FINDING PROJECTIONS ONTO THE INTERSECTION OF CONVEX SETS IN HILBERT SPACES

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Abstract

We present a parallel iterative method to find the shortest distance projection of a given point onto the intersection of a finite number of closed convex sets in a real Hilbert space. In the method use is made of weights and a relaxation coefficient which may vary at each iteration step, and which are determined at each step by geometrical conditions.

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1 Introduction

Finding the shortest distance projection of a given point $v_0$ onto the nonempty intersection $C^* = \bigcap_{j=1}^{r} C_j$ of a finite number of given closed convex sets $C_j (j = 1, \ldots, r)$ in a real Hilbert space $H$, is a problem that often arises in applied mathematics. By way of example, we refer to [3] for applications in signal recovery, where $v_0$ is a given reference signal, and where

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the true signal should be an element in $C^*$ which best approximates $v_0$; for applications in multivariate data analysis we refer to [13].

The stated problem belongs to the more general class of so-called convex feasibility problems, in which in an iterative manner one tries to find "a" point in $C^*$ starting from "some" point $v_0$. Both sequential and parallel methods, using the projections onto the different sets $C_j$ ($j = 1, \ldots, r$), have been invented to obtain a point in $C^*$; we refer to [2], [5], [6] and [8] for some more recent methods; an overview with a more extensive list of references is given in [4]. Although under supplementary conditions about the sets $C_j$ a procedure for solving a general convex feasibility problem may lead to the projection of the starting point $v_0$ onto $C^*$ (e.g., see [16]), this is not the general rule. Hence, to find the projection of $v_0$ onto $C^*$, existing algorithms for solving general convex feasibility problems have to be adapted; some of those methods are given in [1], [9], [10] and [12].

Due to the iterative nature of the above-mentioned procedures, it is clear that in recent years most interest is focused on parallel methods, involving at each iteration step a number of variable weights and a variable relaxation coefficient, which have to be chosen adequately in order to assure convergence. The theoretical base for the underlying paper has been given in [7]; there it has been shown that solving a general convex feasibility problem by a parallel method with variable weights $\mu_i(j)$ ($j = 1, \ldots, r; i = 1, 2 \ldots$) and one variable relaxation coefficient $\lambda_i$ at each iteration step $i$ ($i = 1, 2 \ldots$) in the given Hilbert space $H$, is equivalent to solving another convex feasibility problem by a semi-sequential method in an accompanying Hilbert space $\mathcal{H}$ (more on this will be said in section 2). Adapting this semi-sequential method in $\mathcal{H}$ suitably gives us the possibility to determine the weights and the relaxation coefficients by geometrical conditions.

In section 2 we outline the ideas presented in [7]. In section 3 we construct the transform $\{\tilde{v}_k\}_{k=0}^{\infty}$ of the wanted sequence $\{v_k\}_{k=0}^{\infty}$ (in $H$) in the accompanying space $\mathcal{H}$. In section 4 we investigate the convergence of a subsequence of the sequence $\{\tilde{v}_k\}_{k=0}^{\infty}$ in $\mathcal{H}$. Finally, in section 5 we show that the sequence $\{v_k\}_{k=0}^{\infty}$ in $H$ really is norm convergent to the projection of $v_0$ onto $C^*$.

2 Viewing a parallel projection method in one Hilbert space as a semi-sequential one in another Hilbert space

In this section we sketch the most important topics of the theoretical base as presented in [7]; for technical details we refer to that paper.
Let $H$ be a real Hilbert space with inner product $\langle \cdot , \cdot \rangle$ and norm $\| \cdot \|$ derived from $\langle \cdot , \cdot \rangle$. Suppose that in $H$ close convex sets $C_j (j = 1, \ldots , r)$ are given with nonempty intersection $C^* = \cap_{j=1}^{r} C_j$; the projection operator onto $C_j$ is denoted by $P_j$.

Assume that, during some iteration process, we obtained at step $k$ a point $v_k$, and that to obtain the next iterate $v_{k+1}$ we need an "intermediate" point $x_{k+1}$ given by

$$x_{k+1} = v_k + \lambda_{k+1} \left( \sum_{j=1}^{r} \mu_{k+1}(j) P_j v_k - v_k \right),$$

in which $\lambda_{k+1}$ is a variable positive relaxation coefficient, and in which $\mu_{k+1}(j) (j = 1, \ldots , r)$ are variable nonnegative weights, i.e., $\mu_{k+1}(j) \geq 0$ for each $j$, and $\sum_{j=1}^{r} \mu_{k+1}(j) = 1$. We investigate this intermediate step (1) in a newly defined Hilbert space $\mathcal{H}$.

The new Hilbert space $\mathcal{H}$ with inner product $\langle \cdot , \cdot \rangle$ and norm $\| \cdot \|$ consists of functions defined on the closed interval $[0, 1]$ of the real line, with values in $H$, and which are square integrable on $[0, 1]$. The mapping $q$ of $H$ into $\mathcal{H}$, which associates to each $v$ in $H$ the element $q(v)$ in $\mathcal{H}$ defined by $q(v)(t) = v$ for all $t \in [0, 1]$, is an imbedding of $H$ into $\mathcal{H}$.

The image $q(H)$, which we denote by $D$, is a closed linear subspace in $\mathcal{H}$. $D$ is the first interesting subset of $\mathcal{H}$; projection onto $D$ is denoted as $PD$.

Considering the weights $\mu_{k+1}(j) (j = 1, \ldots , r)$ in (1), we can associate with them a subset $F_{\mu_{k+1}}$ in $\mathcal{H}$. Without going into details, we can say that these weights determine a partition (with possible empty subsets when some of the weights are zero) of the interval $[0, 1]$; $F_{\mu_{k+1}}$ then consists of those elements in $\mathcal{H}$ (which are functions from $[0, 1]$ into $H$) on each subinterval take on a constant value in a corresponding set $C_j$; otherwise said, $F_{\mu_{k+1}}$ consists of piecewise constant functions on $[0, 1]$, whose values are lying in some (or all) of the sets $C_j$. As such we connect the convex subsets $C_j (j = 1, \ldots , r)$ in $H$ with $F_{\mu_{k+1}}$ in $\mathcal{H}$.

