DECOMPOSITIONS OF FUNCTIONS AS SUMS OF ELEMENTARY FUNCTIONS

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[Received 8 January 1985]

The purpose of this article is to demonstrate the usefulness of Banach’s closed range theorem for proving decomposition theorems. The author learnt this method from D. H. Luecking, who has used it ([5], Prop. 4.5) to give an elegant proof of the Coifman-Rochberg decomposition theorem for the Bergman space \( L^1_\alpha \) [2]. After a few decomposition theorems had been proved in this way, it became apparent that it would be economical to express the method as a decomposition theorem for vectors in a Banach space. This is done in Theorem 1, in which certain inequalities involving the norms of the elementary vectors and of linear functionals give the decomposition with corresponding inequalities between norms. The proof of the theorem is no more than an exercise on Banach’s closed range theorem.

In Corollary 2, we show that, if \( \{z_k\} \) is a suitably chosen sequence of points in the open unit disc \( D \), then every integrable function on the circle \( \partial D \) can be decomposed as an \( l^1 \)-sum of Poisson kernel functions corresponding to the points \( z_k \). Many analogues and generalizations of this corollary are available, and in particular Corollary 3 is a decomposition theorem for \( L^1(G) \), with \( G \) a locally compact group, in terms of elements of an approximate identity. Corollary 4 is Fefferman’s atomic decomposition theorem for \( \text{Re}H^1 \) in terms of simple atoms, that is atoms taking only two values. In Corollary 5, we show that a suitably chosen sequence of rank one operators suffices to decompose all trace class operators.

Notation. Given a set \( \Delta \), \( l^1(\Delta) \) denotes as usual the Banach space of complex valued functions \( \lambda \) on \( \Delta \) for which

\[
\|\lambda\|_1 = \sum_{\delta \in \Delta} |\lambda(\delta)| < \infty.
\]

Theorem 1. Let \( X \) be a Banach space, let \( M_1, M_2 \) be positive constants, and let \( u \) be a mapping of a set \( \Delta \) into \( X \) such that

(i) \( \|u(\delta)\| \leq M_1 \) (\( \delta \in \Delta \)),

(ii) \( \sup \{ |\psi(u(\delta))| : \delta \in \Delta \} \geq M_2 \|\psi\| \) (\( \psi \in X^* \)).
Then every \( f \in X \) is of the form
\[
f = \sum_{\delta \in \Delta} \lambda(\delta)u(\delta)
\] (1)
with \( \lambda \in l^1(\Delta) \), and also
\[
M_2 \inf \| \lambda \|_1 \leq \| f \| \leq M_1 \inf \| \lambda \|_1,
\] (2)
with the infimum taken over all decompositions (1) of \( f \).

Proof. Let \( T \) be the bounded linear mapping of \( l^1(\Delta) \) into \( X \) defined by
\[
T\lambda = \sum_{\delta \in \Delta} \lambda(\delta)u(\delta).
\]
Given \( \varepsilon \in \Delta \), define \( e_\varepsilon \) on \( \Delta \) by \( e_\varepsilon(\varepsilon) = 1 \), \( e_\varepsilon(\delta) = 0 \) (\( \delta \neq \varepsilon \)). Then \( Te_\varepsilon = u(\varepsilon) \), and so, for all \( \psi \in X^* \), we have
\[
M_2 \| \psi \| \leq \sup \{ |\psi(u(\delta))| : \delta \in \Delta \}
= \sup \{ |\psi(Te_\varepsilon)| : \varepsilon \in \Delta \} \leq \| T^*\psi \|.
\] (3)
Thus \( T^* \) has closed range and zero kernel, and therefore, by Banach's closed range theorem ([1] p. 150), \( Tl^1(\Delta) = X \), every \( f \in X \) is of the form (1).

