Image Recovery by Convex Combinations of Projections

G. CROMBEZ

State University of Ghent, Krijgslaan 281, B-9000 Gent, Belgium

Submitted by C. Foias

Received June 28, 1989

1. INTRODUCTION

The problem of image recovery in a Hilbert space setting by using convex projections (by projection we always mean minimum distance projection onto a closed convex set) may be stated as follows: the original (unknown) image \( f \) is known a priori to belong to the intersection \( C_0 \) of \( r \) well-defined closed convex sets \( C_1, \ldots, C_r \) in a Hilbert space \( H \); hence \( f \in C_0 = \bigcap_{i=1}^{r} C_i \); given only the projection operators \( P_i \) onto the individual sets \( C_i \) \((i = 1, \ldots, r)\), recover \( f \) by an iterative scheme. Practically, the image \( f \) is a function of two real variables, and the Hilbert space \( H \) is the (real or complex) space \( L^2(\Omega) \), where \( \Omega \subset \mathbb{R}^2 \). Since the sets \( C_i \) are only assumed to be convex, the projections \( P_i \) are in general nonlinear.

Iterative methods of finding a common point of sets by means of contraction-like operators can be found in [1, 2, 5]; these methods have been applied to image processing first by Youla and Webb [11]. A procedure with essentially the same ideas has been used in [7] for the extrapolation of bandlimited functions.

In the methods of image recovery used until now [9–12, 7], from the projections \( P_i \) \((i = 1, \ldots, r)\) \( r \) operators \( T_i \) are constructed, and from these the composition operator \( T = T_r T_{r-1} \cdots T_2 T_1 \); starting from an arbitrary element \( x \) of \( H \) the sequence \( \{T^n x\}_{n=0}^{\infty} \) is shown to converge, at least weakly, to an element of \( C_0 \).

Two cases are important: the case where each \( T_i \) equals \( P_i \) (hence \( T = P_r P_{r-1} \cdots P_2 P_1 \)), and the case where each \( T_i \) is given by \( T_i = 1 + \lambda_i (P_i - 1) \), with \( 1 \) the identity operator on \( H \) and \( \lambda_i \) a relaxation parameter with \( 0 < \lambda_i < 2 \) (when \( \lambda_i = 1 \) for all \( i \) this reduces to the foregoing case). So, at each iteration step the computation has to be done sequentially.

When a parallel computer is available the computing process could be speeded up a lot if at each iteration step the contribution of the different
projections could be computed in parallel. In [3] we showed that, using for
t a convex combination of projections, i.e., \( T = \sum_{i=1}^{r} \alpha_i P_i, \alpha_i > 0 \) for all \( \alpha \),
\( \sum_{i=1}^{r} \alpha_i = 1 \), the resulting sequence \( \{ T^n x \}_{n=0}^{\infty} \) still converges weakly to an
element of \( C_0 \). In this paper we extend this result, giving new expressions
for an operator \( T \) such that \( \{ T^n x \}_{n=0}^{\infty} \) converges weakly to an element of
\( C_0 \), and where at each iteration step parallelism in computing may be used.

2. Mathematical Preliminaries

We denote by \( H \) a complex Hilbert space with norm \( \| \cdot \| \), \( C_1, ..., C_r \)
denote \( r \) closed convex sets in \( H \) with nonempty intersection \( C_0 \), and
\( P_i \) \((i = 1, ..., r)\) are the projection operators onto the individual sets \( C_i \).

The following definitions and Theorem 1 are from [12]; \( C \) denotes a
closed convex set in \( H \).

**Definition 1.** A mapping \( T: C \to C \) is said to be nonexpansive iff
\( \|Tx - Ty\| \leq \|x - y\| \), for all \( x, y \in C \).

**Definition 2.** A mapping \( T: C \to C \) is said to be asymptotically regular
iff, for every \( x \in C \), \( T^n x - T^{n+1} x \to 0 \).