When considering at each step $i$ of some iteration process as in (1) the corresponding set of weights $\mu_i(j) (j = 1, \ldots , r; i = 1, 2, \ldots )$, we see that we have in $\mathcal{H}$ an infinite number of subsets $F_{\mu_i}$ $(i = 1, 2, \ldots )$. Each set $F_{\mu_i}$ is a closed convex set in $\mathcal{H}$, and as such for each point $\bar{v}$ in $\mathcal{H}$ there is a unique shortest distance point in $F_{\mu_i}$, which we denote by $P_{F_{\mu_i}} \bar{v}$. These sets $F_{\mu_i}$ $(i = 1, 2, \ldots )$ form, together with $D$, the interesting subsets of $\mathcal{H}$.

The intermediate iteration step (1) in $H$, where from the last obtained point $v_k$ in some iteration we obtain the intermediate point $x_{k+1}$ in a parallel manner, may now be interpreted in the space $\mathcal{H}$ in the following manner. Putting $q(v_k) = \bar{v}_k$, which is a point in $D \subset \mathcal{H}$, we obtain a point $\tilde{x}_{k+1}$ in $D$ in two steps: first, determine $\tilde{y}_{k+1}$ in $\mathcal{H}$ by

$$\tilde{y}_{k+1} = \bar{v}_k + \lambda_{k+1} (P_{F_{\mu_{k+1}}} \bar{v}_k - \bar{v}_k),$$

and then the intermediate point is given by

$$\tilde{x}_{k+1} = \tilde{y}_{k+1}.$$
i.e., $\tilde{y}_{k+1}$ is obtained from $\tilde{u}_k$ by relaxed projection onto $F_{\mu_{k+1}}$ with relaxation parameter $\lambda_{k+1}$; then, determine $\tilde{x}_{k+1}$ by
\begin{equation}
\tilde{x}_{k+1} = P_{D} \tilde{y}_{k+1},
\end{equation}
i.e., $\tilde{x}_{k+1}$ is the result of pure projection of $\tilde{y}_{k+1}$ onto $D \subset \mathcal{H}$. As has been proved in [7], $q(x_{k+1}) = \tilde{x}_{k+1}$. Hence, the parallel step in $H$ given by (1) is equivalent with the semi-sequential step ((2)+(3)) in $\mathcal{H}$. This gives us the possibility to determine the weights $\mu_{k+1}(j) (j = 1, \ldots, r)$ and the relaxation coefficient $\lambda_{k+1}$ in (1) by using ((2)+(3)) in $\mathcal{H}$, for instance, by imposing some geometrical condition in $\mathcal{H}$. Otherwise said, convergence of some iterative process in $H$ will be investigated by using $\mathcal{H}$ as a "server" or "transformed" space, delivering the necessary tools (such as the theoretical background, values of variable parameters, ...). This is possible since both spaces are closely connected to each other. As an example we mention the following expressions for distances, which will be needed later:
\begin{align}
||\tilde{u}_k - P_{F_{\mu_{k+1}}} \tilde{u}_k||^2 &= \sum_{j=1}^{r} \mu_{k+1}(j)||v_k - P_j v_k||^2 \\
||\tilde{u}_k - P_D (P_{F_{\mu_{k+1}}} \tilde{u}_k)||^2 &= ||v_k - \sum_{j=1}^{r} \mu_{k+1}(j) P_j v_k||^2.
\end{align}

3 Construction of the transformed sequence in the server space $\mathcal{H}$

We are faced with the following problem: "In the original Hilbert space $H$ with inner product $< , >$ and norm $|| \cdot ||$, $r$ closed convex sets $C_j$ are given with nonempty intersection $C^* = \bigcap_{j=1}^{r} C_j$, together with a point $v_0$ that doesn't belong to $C^*$. Determine in an iterative manner the shortest distance projection $P_{C^*} v_0$ of $v_0$ onto $C^*$.”

In this section we construct the sequence $\{\tilde{u}_k\}_{k=0}^{+\infty}$ in $D \subset \mathcal{H}$, which will be such that the inverse transform $\{q^{-1}(\tilde{u}_k)\}_{k=0}^{+\infty} \equiv \{v_k\}_{k=0}^{+\infty}$ in $H$ will converge to $P_{C^*} v_0$; this last property will be shown in section 5.

The following definitions and proposition 1 are from [14]:
Consider in $\mathcal{H}$ an ordered pair $(\hat{a}, \hat{b})$ of points. Let
\begin{equation}
S(\hat{a}, \hat{b}) = \{\tilde{u} \in \mathcal{H} : < \tilde{u} - \hat{b}, \hat{b} - \hat{a} > \geq 0\}.
\end{equation}
When $\hat{a} \neq \hat{b}$, $S(\hat{a}, \hat{b})$ is a closed half-space, and $\hat{b}$ is the projection of $\hat{a}$ onto $S(\hat{a}, \hat{b})$.

Given an ordered triple $(\hat{a}, \hat{b}, \hat{c})$ of points of $\mathcal{H}$, denote by $P(\hat{a}, \hat{b}, \hat{c})$ the projection of $\hat{a}$ onto $S(\hat{a}, \hat{b}) \cap S(\hat{b}, \hat{c})$. 

Proposition 1. [14]

Given the ordered triple \((a, b, c)\) of points of \(\mathcal{H}\), denote the following expressions for short by \(\alpha, \beta, \gamma, \delta\) respectively:

\[
\left< \hat{b} - \hat{a}, \hat{c} - \hat{b} \right> \equiv \alpha, ||\hat{c} - \hat{b}||^2 \equiv \beta, ||\hat{a} - \hat{b}||^2 \equiv \gamma, \beta \gamma - \alpha^2 \equiv \delta.
\]

Then we have

(i) When \(\delta = 0, \alpha < 0\), then \(S(a, b) \cap S(b, c) = \emptyset\).