Let \( N = \ker T \), \( Y = l^1(\Delta)/N \), and define \( S \) by \( Sy = T\lambda (\lambda \in y \in Y) \). Then \( S \) is a bounded linear bijection of \( Y \) onto \( X \), and therefore has a bounded inverse. It follows that \( S^* \) is a bounded linear bijection of \( X^* \) onto \( Y^* \). By (3), we have, for all \( \psi \in X^* \),
\[
M_2 \| \psi \| \leq \| T^*\psi \| = \sup \{ |\psi(T\lambda)| : \| \lambda \|_1 \leq 1 \}
= \sup \{ |\psi(Sy)| : \| y \| \leq 1 \} = \| S^*\psi \|.
\]
Therefore \( \| S^{-1} \| = \| (S^*)^{-1} \| \leq M_2^{-1} \), and the inequalities (2) are now clear.

Our first corollary is exceedingly elementary and could have been proved with ease at any time since 1932. However, to our surprise, it does not seem to be known. I am indebted to R. Rochberg for a helpful comment which has substantially improved the result.

Notation. Let \( D = \{ z \in \mathbb{C} : |z| < 1 \} \), \( \partial D = \{ z \in \mathbb{C} : |z| = 1 \} \), and, for \( z \in D \), \( \xi \in \partial D \), let
\[
p_z(\xi) = (1 - |z|^2)/|1 - z\xi|^2,
\] so that \( p_z(e^{i\theta}) \) is the Poisson kernel \( P_z(\theta) \). For \( 1 \leq p \leq \infty \), the space \( L^p(\partial D) \) is defined in terms of the normalized Lebesgue measure \( (2\pi)^{-1} \, d\theta \).

Corollary 2. Let \( \{ z_k \} \) be a sequence of points of \( D \) such that almost every point of \( \partial D \) is the non-tangential limit of a subsequence of \( \{ z_k \} \).
Then $L^1(\partial D)$ is the set of all $f$ of the form

$$f = \sum_{k=1}^{\infty} \lambda_k p_{z_k}$$

with $\lambda_k \in \mathbb{C}$ and $\sum_{k=1}^{\infty} |\lambda_k| < \infty$. Also

$$\|f\|_1 = \inf \sum_{k=1}^{\infty} |\lambda_k|,$$

with the infimum taken over all decompositions (4) of $f$.

Proof. Given $g \in L^\infty(\partial D)$, we have

$$[p_z, g] = \frac{1}{2\pi} \int_{0}^{2\pi} g(e^{i\theta}) p_z(e^{i\theta}) \, d\theta = g(z) \quad (z \in D),$$

where $g(z)$ denotes the harmonic extension of $g$. We have $\|p_z\|_1 = 1$; and, by Fatou's theorem (see Hoffman [3], p. 38), $g(\zeta)$ is the limit of $g(z)$ as $z \to \zeta$ non-tangentially for almost all $\zeta \in \partial D$. Therefore

$$\|g\|_1 = \sup \{\|p_{z_k}, g\|: k \in \mathbb{N}\},$$

and the usual identification of the dual space of $L^1(\partial D)$ with $L^\infty(\partial D)$ gives

$$\|\psi\| = \sup \{\|\psi(p_{z_k})\|: k \in \mathbb{N}\}$$

for all $\psi \in (L^1(\partial D))^*$. Thus Theorem 1 applies with $M_1 = M_2 = 1$.

Remark. A similar corollary holds for $L^1(R)$ with $D$ replaced by the open upper half plane $U$, and with

$$p_z(t) = \frac{1}{\pi} \text{Im} \frac{z}{|z-t|^2} \quad (z \in U, t \in \mathbb{R}).$$

The Poisson kernel can be replaced by other positive kernels. A further generalization is given in Corollary 3.

Since $p_z(\zeta) = (1 - \bar{z}\zeta)^{-1} + z\bar{\zeta}(1 - z\bar{\zeta})^{-1}$, the Fourier series of $p_z$ is

$$1 + \sum_{n=1}^{\infty} (\bar{z}^n e^{in\theta} + z^n e^{-in\theta}).$$

Therefore every Fourier series can be decomposed as an $l^1$-sum of the Fourier series of this form with $z = z_k$. A similar result holds for Fourier transforms.