**Theorem 1.** Let \( T: C \to C \) be an asymptotically regular nonexpansive
map whose set of fixed points \( F \subset C \) is nonempty. Then, for any \( x \in C \) the
sequence \( \{ T^n x \}_{n=0}^{\infty} \) is weakly convergent to an element of \( F \).

Since a Hilbert space is in particular a uniformly convex space with
modulus of convexity \( \delta \), we will also make use of the following lemma
[8, p. 4]:

**Lemma 1.** For given \( \varepsilon > 0, \ d > 0, \ x \in [0, 1], \) the inequalities
\( \|w\| \leq \|v\| \leq d \) and \( \|v - w\| \geq \varepsilon \) imply that
\( \|(1 - x)v + xw\| \leq \|v\| \left[ 1 - 2\delta(\varepsilon/d) \right] \min(x, 1 - x) \].

3. Main Results

Our result will be based on the following general theorem.

**Theorem 2.** Let \( T: H \to H \) be the operator given by \( T = \alpha_0 1 +
\sum_{i=1}^{r} \alpha_i T_i, \ \alpha_i > 0 \) for \( j = 0, 1, ..., r, \ \sum_{j=0}^{r} \alpha_j = 1 \) such that
(i) each $T_i$ is nonexpansive on $H$

(ii) the set of fixed points of $T$ is nonempty

(iii) $Tu = u$ if $T_iu = u$ for all $i$.

Then $T$ is asymptotically regular.

**Proof.** We first remark that condition (i) immediately implies that $T$ itself is nonexpansive. Moreover, for $x_0 = 1$ there is nothing to prove.

The proof follows the same lines of that in [6, Th 2; 4, Lemma 1]. Let $x \in H$, and for $n$ a positive integer let $x_n = T^n x = T(T^{n-1} x)$. Take a fixed point $u$ of $T$ (by (ii)); then by (iii), $T_i u = u$ for all $i$. The sequence $\{\|x_n - u\|\}_{n=1}^{\infty}$ is nonincreasing. Also,

$$x_{n+1} - u = Tx_n - Tu = x_0 (x_n - u) + (1 - x_0) \sum_{i=1}^{r} \frac{x_i}{1 - x_0} (T_i x_n - u),$$

or

$$x_{n+1} - u = x_0 (x_n - u) + (1 - x_0) z_n,$$ (1)

with

$$z_n = \sum_{i=1}^{r} \frac{x_i}{1 - x_0} (T_i x_n - u),$$

from which we also derive that $\|z_n\| \leq \|x_n - u\|$. We now show that the sequence $\{x_n - u - z_n\}_{n=1}^{\infty}$ has a subsequence that converges to zero. If this is not the case, there would exist $\varepsilon > 0$ such that $\|x_n - u - z_n\| \geq \varepsilon$ for all $n$. Then applying Lemma 1 for $w = z_n, v = x_n - u, x = 1 - x_0$, and $d = \|x_1 - u\|$ we would obtain from (1)

$$\|x_{n+1} - u\| \leq \|x_n - u\| \left[ 1 - 2 \delta \left( \frac{\varepsilon}{\|x_1 - u\|} \right) \min(1 - x_0, x_0) \right],$$

which gives by induction

$$\|x_{n+1} - u\| \leq \|x_1 - u\| \left[ 1 - 2 \delta \left( \frac{\varepsilon}{\|x_1 - u\|} \right) \min(1 - x_0, x_0) \right]^{n-1}.$$ 

Since $0 < x_0 < 1$ the right-hand-side converges to zero, and so

$$\lim_{n} \|x_n - u\| = 0 = \lim_{n} \|z_n\|;$$

this is clearly a contradiction with our assumption. Hence, some subsequence $\{x_{n_j} - u - z_{n_j}\}_{j=1}^{\infty}$ converges to zero. Since $x_{n+1} - x_n = (1 - x_0)(u - x_n + z_n)$, also the subsequence $\{x_{n_j+1} - x_{n_j}\}_{j=1}^{\infty}$ will converge to zero. However, $\|x_{n+1} - x_n\| = \|Tx_n - Tx_{n-1}\| \leq \|x_n - x_{n-1}\|$, and so the sequence $\{\|x_{n+1} - x_n\|\}_{n=1}^{\infty}$ is
nonincreasing, bounded, and has a subsequence convergent to zero; this means that the sequence \(\{x_{n+1} - x_n\}_{n=1}^\infty\) itself is convergent to zero. Since \(x_{n+1} - x_n = T^{n+1}x - T^nx\), the operator \(T\) is asymptotically regular.