(ii) When \(\delta = 0, \alpha \geq 0\), then \(P(a, b, c) = c\).

(iii) When \(\delta \neq 0, \alpha \beta - \delta \geq 0\), then \(P(a, b, c) = a + (1 + \frac{\beta}{\delta})(c - b)\).

(iv) When \(\delta \neq 0, \alpha \beta - \delta < 0\), then \(P(a, b, c) = b + \frac{\delta}{\gamma}(\gamma(c - b) - \alpha(c - a))\).

We remark that, in what follows, case (i) can not happen; this will be shown further on.

The sequence \(\{u_k\}_{k=0}^{\infty}\) in \(D \subset \mathcal{H}\) is constructed as follows.

Choose as starting element the point \(v_0 \equiv q(v_0)\). Construct

\[
x_1 = v_0 + \lambda_1 (P_D(P_{F_{u_1}} v_0) - v_0),
\]

in which \(\lambda_1\) and \(F_{u_1}\) are determined according to some criterion to be explained further on, and put \(v_1 \equiv x_1\). Construct

\[
x_2 = v_1 + \lambda_2 (P_D(P_{F_{u_2}} v_1) - v_1),
\]

in which again the parameters have to be chosen properly. Consider the half spaces \(S(v_0, v_1)\) and \(S(v_1, x_2)\), and take as \(v_2\) the projection of \(v_0\) onto \(S(v_0, v_1) \cap S(v_1, x_2)\), i.e., put \(v_2 = P(v_0, v_1, x_2)\).

In a general manner, when \(v_0, v_1, \ldots, v_k\) have already been obtained, the next iterate \(v_{k+1}\) is determined in two steps:

\[
x_{k+1} = v_k + \lambda_{k+1} (P_D(P_{F_{u_{k+1}}} v_k) - v_k)
\]

\[
\hat{v}_{k+1} = P(v_0, v_k, x_{k+1}).
\]

Since \(v_0 \in D \subset \mathcal{H}\), and taking into account proposition 1, we conclude from (6) and (7)
that the sequence \( \{ v_k \}_{k=0}^{\infty} \) is a sequence in \( D \); hence, \( \{ q^{-1}(v_k) \}_{k=0}^{\infty} = \{ u_k \}_{k=0}^{\infty} \) will be a sequence in \( H \). Remark that step (6) is exactly the combination of ((2)+(3)) in section 2.

At each iteration step \( i \) (\( i := 1, 2, \ldots \)), the weights \( \mu_i(j) \) (\( j = 1, \ldots, r \)) and the relaxation coefficient \( \lambda_i \) are determined according to the same fixed principles, which we now explain.

Suppose that \( v_k \) has been obtained. In order to obtain \( x_{k+1} \) as in (6), we choose \( F_{\mu_{k+1}} \) such that the distance from \( v_k \) to \( F_{\mu_{k+1}} \) is maximal among all distances from \( v_k \) to the different sets \( F_{\mu_n} \) (an uncountable infinite number) which may be involved, i.e., \( \| v_k - P_{F_{\mu_{k+1}}} v_k \| \geq \| v_k - P_{F_{\mu_{m}}} v_k \| \).

In view of (4), this means that the weights \( \mu_{k+1}(j) \) have to be determined such that \( \sum_{j=1}^{r} \mu_{k+1}(j) \| v_k - P_j v_k \|^2 \) is maximal, in which \( v_k = q^{-1}(v_k) \). So, it is sufficient to check in \( H \) the distances from \( v_k \) to the different sets \( C_j \); when for some \( j_0 \) with \( 1 \leq j_0 \leq r \) we have that \( \| v_k - P_{j_0} v_k \| > \| v_k - P_j v_k \| \) for \( j \neq j_0 \), then \( \mu_{k+1}(j_0) = 1 \) and \( \mu_{k+1}(j) = 0 \) for \( j \neq j_0 \); when equal maximal distances from \( v_k \) to the different \( C_j \) are appearing, say for the indices \( j_0, j_1, \ldots, j_s \), then \( \mu_{k+1}(j_0), \mu_{k+1}(j_1), \ldots, \mu_{k+1}(j_s) \) may be chosen different from zero such that their sum adds up to 1, and \( \mu_{k+1}(j) = 0 \) for indices \( j \notin \{ j_0, j_1, \ldots, j_s \} \).

Having determined \( F_{\mu_{k+1}} \) in (6) as (one of) the most remote set(s) from \( v_k \), we now turn our attention to the determination of \( \lambda_{k+1} \) in (6).

We denote by \( P_{k+1} \) the hyperplane going through the point \( P_{F_{\mu_{k+1}}} v_k \) and which is orthogonal to \( P_{F_{\mu_{k+1}}} v_k - v_k \); i.e., \( P_{k+1} \) is given by

\[
P_{k+1} = \{ \bar{v} \in H : \langle \bar{v} - P_{F_{\mu_{k+1}}} v_k, P_{F_{\mu_{k+1}}} v_k - v_k \rangle = 0 \}.
\]

We now take for \( \lambda_{k+1} \) that value such that \( x_{k+1} \) (which always belongs to \( D \)) also belongs to the hyperplane \( P_{k+1} \).

**Lemma 1.**

When \( \lambda_{k+1} \) in (6) has the value

\[
\lambda_{k+1} = \frac{\| P_{F_{\mu_{k+1}}} v_k - v_k \|^2}{\| P_D(P_{F_{\mu_{k+1}}} v_k - v_k) \|^2},
\]

then \( x_{k+1} \) belongs to \( P_{k+1} \).

