha^2s^2}{(1 - \alpha^2s^2)^3} \, v(\sqrt{s}) \, ds.
\]
In the above equation, replace \( \alpha \) with \( \sqrt{t} \) and define a function \( V \) on \( [0, 1) \) by \( V(t) = v(\sqrt{t}) \), transforming the above equation into (7). Hence \( V \) is constant on \( [0, 1) \), and thus so is \( v \), as claimed.

Let \( H^0(D) \) denote the usual Hardy space on the disk. It is well known that \( H^1(D) \subset L^2 \) (the proof follows easily from Hardy’s inequality). In the proof of Theorem 1 we will use, without comment, the following consequence: If \( f, g \in H^2(D) \), then \( fg \in L^2 \), and thus \( \int f \bar{g} \in L^2(D, dA) \).

We have now assembled all the ingredients needed to prove Theorem 1.
PROOF OF THEOREM 1: We begin with the easy direction. First suppose that (1.1) holds, so that \( \varphi \) and \( \psi \) are analytic on \( D \), which means that \( T_\varphi \) and \( T_\psi \) are, respectively, the operators on \( L^2_D \) of multiplication by \( \varphi \) and \( \psi \). Thus \( T_\varphi T_\psi = T_\psi T_\varphi \).

Now suppose that (1.2) holds, so that \( \varphi \) and \( \psi \) are analytic on \( D \). By the paragraph above, \( T_\varphi T_\psi = T_\psi T_\varphi \). Take the adjoint of both sides of this equation, and use the identity \( T_\varphi^* = T_\varphi \) to conclude that \( T_\varphi T_\psi = T_\psi T_\varphi \).

Finally (for the easy direction) suppose that (1.3) holds, so there exist constants \( a, b \in C \), not both \( 0 \), such that \( \varphi = a \psi + b \) on \( D \). Let \( h \in \text{Aut}(D) \). Then \( T_\varphi T_\psi = T_\psi T_\varphi \) because \( \psi \) and \( \psi \) commute.

To prove the other direction of Theorem 1, suppose now that \( \varphi \) and \( \psi \) are bounded harmonic functions on \( D \) such that \( T_\varphi T_\psi = T_\psi T_\varphi \). Because \( \varphi \) and \( \psi \) are harmonic on \( D \), there exist functions \( f_1, f_2, g_1, \) and \( g_2 \) such that
\[
\varphi = f_1 + f_2 \quad \text{and} \quad \psi = g_1 + g_2 \quad \text{on} \quad D. \tag{10}
\]

Because \( \varphi \) and \( \psi \) are bounded on \( D \), the functions \( f_1, f_2, g_1, \) and \( g_2 \) must be in \( H^2(D) \).

Let \( 1 \) denote the constant function 1 on \( D \). Then
\[
T_\varphi T_\psi 1 = T_\psi (P_\varphi 1) = P_\varphi (T_\psi 1) = P_\varphi (g_1 + g_2).
\]

Using the identities \( T_\varphi = P_\varphi - \beta V \) and \( T_\psi = P_\psi - \gamma V \) (where \( \beta \) and \( \gamma \) are constants), we obtain
\[
T_\varphi T_\psi 1 = (P_\varphi - \beta V)(P_\psi - \gamma V) 1 = \gamma P_\varphi 1 = \gamma T_\psi 1.
\]

Thus
\[
<T_\varphi T_\psi 1, 1> = <f_1 g_1 + f_2 g_2, f_1 + f_2 g_2> = \int_D f_1 g_1 + f_2 g_2 da. \tag{11}
\]

A similar formula (interchanging the \( f \)'s and the \( g \)'s) can be obtained for \( <T_\psi T_\varphi 1, 1> \).

Because \( T_\varphi T_\psi = T_\psi T_\varphi \), we can set the right-hand side of (11) equal to the corresponding formula for \( <T_\psi T_\varphi 1, 1> \), getting
\[
\int_D f_1 g_1 + f_2 g_2 da = \int_D f_1 g_1 + f_2 g_2 da. \tag{12}
\]

Let \( h \in \text{Aut}(D) \). Multiplying both sides of the equation \( T_\varphi T_\psi = T_\psi T_\varphi \) by \( U_h \) on the left and by \( U_h^* \) on the right, and recalling that \( U_h^* U_h = I \), we get
\[
U_h T_\varphi U_h^* U_h T_\psi U_h^* = U_h T_\psi U_h^* U_h T_\varphi U_h^*. \tag{13}
\]

Lemma 8 now shows that
\[
T_{\varphi h} T_{\psi h} = T_{\psi h} T_{\varphi h}. \tag{14}
\]

Composing both sides of the equations in (10) with \( h \) expresses each of the bounded harmonic functions \( \varphi h \) and \( \psi h \) as the sum of an analytic function and a conjugate analytic function:
\[
\varphi h = f_1 h + f_2 h \quad \text{and} \quad \psi h = g_1 h + g_2 h \quad \text{on} \quad D. \tag{15}
\]

Equation (12) was derived under the assumption that \( T_\varphi T_\psi = T_\psi T_\varphi \); thus (13), combined with (14), says that (12) is still valid when we replace each function in it by its composition with \( h \). In other words,
\[
\int_D (f_1 g_1 + f_2 g_2) h d\alpha = \int_D (f_1 h)(g_1 h) + (f_2 h)(g_2 h) d\alpha.
\]

Lifting
\[
u = \frac{f_2 g_1 - f_1 g_2}{2},
\]
the above equation becomes
\[
\int_D u h d\alpha = u(h(0)).
\]

In other words, \( u \) has the area version of the invariant mean value property.