Notation. Let $G$ be a locally compact group, and let $L^1(G)$ denote the group algebra with respect to a left invariant Haar integral (Loomis [4]).
Thus, for \( f, g \in L^1(G) \), the product \( f \ast g \) is given by
\[
(f \ast g)(x) = \int f(y)g(y^{-1}x) \, dy.
\]

Let \( \{ v_\gamma : \gamma \in \Gamma \} \) be a right approximate identity of unit vectors in \( L^1(G) \); that is \( \Gamma \) is a directed set, \( \|v_\gamma\|_1 = 1 \), and, given \( f \in L^1(G) \) and \( \varepsilon > 0 \), there exists \( \gamma_0 \in \Gamma \) such that \( \|f - f \ast v_\gamma\|_1 < \varepsilon \) whenever \( \gamma \geq \gamma_0 \).

Let \( \Delta = \Gamma \times G \), and for \( \delta = (\gamma, y) \in \Delta \), let
\[
u_\delta(x) = v_\gamma(y^{-1}x) \quad (x \in G).
\]

**Corollary 3.** Each \( f \in L^1(G) \) is of the form
\[
f = \sum_{\delta \in \Delta} \lambda(\delta)u_\delta
\]
with \( \lambda \in l^1(\Delta) \), and
\[
\|f\|_1 = \inf \|\lambda\|_1
\]
with the infimum taken over all such decompositions of \( f \).

**Proof.** Let \( h \in L^\infty(G) \). By left invariance of the Haar integral, we have \( \|u_\delta\|_1 = 1 \), and so it is enough to prove that
\[
\|h\|_\infty \leq \sup \{\|[u_\delta, h]\| : \delta \in \Delta\}.
\]  
(5)

For \( g \in L^1(G) \), let \( g^\dagger(x) = g(x^{-1}) \). Then, for \( f, g \in L^1(G) \), we have
\[
[f \ast g, h] = \int \left\{ \int f(y)g(y^{-1}x) \, dy \right\} h(x) \, dx
\]
\[
= \int f(y) \left\{ \int h(x)g(y^{-1}x) \, dx \right\} dy
\]
\[
= [f, h \ast g^\dagger].
\]

Given \( \varepsilon > 0 \), we choose \( f \in L^1(G) \) with \( \|f\|_1 = 1 \) and
\[\|[f, h]\| > \|h\|_\infty - \varepsilon,\]
and then choose \( \gamma_0 \) such that, for \( \gamma \geq \gamma_0 \),
\[\|[f \ast v_\gamma, h]\| > \|h\|_\infty - \varepsilon.\]
Then for \( \gamma \geq \gamma_0 \),
\[
\|h \ast v_\gamma\|_\infty \geq \|[f, h \ast v_\gamma]\|
\]
\[
= \|[f \ast v_\gamma, h]\| > \|h\|_\infty - \varepsilon.
\]
But
\[ [u(y,y), h] = \int h(x)v_y(y^{-1}x) \, dx \]
\[ = \int h(x)v_y(x^{-1}y) \, dx = (h * v_y)(y). \]

Therefore, for \( \gamma \geq \gamma_0 \),
\[ \sup \{ [u(y,y), h] : y \in G \} > \| h \|_\infty - \epsilon, \]
and the inequality (5) is proved.

Our next corollary is Fefferman's atomic decomposition of \( \text{Re}H^1 \).
Several authors have remarked that this follows easily from Fefferman's duality theorem, but without indicating a method. The present proof gives the result with atoms of a special kind that we call simple atoms.