**Proof.**

Using the fact that \( P_D \) is an orthogonal projection onto the closed linear subspace \( D \), and hence that the vector \( P_{F_{\mu_{k+1}}} v_k - P_D(P_{F_{\mu_{k+1}}} v_k) \) is orthogonal, in particular, to the vector \( P_D(P_{F_{\mu_{k+1}}} v_k - v_k) \), we first get that
Combining this result with (6) leads to
\[ \langle \bar{x}_{k+1} - P_{H\ast_{k+1}} \bar{v}_k, P_{H\ast_{k+1}} \bar{v}_k - \bar{v}_k \rangle \]
\[ = \langle \bar{x}_{k+1} - \bar{v}_k + \bar{v}_k - P_{H\ast_{k+1}} \bar{v}_k, P_{H\ast_{k+1}} \bar{v}_k - \bar{v}_k \rangle \]
\[ = \lambda_{k+1} \langle P_{H\ast_{k+1}} \bar{v}_k - \bar{v}_k, P_{H\ast_{k+1}} \bar{v}_k - \bar{v}_k \rangle + \langle \bar{v}_k - P_{H\ast_{k+1}} \bar{v}_k, P_{H\ast_{k+1}} \bar{v}_k - \bar{v}_k \rangle \]
\[ = \lambda_{k+1} \| P_{H\ast_{k+1}} \bar{v}_k - \bar{v}_k \|^2 - \| \bar{v}_k - P_{H\ast_{k+1}} \bar{v}_k \|^2, \]
and replacing \( \lambda_{k+1} \) by its value (9) we see that the result is zero; hence \( \bar{x}_{k+1} \in P_{k+1} \).

We remark that, since the sequence \( \{v_k\} \) that will do the final job is a sequence in \( H \), we also have to check how the value of \( \lambda_{k+1} \) is computed by elements of \( H \); but this is easy, since the expressions appearing on the right-hand side of (9) are already known by (4) and (5).

Although the procedure for constructing the sequence \( \{\bar{v}_k\} \) in \( H \), as given in (6) and (7), is now completely described since we determined how to choose at each step the weights and the relaxation coefficient, we still need some other results before considering convergence properties of the sequence. To this end, we will use the vector \( \tilde{y}_{k+1} \) which we introduced in (2), and so \( \tilde{z}_{k+1} \) may be written either by its expression ((2)+(3)) or by its expression (6).

We denote by \( R_{k+1} \) the hyperplane going through the point \( \tilde{z}_{k+1} \) and which is orthogonal to \( \bar{x}_{k+1} - \bar{v}_k \); i.e., \( R_{k+1} \) is given by

\[ R_{k+1} = \{ \bar{w} \in H : \langle \bar{w} - \tilde{z}_{k+1}, \tilde{x}_{k+1} - \bar{v}_k \rangle = 0 \}. \]

Lemma 2.

The following statements are equivalent:

(i) \( \bar{w} \in D \cap P_{k+1} \)

(ii) \( \bar{w} \in D \cap R_{k+1} \)

Proof.

Since both implications are shown in an analogous manner, we just show the implication...
(ii) ⇒ (i). So, let \( \hat{w} \) be an element in \( D \) such that \( \langle \hat{w} - \hat{x}_{k+1}, \hat{x}_{k+1} - \hat{v}_k \rangle = 0 \). By (2) we have that \( P_{F_{\mu_{k+1}}} \hat{v}_k - \hat{v}_k = \frac{1}{\lambda_{k+1}} (y_{k+1} - \hat{v}_k) \), and hence
\[
\langle \hat{w} - \hat{x}_{k+1}, P_{F_{\mu_{k+1}}} \hat{v}_k - \hat{v}_k \rangle
\]
\[
= \frac{1}{\lambda_{k+1}} \langle \hat{w} - \hat{x}_{k+1}, y_{k+1} - \hat{x}_{k+1} + \hat{x}_{k+1} - \hat{v}_k \rangle
\]
\[
= \frac{1}{\lambda_{k+1}} \langle \hat{w} - \hat{x}_{k+1}, y_{k+1} - \hat{x}_{k+1} \rangle + \frac{1}{\lambda_{k+1}} \langle \hat{w} - \hat{x}_{k+1}, \hat{x}_{k+1} - \hat{v}_k \rangle
\]
\[= 0\]
since both terms are zero, the last one by definition of \( R_{k+1} \), the first one since \( P_D \) is an orthogonal projection. Using this we get
\[
\langle \hat{w} - P_{F_{\mu_{k+1}}} \hat{v}_k, P_{F_{\mu_{k+1}}} \hat{v}_k - \hat{v}_k \rangle
\]
\[
\langle \hat{w} - \hat{x}_{k+1} + \hat{x}_{k+1} - P_{F_{\mu_{k+1}}} \hat{v}_k, P_{F_{\mu_{k+1}}} \hat{v}_k - \hat{v}_k \rangle
\]
\[
= \langle \hat{x}_{k+1} - P_{F_{\mu_{k+1}}} \hat{v}_k, P_{F_{\mu_{k+1}}} \hat{v}_k - \hat{v}_k \rangle,
\]
and this is zero since by our choice of \( \lambda_{k+1} \) the point \( \hat{x}_{k+1} \) belongs to \( P_{k+1} \).

We now show geometrically that, in our situation, case (i) of proposition 1 cannot happen, i.e., it cannot be true that for some index \( k \) we have at the same time that

\begin{align*}
(11) & \quad |||\hat{x}_{k+1} - \hat{v}_k|||^2 |||\hat{v}_0 - \hat{v}_k|||^2 - \langle \hat{v}_k - \hat{v}_0, \hat{x}_{k+1} - \hat{v}_k \rangle^2 = 0 \\
(12) & \quad \langle \hat{v}_k - \hat{v}_0, \hat{x}_{k+1} - \hat{v}_k \rangle < 0.
\end{align*}

Indeed, due to the manner in which \( \hat{v}_k \) is constructed from \( \hat{v}_{k-1} \) and \( \hat{x}_k \), and from lemma 2, we conclude that \( \left( \bigcap_{i=1}^{\infty} F_{\mu_i} \right) \cap D \) is situated on one side of the "intersection line" \( D \cap R_k \) which also passes through \( \hat{v}_k \). Now, according to the Cauchy-Bunyakowski-Schwarz inequality it would follow from (11) that the vectors \( \hat{v}_k - \hat{v}_0 \) and \( \hat{x}_{k+1} - \hat{v}_k \) are linearly dependent, and from (12) it would then result that \( \hat{x}_{k+1} \) is not only different from \( \hat{v}_k \), but is situated on the same side of the line \( D \cap R_k \) as \( \hat{v}_0 \), and this is the opposite side of where \( \left( \bigcap_{i=1}^{\infty} F_{\mu_i} \right) \cap D \) is situated. As such the distance of any point \( \hat{z} \) in \( \left( \bigcap_{i=1}^{\infty} F_{\mu_i} \right) \cap D \) to \( \hat{x}_{k+1} \) would be strictly greater than the distance of \( \hat{z} \) to \( \hat{v}_k \). But this is impossible, since we necessarily must have that \( |||\hat{v}_k - \hat{z}||| \geq |||\hat{x}_{k+1} - \hat{z}||| \) for any such \( \hat{z} \), as is shown in the following lemma.