We want to show that \( u \) is harmonic on \( D \). By the above equation and Lemma 2, we need only show that \( R(u h) \in C(D) \). To do this, represent the analytic functions \( f_2 h \) and \( g_1 h \) as Taylor series:
\[
(f_2 h)(z) = \sum_{n=0}^\infty \alpha_n z^n \quad \text{and} \quad (g_1 h)(z) = \sum_{n=0}^\infty \beta_n z^n \quad \text{for all} \quad z \in D.
\]

Because \( \varphi h \) and \( \psi h \) are bounded harmonic functions on \( D \), (14) implies that \( f_2 h \) and \( g_1 h \) are in \( H^2(D) \), so
\[
\sum_{n=0}^\infty |\alpha_n|^2 < \infty \quad \text{and} \quad \sum_{n=0}^\infty |\beta_n|^2 < \infty. \tag{15}
\]

Now for \( z \in D \) we have
\[
R((f_2 g_1 - f_1 g_2) h)(z) = \int_0^{2\pi} (f_2 h)(z e^{i\theta}) (g_1 h)(z e^{i\theta}) \frac{d\theta}{2\pi} = \sum_{n=0}^\infty \alpha_n \beta_n |z^n|^2.
\]
Thus, because derivatives and that desired. Thus at this stage of the proof we know that holds. If is analytic, then the Cauchy-Riemann equations show that would be constant on . Hence the inequalities in (15) imply that 

\[ 0 = 4 \frac{\partial^2}{\partial z^2} \left( \frac{\partial u}{\partial z} \right) = 4 \frac{\partial^2}{\partial z^2} \left( \frac{\partial f_1 g_1 - f_1 g_2}{\partial z} \right) = 4 \frac{\partial}{\partial z} \left( T_1 g_1 - f_1 g_2 \right) \]

on . Hence

\[ f_1 g_2 = f_1 g_1' \quad \text{on } D. \]  

We finish the proof by showing that the above equation implies that (1.1), (1.2), or (1.3) holds. If \( g_1' \) is identically \( 0 \) on \( D \), then (16) shows that either \( g_2' \) is identically \( 0 \) on \( D \) (so \( \psi \) would be constant on \( D \) and (1.3) would hold) or \( f_1' \) is identically \( 0 \) on \( D \) (so both \( \bar{\psi} \) and \( \bar{\psi} \) would be analytic on \( D \) and (1.2) would hold). Similarly, if \( g_2' \) is identically \( 0 \) on \( D \), then (16) shows that either (1.3) or (1.1) would hold. Thus we may assume that neither \( g_1' \) nor \( g_2' \) is identically \( 0 \) on \( D \), and so (16) shows that

\[ f_1' g_1' = \frac{f_2' g_2'}{g_2} \quad \text{at all points of } D \]

except the countable set consisting of the zeroes of \( g_1' g_2' \). The left-hand side of the above equation is an analytic function (on \( D \) with the zeroes of \( g_1' g_2' \) deleted), and the right-hand side is the complex conjugate of an analytic function on the same domain, and so both sides must equal a constant \( c \in \mathbb{C} \). Thus \( f_1' = c g_1' \) and \( f_2' = \bar{c} g_2' \) on \( D \). Hence \( f_1 - c g_1 \) and \( f_2 - c g_2 \) are constant on \( D \), and so their sum, which equals \( \psi - c \psi \), is constant on \( D \); in other words, (1.3) holds and the proof of Theorem 1 is complete. 

Recall that an operator is called normal if it commutes with its adjoint. We can use Theorem 1 to prove the following corollary, which states that for \( \varphi \) a bounded harmonic function on \( D \), the Toeplitz operator \( T_\varphi \) is normal only in the obvious case.

**Corollary 17:** Suppose that \( \varphi \) is a bounded harmonic function on \( D \). Then \( T_\varphi \) is a normal operator if and only if \( \varphi(D) \) lies on some line in \( C \).

**Proof:** First suppose that \( \varphi(D) \) lies on some line in \( C \). Then there exist constants \( \alpha, \beta \in \mathbb{C} \), with \( \alpha \neq 0 \), such that \( \alpha \varphi + \beta \) is real valued on \( D \). Thus \( T_\varphi \alpha \varphi + \beta \) is a self-adjoint operator, and hence \( T_\varphi \varphi \) which equals \( \alpha^{-1}(\alpha \varphi + \beta - \beta) \) is a normal operator. 

To prove the other direction, suppose now that \( T_\varphi \) is a normal operator. Thus \( T_\varphi T_\varphi^* = T_\varphi^* T_\varphi \) and so Theorem 1 implies that \( \varphi \) and \( \varphi \) are both analytic on \( D \) (in which case \( \varphi \) is constant, so we are done) or there are constants \( a, b \in \mathbb{C} \) not both \( 0 \), such that \( a \varphi + b \varphi \) is constant on \( D \). The latter condition implies that \( \varphi(D) \) lies on a line. 

We conclude with a question. Does Theorem 1 remain true if we replace the disk by an arbitrary connected region in the plane? (So the Toeplitz operators now act on the Bergman space of this new region.) The statement of Theorem 1 certainly makes sense in this context and is plausible. However, the proof of Theorem 1 made extensive use of the set of analytic automorphisms of the disk. This approach will not work in general, because non-simply-connected regions have too few analytic automorphisms to provide any useful information.

**References**