**Notation.** \( L^p_R \) will denote the real valued functions in \( L^p(\partial D) \), and \( \text{Re}H^1 \) the set of \( f \in L^1_R \) such that the harmonic conjugate \( \tilde{f} \) also belongs to \( L^1_h \). (Plainly, \( \text{Re}H^1 \) is also the set of real parts of functions in \( H^1 \).) With the norm given by
\[ \| f \| = \| f \|_1 + \| \tilde{f} \|_1, \]
\( \text{Re}H^1 \) becomes a Banach space. Given \( f \in L^1_R \) and an arc \( I \subset \partial D \) with length \( |I| \neq 0 \),
\[ I(f) = |I|^{-1} \int_I f(e^{i\theta}) \, d\theta. \]
Then \( \text{BMO} \) denotes the space of functions \( f \in L^1_R \) with bounded mean oscillation, that is with
\[ \| f \|_* = \sup_I I(|f - I(f)|) < \infty. \]
The norm \( \| f \|_{\text{BMO}} \) is defined by \( \| f \|_{\text{BMO}} = \| f \|_* + |\tilde{f}(0)| \).

A simple atom (on an arc \( I \)) is a function \( a \) of the form
\[ a = |I|^{-2} (|Q|\chi_P - |P|\chi_Q), \]
where \( P, Q \) are measurable subsets of \( I \) with \( P \cup Q = I \), \( P \cap Q = \emptyset \), and \( \chi_E \) denotes the characteristic function of a set \( E \). Let \( A \) denote the set of all simple atoms together with the function on \( \partial D \) with the constant value \((2\pi)^{-1}\).

**Corollary 4.** \( \text{Re}H^1 \) is the set of functions \( f \) of the form
\[ f = \sum_{k=1}^{\infty} \lambda_k a_k, \]
with \( \lambda_k \in \mathbb{R}, \sum_{k=1}^{\infty} |\lambda_k| < \infty, \) and \( a_k \in A \).
Proof. By Fefferman’s duality theorem, there exists a positive constant C such that, given \( \psi \in (\text{ReH}^1)^* \), there exists \( g \in \text{BMO} \) with

\[
\psi(f) = [f, g] \quad (f \in \text{ReH}^1), \quad \|g\|_{\text{BMO}} \leq C \|\psi\|.
\]

We note that if \( a \) is a simple atom on \( I \) and \( g \in \text{BMO} \), then

\[
[a, g] = (2\pi)^{-1} \int_I a(e^{i\theta})(g(e^{i\theta}) - I(g)) \, d\theta;
\]

and therefore

\[
\|a, g\| \leq (2\pi)^{-1} \|g\|_{\text{BMO}} (a \in A, g \in \text{BMO}).
\]

By (6), (8) and the Hahn–Banach theorem,

\[
\|a\| \leq (2\pi)^{-1} C \quad (a \in A).
\]

By taking \( \Delta = A \) and \( u(a) = a \), we obtain a mapping of \( \Delta \) into a bounded subset of \( \text{ReH}^1 \), and it now suffices to prove the inequality

\[
\|g\|_{\text{BMO}} \leq 6\pi \sup \{\|a, g\|: a \in A\} \quad (g \in \text{BMO}).
\]

Let \( g \in \text{BMO} \), let \( I \) be an arc in \( \partial D \), and let

\[
a = |I|^{-2} \{ |Q| \chi_P - |P| \chi_Q \},
\]

where

\[
P = \{ \theta \in I: g(e^{i\theta}) - I(g) \geq 0 \}, \quad Q = \{ \theta \in I: g(e^{i\theta}) - I(g) < 0 \}.
\]

We note that

\[
\int_P |g - I(g)| \, d\theta = \int_Q |g - I(g)| \, d\theta,
\]

and therefore, by (7),

\[
[a, g] = (2\pi)^{-1} |I|^{-2} \left\{ |Q| \int_P |g - I(g)| \, d\theta + |P| \int_Q |g - I(g)| \, d\theta \right\}
\]

\[
= (4\pi)^{-1} I(|g - I(g)|),
\]

and the inequality (9) follows easily.