Lemma 3.

For any index \( k \), and for any \( \hat{z} \) in \( \left( \bigcap_{i=1}^{\infty} F_{\mu_i} \right) \cap D \) we have
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\[ |||\mathbf{z}_{k+1} - \mathbf{z}||| \leq |||\mathbf{z}_k - \mathbf{z}|||.\]

Proof.

According to the construction of \( \mathbf{z}_{k+1} \) from \( \mathbf{z}_k \) and using lemma 2, we have that

\[ \langle \mathbf{z} - \mathbf{z}_{k+1}, \mathbf{z}_{k+1} - \mathbf{z}_k \rangle \geq 0 \quad \text{for all } \mathbf{z} \text{ in } \bigcap_{i=1}^{\infty} F_{F_{k+1}} \cap D. \]

Hence,

\[
|||\mathbf{z}_k - \mathbf{z}|||^2 = |||(\mathbf{z}_k - \mathbf{z}_{k+1}) + (\mathbf{z}_{k+1} - \mathbf{z})|||^2
= |||\mathbf{z}_k - \mathbf{z}_{k+1}|||^2 + |||\mathbf{z}_{k+1} - \mathbf{z}|||^2 + 2 \langle \mathbf{z}_k - \mathbf{z}_{k+1}, \mathbf{z}_{k+1} - \mathbf{z} \rangle \geq 0,
\]

from which we conclude that

\[ |||\mathbf{z}_k - \mathbf{z}|||^2 \geq |||\mathbf{z}_k - \mathbf{z}_{k+1}|||^2 + |||\mathbf{z}_{k+1} - \mathbf{z}|||^2.\]

\[ \square \]

From now on, we are sure that \( \mathbf{z}_{k+1} = F(\mathbf{z}_0, \mathbf{z}_k, \mathbf{z}_{k+1}) \) as given by (7) is well-defined, by using in proposition 1 the cases (ii), (iii), (iv), replacing \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) by \( \mathbf{z}_0, \mathbf{z}_k, \mathbf{z}_{k+1} \) respectively.

4 Convergence of a subsequence of the sequence \( \{\mathbf{z}_k\}_{k=0}^{+\infty} \) in \( \mathcal{H} \)

Remembering the fact that, when \( \mathbf{z}_k \) has been obtained, \( \mathbf{z}_{k+1} \) is a point in the half-space \( S(\mathbf{z}_0, \mathbf{z}_k) \) and that \( \mathbf{z}_k \) is the projection of \( \mathbf{z}_0 \) onto \( S(\mathbf{z}_0, \mathbf{z}_k) \), we may conclude that \( |||\mathbf{z}_0 - \mathbf{z}_k||| \leq |||\mathbf{z}_0 - \mathbf{z}_{k+1}||| \). Hence, the sequence \( \{|||\mathbf{z}_k - \mathbf{z}_0|||\}_{k=1}^{+\infty} \) is non-decreasing. This sequence is also bounded from above; indeed, since \( \bigcap_{i=1}^{\infty} F_{F_{k+1}} \cap D \) is situated on one side of \( D \cap F_{k+1} \) (which is orthogonal to \( \mathbf{z}_{k+1} - \mathbf{z}_k \)), and since \( \mathbf{z}_{k+1} \) also belongs to the half-space \( S(\mathbf{z}_k, \mathbf{z}_{k+1}) \) (which means in particular that \( \mathbf{z}_{k+1} \) is the projection of \( \mathbf{z}_k \) onto \( S(\mathbf{z}_k, \mathbf{z}_{k+1}) \)), we conclude that for any \( \mathbf{z} \) in \( \bigcap_{i=1}^{\infty} F_{F_{k+1}} \cap D \) and for any index \( k \) we have that \( |||\mathbf{z}_{k+1} - \mathbf{z}_0||| \leq |||\mathbf{z} - \mathbf{z}_0|||. \)

So, the sequence \( \{|||\mathbf{z}_k - \mathbf{z}_0|||\}_{k=1}^{+\infty} \) is convergent, say to some number \( d \).

From this we immediately derive some results which will be used further on.

First of all, since \( |||\mathbf{z}_k - \mathbf{z}_0||| = |||(\mathbf{z}_k - \mathbf{z}_{k-1}) + (\mathbf{z}_{k-1} - \mathbf{z}_0)||| \), we conclude, on squaring the norms and using the fact that both sequences \( \{|||\mathbf{z}_k - \mathbf{z}_0|||^2\}_{k=1}^{+\infty} \) and \( \{|||\mathbf{z}_{k-1} - \mathbf{z}_0|||^2\}_{k=2}^{+\infty} \) are convergent to \( d^2 \), that

\[ \lim_{k \to +\infty} |||\mathbf{z}_k - \mathbf{z}_{k-1}|||^2 = 0. \]
In the same manner, writing $||v_{k+1} - \bar{v}_k|| = ||(v_{k+1} - \bar{x}_{k+1}) + (\bar{x}_{k+1} - \bar{v}_k)||$, squaring the norms and using (13) leads to

$$\lim_{k \to +\infty} ||v_{k+1} - \bar{x}_{k+1}||^2 = 0 = \lim_{k \to +\infty} ||\bar{x}_{k+1} - \bar{v}_k||^2. \tag{14}$$