We consider expressions for \(T\) as in Theorem 2, for a particular choice of operators \(T_i\), and we want to apply to this \(T\) Theorem 1 with \(C = H\). As remarked at the beginning of the proof of Theorem 2, if each \(T_i\) is nonexpansive the same will be true for \(T\). Moreover, from the expression of \(T\) it also follows that a common fixed point of all \(T_i\) is also a fixed point of \(T\). Hence, since for our choice of \(T\), the set of common fixed points of all \(T_i\) will always be \(C_0\) (which is nonempty since \(f \in C_0\)), condition (ii) of Theorem 2 will always be fulfilled. So the common verification of the conditions of Theorems 1 and 2 may then be restricted to the verification of the following facts:

1. (\(\alpha\)) each \(T_i\) is nonexpansive on \(H\)
2. (\(\beta\)) \(C_0\) is the set of fixed points of \(T\), and the set of common fixed points of all \(T_i\).

In the expression of \(T\) in Theorem 2 we now take

\[
T_i = 1 + \lambda_i(P_i - 1)
\]

with \(\lambda_i\) scalars. We investigate the conditions (\(\alpha\)) and (\(\beta\)) in Lemmas 2 and 4.

In the proof of Lemma 2, use will be made of two inequalities which are characteristic to a minimum distance projection. If \(C\) is any closed convex set in \(H\) and \(P\) is the projection operator onto \(C\), then we first have the well-known inequality

\[
\Re\langle x - Px, y - Px \rangle \leq 0 \quad \text{for} \quad x \in H, \ y \in C,
\]

from which the following inequality may easily be derived for arbitrary \(x\) and \(z\) in \(H\)

\[
\|Px - Pz\|^2 \leq \Re\langle x - z, Px - Pz \rangle.
\]

**Lemma 2.** For \(0 < \lambda_i < 2\), \(T_i\) is nonexpansive.

**Proof.** The proof of this is given in [12], but further on we need two inequalities from the proof; so we state it here shortly. Let \(x \in H, \ z \in H\), and denote by \(\Re\) the real part of a complex number. For \(0 < \lambda_i < 1\) we have

\[
\|T_i x - T_i z\| = \|(1 - \lambda_i)(x - z) + \lambda_i(P_i x - P_i z)\|
\]

\[
\leq (1 - \lambda_i) \|x - z\| + \lambda_i \|P_i x - P_i z\|
\]
\[ \leq (1 - \lambda_i) \| x - z \| + \lambda_i \| x - z \| \]
\[ = \| x - z \|. \]

For \( \lambda_i = 1 \) the result is trivial since then \( T_i = P_i \). For \( 1 < \lambda_i < 2 \) we have
\[
\| T_i x - T_i z \|^2 = \| (1 - \lambda_i)(x - z) + \lambda_i(P_i x - P_i z) \|^2 \\
= (1 - \lambda_i)^2 \| x - z \|^2 + 2\lambda_i(1 - \lambda_i) \\
\times \text{Re} \langle x - z, P_i x - P_i z \rangle + \lambda_i^2 \| P_i x - P_i z \|^2.
\]

Since \( \| P_i x - P_i z \|^2 \leq \text{Re} \langle x - z, P_i x - P_i z \rangle \), and \( \| P_i x - P_i z \| \leq \| x - z \| \), we derive
\[
\| T_i x - T_i z \|^2 \leq (1 - \lambda_i)^2 \| x - z \|^2 + \lambda_i(2 - \lambda_i) \| x - z \|^2 \\
\leq (1 - \lambda_i)^2 \| x - z \|^2 + \lambda_i(2 - \lambda_i) \| x - z \|^2 \\
= \| x - z \|^2.
\]