Notation. Let \( \mathcal{C}_1 \) denote the Banach space of all trace class operators on a Hilbert space \( H \). It is well known (see Ringrose [6] p. 99) that the dual space \((\mathcal{C}_1)^*\) is isometrically isomorphic to the space \( BL(H) \) of all bounded linear operators by way of the mapping \( T \mapsto f_T: BL(H) \to (\mathcal{C}_1)^* \), where

\[
f_T(S) = \text{tr} (ST) \quad (S \in \mathcal{C}_1).
\]

Corollary 5. Let \( \eta > 0 \), let \( \Gamma \) be a set of unit vectors in \( H \) such that every unit vector \( x \) satisfies \( \|x - v\| \leq \eta \) for some \( v \in \Gamma \), and let \( \Delta = \Gamma \times \Gamma \).
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(i) If \( \eta \leq \frac{1}{4} \), then every \( S \in C_1 \) is of the form
\[
S = \sum_{\lambda \in \Gamma} \lambda(v) v \otimes v
\]
with \( \lambda \in l^1(\Gamma) \), and
\[
\|S\|_1 \geq (\frac{1}{2} - 2\eta) \inf \|\lambda\|_1.
\]

(ii) If \( \eta < \frac{2}{3} - 1 \), then every \( S \in C_1 \) is of the form
\[
S = \sum_{(u, v) \in \Delta} \lambda(u, v) u \otimes v,
\]
with \( \lambda \in l^1(\Delta) \), and
\[
\|S\|_1 \geq (1 - 2\eta - \eta^2) \inf \|\lambda\|_1.
\]

**Proof.** (i). Given \( T \in BL(H) \) and \( \varepsilon > 0 \), there exists a unit vector \( x \) with \( |(Tx, x)| > \frac{1}{2} \|T\| - \varepsilon \). Choose \( u \in \Gamma \) such that \( x = v + x_1 \) with \( \|x_1\| \leq \eta \). Then
\[
(Tx, x) = (Tv, v) + (Tv, x_1) + (Tx_1, x),
\]
and \( |(Tv, x_1) + (Tx_1, x)| \leq 2\eta \|T\| \). Therefore
\[
|(Tv, v)| > (\frac{1}{2} - 2\eta) \|T\| - \varepsilon,
\]
\[
\sup \{|(Tv, v)| : v \in \Gamma\} \geq (\frac{1}{2} - 2\eta) \|T\|.
\]
Since \((v \otimes v)T = v \otimes T^* v \) and \( \text{tr} (x \otimes y) = (x, y) \), we have
\[
\text{tr} ((v \otimes v)T) = (v, T^* v) = (Tv, v).
\]
Thus, for every \( \psi \in (C_1)^* \),
\[
\sup \{|\psi(v \otimes v)| : v \in \Gamma\} \geq (\frac{1}{2} - 2\eta) \|\psi\|,
\]
and (i) follows from Theorem 1.

(ii). Given \( T \in BL(H) \) and \( \varepsilon > 0 \), there exist unit vectors \( x, y \) with \( |(Tx, y)| > \|T\| - \varepsilon \). Choose \( u, v \in \Gamma \) with \( x = u + x_1, y = v + y_1 \), and \( \|x_1\|, \|y_1\| \leq \eta \). Then
\[
(Tx, y) = (Tu, v) + (Tx_1, v) + (Tu, y_1) + (Tx_1, y_1),
\]
\[
|\langle Tx_1, v \rangle + (Tu, y_1) + (Tx_1, y_1)\| \leq \|T\| (2\eta + \eta^2),
\]
\[
|(Tu, v)| > (1 - 2\eta - \eta^2) \|T\| - \varepsilon.
\]
Thus, as above, for every \( \psi \in (C_1)^* \),
\[
\sup \{|\psi(u \otimes v)| : (u, v) \in \Delta\} \geq (1 - 2\eta - \eta^2) \|\psi\|,
\]
and (ii) follows from Theorem 1.

**Remark.** If \( \Gamma \) is norm dense in the unit sphere of \( H \), we have \( \|S\|_1 \geq \frac{1}{2} \inf \|\lambda\|_1 \) in (i), and \( \|S\|_1 = \inf \|\lambda\|_1 \) in (ii).
REFERENCES


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