Since

$$||\bar{v}_k - \bar{x}_{k+1}||^2 = ||\bar{v}_k - P_{F_{n+1}} \bar{v}_k||^2 + ||P_{F_{n+1}} \bar{v}_k - \bar{x}_{k+1}||^2,$$

we conclude by using (14) that also

$$\lim_{k \to +\infty} ||\bar{v}_k - P_{F_{n+1}} \bar{v}_k||^2 = 0. \tag{15}$$

By construction, $P_{F_{n+1}} \bar{v}_k$ denotes projection of $\bar{v}_k$ onto (one of) the most remote set(s) among all $\{F_n\}_{n=1}^{\infty}$ appearing in the construction; so we have

$$||\bar{v}_k - P_{F_{n+1}} \bar{v}_k|| \geq ||\bar{v}_k - P_{F_n} \bar{v}_k|| \tag{16}$$

for all indices $i$. Combining this result with (15) we conclude that

$$||\bar{v}_k - P_{F_n} \bar{v}_k|| \to 0 \tag{17}$$

for $k \to +\infty$, for each index $i$.

Finally, we remark that the sequence $\{\bar{v}_k\}_{k=0}^{\infty}$ is bounded; indeed, we have

$$||\bar{v}_k|| \leq ||\bar{v}_k - \bar{v}_0|| + ||\bar{v}_0|| \leq d + ||\bar{v}_0||.$$

We now have all elements to prove the convergence result for a subsequence of the sequence $\{\bar{v}_k\}_{k=0}^{\infty}$ in $D \subset \mathcal{H}$. We only need its weak convergence.

Theorem 1.

When $\left( \bigcap_{i=1}^{\infty} F_{\mu_i} \right) \cap D \neq \emptyset$ in $\mathcal{H}$, and when, starting from an arbitrary point $\bar{v}_0$ in $D$, the sequence $\{\bar{v}_n\}_{n=0}^{\infty}$ in $D$ is constructed by

$$\bar{x}_1 = \bar{v}_0 + 1(P_{D}(P_{F_{\mu_1}} \bar{v}_0) - \bar{v}_0)$$

and, for $n \geq 1$

$$\left\{ \begin{array}{ll}
\bar{x}_{n+1} = \bar{v}_n + \lambda_n (P_{D}(P_{F_{\mu_{n+1}}} \bar{v}_n) - \bar{v}_n) \\
\bar{v}_{n+1} = P(\bar{v}_n, \bar{v}_n, \bar{x}_{n+1})
\end{array} \right. \tag{18}$$

with (for $n \geq 0$) $P_{F_{\mu_{n+1}}} \bar{v}_n$ denoting projection of $\bar{v}_n$ onto (one of) the most remote set(s) among all $F_{\mu_i}$, and with $\lambda_n$ given by

$$\lambda_n = \frac{||P_{F_{\mu_{n+1}}} \bar{v}_n - \bar{v}_n||^2}{||P_{D}(P_{F_{\mu_{n+1}}} \bar{v}_n) - \bar{v}_n||^2},$$

then the sequence $\{\bar{v}_n\}_{n=0}^{\infty}$ has a subsequence that is weakly convergent in $\mathcal{H}$ to a point in
(\bigcap_{i=1}^{\infty} F_{\mu_i}) \cap D.

**Proof.**

The sequence \( \{v_n\}_{n=0}^{\infty} \) is bounded; hence there exists a subsequence \( \{v_{n_k}\}_{k=0}^{\infty} \) of it and a point \( \hat{a} \) in \( \mathcal{H} \) such that \( \{v_{n_k}\}_{k=0}^{\infty} \) is weakly convergent to \( \hat{a} \). Let \( i \) be a fixed index. For each \( \hat{b} \in \mathcal{H} \) we then have

\[
| \langle \langle P_{F_{\mu_i}} v_{n_k} - \hat{a}, \hat{b} \rangle \rangle | = | \langle \langle P_{F_{\mu_i}} v_{n_k} - v_{n_k}, \hat{b} \rangle \rangle | + | \langle \langle v_{n_k} - \hat{a}, \hat{b} \rangle \rangle |
\]

Both terms on the right-hand side tend to zero, the last one by weak convergence of \( \{v_{n_k}\}_{k=0}^{\infty} \) to \( \hat{a} \), the first one by (17). We conclude that also the sequence \( \{P_{F_{\mu_i}} v_{n_k}\}_{k=0}^{\infty} \) is weakly convergent to \( \hat{a} \) for each index \( i \). This last sequence belongs to \( F_{\mu_i} \), which is a weakly closed set; so also \( \hat{a} \) belongs to \( F_{\mu_i} \), and this is true for all \( i \). But \( \hat{a} \) also belongs to \( D \), since each \( v_{n_k} \) belongs to \( D \) and \( D \) is also weakly closed. Hence, \( \hat{a} \in \left( \bigcap_{i=1}^{\infty} F_{\mu_i} \right) \cap D \).

We remark that, when \( \{v_{n_{n_k}}\}_{k=0}^{\infty} \) is another subsequence of \( \{v_n\}_{n=0}^{\infty} \) that is weakly convergent to some point \( \tilde{a}' \in \mathcal{H} \), then just as before its weak limit point \( \tilde{a}' \) also belongs to \( \left( \bigcap_{i=1}^{\infty} F_{\mu_i} \right) \cap D \).

5 **Convergence of the sequence** \( \{v_k\}_{k=0}^{\infty} \) **in** \( H \) **to** \( P_{C^*} v_0 \)

In this last section we show that the sequence \( \{v_k\}_{k=0}^{\infty} \) in \( H \), with \( v_k = q^{-1}(\tilde{v}_k) \), is norm convergent to \( P_{C^*} v_0 \).

We first repeat some results from [7], which will be needed in the sequel.

(i) When \( C^* = \bigcap_{j=1}^{n} C_j \) is nonempty, then also \( (\bigcap_{j=1}^{n} F_{\mu_j}) \cap D \) is nonempty, since \( q(C^*) \subset (\bigcap_{j=1}^{n} F_{\mu_j}) \cap D \). Since by our assumption \( C^* \neq \phi \), we conclude in particular that in section 4 \( \left( \bigcap_{i=1}^{\infty} F_{\mu_i} \right) \cap D \neq \phi \).