From Lemma 2 we conclude that condition (a) is true for \( 0 < \lambda_i < 2 \) for all \( i \); in the sequel we suppose this to hold. When \( u \in C_0 \), then \( P_i u = u \), and so \( T_i u = u + \lambda_i(P_i u - u) = u \); conversely, if \( T_i u = u \) for all \( i \) then \( P_i u = u \), which leads to \( u \in C_0 \). So, one part of (β) is true. The other part of condition (β) will follow from Lemmas 3 and 4.

**Lemma 3.** For \( x \in H \), \( y \in C_0 \) we have the inequality
\[
\| x - P_i x \|^2 \leq \| x - y \|^2 - \| P_i x - y \|^2.
\]

**Proof.** \( \| P_i x - y \|^2 = \| x - y \|^2 + 2 \text{Re} \langle x - y, P_i x - x \rangle + \| P_i x - x \|^2 \), and
\[
\text{Re} \langle x - y, P_i x - x \rangle = \text{Re} \langle x - P_i x, y - P_i x + x + P_i x \rangle \\
= \text{Re} \langle x - P_i x, y - P_i x \rangle - \| x - P_i x \|^2.
\]
Hence, \( \| P_i x - y \|^2 = \| x - y \|^2 + 2 \text{Re} \langle x - P_i x, y - P_i x \rangle - \| x - P_i x \|^2 \), and the result follows since \( \text{Re} \langle x - P_i x, y - P_i x \rangle \leq 0 \).

**Lemma 4.** The set of fixed points of \( T \) coincides with \( C_0 \).

**Proof.** For \( x \in C_0 \) we have \( P_i x = x \) for all \( i \), hence \( T_i x = x \), and also \( T x = x \). Conversely, suppose that \( T x = x \). For \( y \in C_0 \) we have
\[
\| x - y \| = \| T x - T y \| \\
\leq a_0 \| x - y \| + \sum_{i=1}^{r} \alpha_i \| T_i x - T_i y \|
\]
This can only be true if \( |T_i x - T_i y| = |x - y| \) for all \( i \). To prove that this leads to \( x \in C_0 \) we first take the case that \( \lambda_i = 1 \) for all \( i \); then \( |P_i x - P_i y| = |x - y| \) for all \( i \). Since \( P_i y = y \) we obtain from Lemma 3 that \( |x - P_i x| = 0 \), hence \( P_i x = x \) for all \( i \), and so \( x \) belongs to \( C_0 \).

For \( \lambda_i \neq 1 \) for at least one \( i \) we remark that, when in the proof of Lemma 2 we change \( x \) into \( y \) (with \( y \in C_0 \)), our conditions \( |T_i x - T_i y| = |x - y| \) for all \( i \) imply that we should have equality signs in that proof. Doing this in both cases \( 0 < \lambda_i < 1 \) and \( 1 < \lambda_i < 2 \) we obtain for both that \( |P_i x - P_i y| = |x - y| \) for all \( i \). Again we derive from Lemma 3 that \( x \in C_0 \).

As a result we obtain

**Theorem 3.** Let \( T = x_0 I + \sum_{i=1}^{r} x_i T_i \) with \( T_i = I + \lambda_i (P_i - I) \) for all \( i \), \( 0 < \lambda_i < 2 \), \( x_j > 0 \) for \( j = 0, 1, \ldots, r \), \( \sum_{j=0}^{r} x_j = 1 \). Then starting from an arbitrary element \( x \) the sequence \( \{T^n x\}_{n=0}^{\infty} \) converges weakly to an element of \( C_0 \).

Since the \( \lambda_i \) may vary freely between 0 and 2, in practical cases a suitable choice of these relaxation parameters may speed up the convergence.

We note that in Theorem 3 the weak limit \( Q x = \lim_{n \to \infty} T^n x \) defines a nonexpansive idempotent with range \( C_0 \).

**References**