(ii) When a sequence \( \{\tilde{w}_n\}_{n=0}^{\infty} \) in \( D \subset H \) is weakly convergent to some point \( \tilde{w} \in D \), then by putting \( w_n = q^{-1}(\tilde{w}_n) \) and \( w = q^{-1}(\tilde{w}) \) we conclude that the sequence \( \{w_n\}_{n=0}^{\infty} \) in \( H \) is weakly convergent in \( H \) to \( w \in H \). In particular, from the weak
convergence of the subsequence \( \{u_k\}_{k=0}^{+\infty} \) to \( \alpha \), as proved in theorem 1, we conclude that the subsequence \( \{u_n\}_{n=0}^{+\infty} \) of \( \{u_n\}_{n=0}^{+\infty} \) in \( H \) is weakly convergent in \( H \) to some point \( a \in H \), with \( a = q^{-1}(\alpha) \).

(iii) For points \( x \) and \( y \) in \( H \) with images \( \hat{x} \equiv q(x) \) and \( \hat{y} \equiv q(y) \) (in \( D \subset H \)) we have

\[
\langle \hat{x}, \hat{y} \rangle = \langle x, y \rangle, \text{ and}
\]

\[
||| \hat{x} - \hat{y} ||| = || x - y ||.
\]

Using (i) and (iii) we get in particular from section 4, for any point \( z \) in \( C^* \).

\[
\|v_n - v_0\| \leq \|v_{n+1} - v_0\| \leq ||z - v_0||.
\]

From (ii) we know that the subsequence \( \{u_n\}_{n=0}^{+\infty} \) of \( \{u_n\}_{n=0}^{+\infty} \) in \( H \) is weakly convergent to some point \( a \in H \), with \( a = q^{-1}(\alpha) \).

Lemma 4.

\( a \in \bigcap_{j=1}^{r} C_j \equiv C^* \).

Proof.

Combining (4) and (15) we obtain for the sequence \( \{u_n\}_{n=0}^{+\infty} \)

\[
\sum_{j=1}^{r} \mu_{n+1}(j)\|v_n - P_jv_n\|^2 \to 0 \text{ when } n \to +\infty.
\]

Remembering the fact that the numbers \( \{\mu_{n+1}(j)\}_{j=1}^{r} \) where chosen such that the left-hand side of (21) was maximal, we conclude that also

\[
\sum_{j=1}^{r} \frac{1}{r}\|v_n - P_jv_n\|^2 \to 0 \text{ when } n \to +\infty
\]

So, for each \( j \in \{1, \ldots, r\} \) we get in particular

\[
\|v_n - P_jv_n\| \to 0 \text{ when } n \to +\infty.
\]

For any \( b \in H \) we have, for any fixed index \( j \in \{1, \ldots, r\} \):

\[
| < P_jv_n, a > - b |
\]

\[
= | < P_jv_n, b > - < v_n, a > - b |
\]

\[
\leq ||P_jv_n - v_n|| \|b\| + | < v_n, a > - b |.
\]
Both terms on the right-hand side tend to zero for $k \to +\infty$, the first one by (22), the last one by weak convergence of $\{v_{n_k}\}_{k=0}^{\infty}$ to $a$. So, for each $j \in \{1, \ldots, r\}$ the sequence $\{P_jv_{n_k}\}_{k=0}^{\infty}$ is weakly convergent to $a$. Since for fixed $j$ all elements $P_jv_{n_k}$ belong to $C_j$, also $a$ belongs to $C_j$. Hence $a \in \bigcap_{j=1}^{r} C_j$.

The subsequence $\{v_{n_k} - v_0\}_{k=0}^{+\infty}$ is weakly convergent to $a - v_0$; by (20) the subsequence $\{\|v_{n_k} - v_0\|\}_{k=0}^{+\infty}$ is non-decreasing and bounded from above by $\|z - v_0\|$ for any $z$ in $C^*$. Applying the weak lower semi-continuity of the norm function $\| \|$ [15, p. 134] to the subsequence $\{\|v_{n_k} - v_0\|\}_{k=0}^{+\infty}$ and denoting the limit of the converging subsequence of numbers $\{\|v_{n_k} - v_0\|\}_{k=0}^{+\infty}$ by $d$, we obtain

\[(23) \quad \|a - v_0\| \leq \liminf_{k \to +\infty} \|v_{n_k} - v_0\| = \lim_{k \to +\infty} \|v_{n_k} - v_0\| = d,\]

and $d \leq \|z - v_0\|$ for all $z$ in $\bigcap_{j=1}^{r} C_j$. Since $a \in \bigcap_{j=1}^{r} C_j$, we conclude that the distance of $v_0$ to $a$ is the infimum of the distances of $v_0$ to all points $z$ in $\bigcap_{j=1}^{r} C_j$; hence $a = P_{C^*} v_0$.

Again using the inequality

\[\|v_{n_k} - v_0\| \leq \|z - v_0\|, \text{ for each } z \text{ in } C^*, \text{ for each } k,\]

and the fact that the subsequence $\{\|v_{n_k} - v_0\|\}_{k=0}^{+\infty}$ is convergent, we conclude

\[(24) \quad \lim_{k \to +\infty} \|v_{n_k} - v_0\| \leq \inf_{z \in C^*} \|v_0 - z\| = \|v_0 - P_{C^*} v_0\| = \|v_0 - a\|,\]

Combining (23) and (24) we have

\[(25) \quad \lim_{k \to +\infty} \|v_{n_k} - v_0\| = \|a - v_0\|.\]

The norm convergence of $\{v_{n_k}\}_{k=0}^{+\infty}$ to $a \equiv P_{C^*} v_0$ is now readily obtained. Indeed, since the subsequence $\{v_{n_k} - v_0\}_{k=0}^{+\infty}$ is weakly convergent to $a - v_0$, and the subsequence $\{\|v_{n_k} - v_0\|\}_{k=0}^{+\infty}$ is convergent to $\|a - v_0\|$, it follows from [11, p. 233] that the subsequence $\{v_{n_k}\}_{k=0}^{+\infty}$ is norm convergent to $a \equiv P_{C^*} v_0$.

Finally, we have to settle the norm convergence in $H$ of the whole sequence $\{v_{n}\}_{n=0}^{+\infty}$ to $P_{C^*} v_0$; this goes as follows. In this section we showed that, whenever a subsequence $\{v_{n_k}\}_{k=0}^{+\infty}$ of the sequence $\{v_{n}\}_{n=0}^{+\infty}$ in $D \subset H$ is weakly convergent to some point $\widehat{a} \in D$, then the "inverse transformed" subsequence $\{v_{n_k}\}_{k=0}^{+\infty}$ is norm convergent to $a \equiv q^{-1}(\widehat{a})$ with $a \equiv P_{C^*} v_0$. As such, any other subsequence $\{v_{n_{k'}}\}_{k'=0}^{+\infty}$ of $\{v_{n}\}_{n=0}^{+\infty}$
that is weakly convergent to some point \( \hat{a} \in D \), will be such that \( \{v_n\}_{n=0}^{+\infty} \) will be norm convergent in \( H \) to \( a' \equiv q^{-1}(\hat{a}) \), again with \( a' = P_{C^*} v_0 \). Due to the isomorphism \( q \) between \( H \) and \( D \) we conclude that \( \hat{v} = \hat{a} \). Otherwise said, whenever a subsequence of the sequence \( \{v_n\}_{n=0}^{+\infty} \) in \( \mathcal{H} \) is weakly convergent, it is weakly convergent to the same point. Hence the whole sequence \( \{v_n\}_{n=0}^{+\infty} \) in \( \mathcal{H} \) is weakly convergent. Applying then the methods of this section to the whole weakly converging sequence \( \{v_n\}_{n=0}^{+\infty} \) in \( H \), we again will get the result that this sequence \( \{v_n\}_{n=0}^{+\infty} \) is in fact norm convergent to \( P_{C^*} v_0 \).

To sum up the described method as an algorithm for the construction of the sequence \( \{v_n\}_{n=0}^{+\infty} \) in \( H \) (that we shall state as theorem 2), we still have to transform the determination of \( \hat{v}_{n+1} \equiv P(\hat{v}_0, \hat{v}_n, \hat{x}_{n+1}) \) in \( \mathcal{H} \), as mentioned in theorem 1 and described in proposition 1, to the Hilbert space \( H \). Since the points \( \hat{v}_0, \hat{v}_n, \hat{x}_{n+1} \) all belong to \( D \subset \mathcal{H} \) we have, in view of (18) and (19), the following procedure (P):

**Procedure (P)**

Given the ordered triple \((v_0, v_n, x_{n+1})\) of points in \( H \), denote for any integer \( n \geq 1 \) the following expressions for short by \( \alpha_{n+1}, \beta_{n+1}, \gamma_{n+1}, \delta_{n+1} \) respectively:

\[
\begin{align*}
\alpha_{n+1} & \equiv < v_n - v_0, x_{n+1} - v_n > \\
\beta_{n+1} & \equiv \|x_{n+1} - v_n\|^2 \\
\gamma_{n+1} & \equiv \|v_0 - v_n\|^2 \\
\delta_{n+1} & = \beta_{n+1} \gamma_{n+1} - \alpha_{n+1}^2.
\end{align*}
\]

Then put

(i) for \( \delta_{n+1} = 0, \alpha_{n+1} \geq 0 \)

\[P(v_0, v_n, x_{n+1}) = x_{n+1}\]

(ii) for \( \delta_{n+1} \neq 0, \alpha_{n+1} \beta_{n+1} - \delta_{n+1} \geq 0 \)

\[P(v_0, v_n, x_{n+1}) = v_0 + (1 + \frac{\alpha_{n+1}}{\beta_{n+1}})(x_{n+1} - v_n)\]

(iii) for \( \delta_{n+1} \neq 0, \alpha_{n+1} \beta_{n+1} - \delta_{n+1} < 0 \)

\[P(v_0, v_n, x_{n+1}) = v_n + \frac{\beta_{n+1}}{\delta_{n+1}}(\gamma_{n+1}(x_{n+1} - v_n) - \alpha_{n+1}(v_n - v_0)).\]

The algorithm to obtain \( P_{C^*} v_0 \) may then be described as follows

**Theorem 2.**

Let \( v_0 \in H \), and suppose \( C^* \equiv \bigcap_{j=1}^{\infty} C_j \neq \emptyset \). Construct the sequence \( \{v_n\}_{n=0}^{+\infty} \) as follows:
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\[ v_1 \equiv x_1 = v_0 + \lambda_1 (\sum_{j=1}^{r} \mu_1(j) P_j v_0 - v_0) \]

and, for \( n \geq 1, \)

\[
\begin{align*}
x_{n+1} &= v_n + \lambda_{n+1} (\sum_{j=1}^{r} \mu_{n+1}(j) P_j v_n - v_n) \\
v_{n+1} &= P(v_0, v_n, x_{n+1})
\end{align*}
\]

in which, for each integer \( k \geq 0, \)

(i) \( \{\mu_{k+1}(j)\}_{j=1}^{r} \) are determined such that \( \sum_{j=1}^{r} \mu_{k+1}(j) \| v_k - P_j v_k \|^2 \) is maximal.

(ii) \( \lambda_{k+1} \) is then determined by

\[
\lambda_{k+1} = \frac{\sum_{j=1}^{r} \mu_{k+1}(j) \| v_k - P_j v_k \|^2}{\| v_k - \sum_{j=1}^{r} \mu_{k+1}(j) P_j v_k \|^2}
\]

and,

(iii) for each \( n \geq 1, P(v_0, v_n, x_{n+1}) \) is determined by procedure (P).

Then the sequence \( \{v_n\}_{n=0}^{\infty} \) is norm convergent to the projection \( P_{C^*} v_0 \) of \( v_0 \) onto \( C^* \).

\[ \square \]

References


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