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Modulation Spaces: Looking Back and Ahead

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Abstract

This note provides historical perspectives and background on the motivations which led to the invention of the *modulation spaces* by the author almost 25 years ago, as well as comments about their role for ongoing research efforts within time-frequency analysis. We will also describe the role of modulation spaces within the more general coorbit theory developed jointly with Karlheinz Gröchenig, and which eventually led to the development of the concept of Banach frames and more recently to the so-called localization theory for frames. A comprehensive bibliography is included.

Key words and phrases: modulation spaces, Wiener amalgam spaces, Gabor analysis, time-frequency analysis, Heisenberg group, time-frequency lattice, Fourier transform, localization operator, Banach frames, localization of frames, pseudo-differential operators, Gelfand triples, Wilson bases.

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1 Introduction

By now the *modulation spaces* have fixed their place in the zoo of Banach spaces of functions and distributions, as described in the preface to this issue by Karlheinz Gröchenig and Christopher Heil. They have specific relevance to almost all of the central topics of time-frequency analysis, and in particular to Gabor analysis. According to Gröchenig's descriptions of this field (see [115]) *time-frequency analysis* can be characterized as that part of mathematical analysis for which the use of *time-frequency shift operators* plays a central role. Time-frequency shift operators are defined to be the unitary operators on $\boldsymbol{L}^2(\mathbb{R}^d)$ (or on $\boldsymbol{L}^2(G)$, for a LCA group G) given by $\pi(\lambda) = \pi(t, \omega) = M_\omega T_t$, with

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$[M_\omega f](z) := \exp(2\pi i\omega \cdot z)f(z)$ and $[T_t f](z) := f(z-t)$, and where $\lambda = (t,\omega)$ is a point in the time-frequency place $\mathbb{R}^{2d}$ (which is also sometimes called *phase space*). The single most important tool within TF-analysis is certainly the short-time Fourier transform, or STFT. The STFT (which is often called the sliding window Fourier transform, see [1]), of a function f with respect to a window g is given (first for $f, g \in \boldsymbol{L}^2(\mathbb{R})$) by

$$S_g f(x,y) := \int_{\mathbb{R}} e^{-2\pi i y\cdot z}\, \bar{g}(z-x)f(z)dz\langle f, M_y T_x g\rangle \quad \text{for } (x,y) \in \mathbb{R}^2. \tag{1}$$

Note that the STFT is symmetric with respect to f and g, i.e.

$$S_g f(x,y) = e^{-\pi i x\cdot y} \cdot \overline{S_f g(-x,-y)}\ . \tag{2}$$

This is important as it shows that decay properties of $S_g f$ are joint properties of f *and* g. Due to its good time-frequency concentration and its invariance under the Fourier transform, the Gauss kernel $g_0(x) := e^{-\pi x^2}$ is a very good choice of window. In fact, Dennis Gabor argued that it attains equality in the Heisenberg uncertainty relation. The formula $S_g f(x,y) = e^{-2\pi i x\cdot y} \cdot S_{\hat{g}} \hat{f}(y,-x)$ implies

$$|S_{g_0} f(x,y)| = |S_{g_0} \hat{f}(y,-x)| \quad \text{for } f \in \boldsymbol{L}^2(\mathbb{R})\ . \tag{3}$$

Thus the behavior of $S_{g_0}\hat{f}$ is exactly the same as that of $S_{g_0} f$, rotated by 90° in the TF-plane. Since g_0 is a Schwartz function, the STFT with window g_0 can be extended to the space of all tempered distributions (which have an STFT of at most polynomial growth), and furthermore even to spaces of ultra-distributions since g_0 has exponential decay (see [36, 115]). It is nowadays a well-established principle, that, for example, the behavior of a function or distribution in terms of variable smoothness over time can be qualitatively well-described by means of the decay and summability properties of its STFT (with respect to a window g that has good TF-concentration). This is a very natural concept, and the modulation spaces can be described in such terms. Let us recall now the "modern" description of modulation spaces.

Given a non-zero window $g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, and $1 \le p,\ q \le \infty$, the *modulation space* $\boldsymbol{M}^s_{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \boldsymbol{\mathcal{S}}'(\mathbb{R}^d))$ such that $V_g f \in \boldsymbol{L}^s_{p,q}(\mathbb{R}^{2d})$ (a weighted mixed-norm space). The norm on $\boldsymbol{M}^s_{p,q}$ is

$$\|f\|_{\boldsymbol{M}^s_{p,q}} = \|V_g f\|_{\boldsymbol{L}^s_{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p\, dx \right)^{q/p} (1+|\omega|)^{sq}\, d\omega \right)^{1/p} .$$

In fact, we believe that such norms quantifiably describe properties of functions which are at least as relevant as those described by the standard $\boldsymbol{L}^p$-norms. Just recall that $\boldsymbol{L}^p$-norms depend only on the distribution of values taken by a function, independently of where those values are taken. Hence, any measure-preserving transformation acts isometrically on $\boldsymbol{L}^p(\mathbb{R}^d)$, yet typically destroys smoothness and decay properties. In contrast, a modulation space norm would distinguish between smooth rearrangements and non-smooth ones of a given function.

Another aspect of TF-analysis is the fact that additional insight can be gained by studying the properties of linear operators (e.g., time-varying systems) by analyzing not only their (distributional) kernels, but also their (distributional) spreading functions, which describe the amount of TF-shifts "occurring within a given operator" (cf. [146, 85]).

Gabor analysis, in turn, can be roughly described as that part of TF-analysis which operates with a sampled version of $S_g f$, and is typically concerned with properties of so-called Gabor or Weyl–Heisenberg systems which arise in the form $(\pi(\lambda)g)_{\lambda\in\Lambda}$, where $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ is a lattice. The terminology refers to Dennis Gabor's seminal paper of 1946 [107], where he proposed a kind of "atomic decomposition" of arbitrary functions as a double series using such a system with $\Lambda = \mathbb{Z}^{2d} \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ and $g = g_0$.

Without going into further details let us mention here that many questions in TF-analysis make use (explicitly and more often implicitly) of modulation spaces in one form or the other. The reason for today's wide-spread interest in TF-analysis is its many applications, for example in the modeling of wireless channels, or for the analysis of linear operators as mentioned above.

The present paper tries to provide a little bit of background and a look behind the scenes at how the modulation spaces came to life more than twenty years ago, and what further developments have been spurred by their study in the meantime. Although we have built a rather lengthy bibliography containing virtually all references to papers explicitly referring to modulation spaces, we are sure that there are many others not known to us, using modulation-space conditions only implicitly. Contributors to the field are encouraged to send updates to the author in order to have their contributions integrated into the NuHAG bibliography system and subsequently to the Gabor server.[1]

2 The Motivation for Modulation Spaces in 1983

I start my narration with a description of my background and interests at the time of the development of the theory of modulation spaces, which certainly started to take shape in the spring of 1980, when I spent a memorable semester (my first visiting position) at the applied mathematics department of the University of Heidelberg at the invitation of Wolf Beiglböck and Michael Leinert. I had just done my Habilitation at the University of Vienna in the spring of

[1] `http://www.univie.ac.at/nuhag-php/home/gabor.php`

1979, on the subject of Banach convolution algebras (see, e.g., [51]). It took place shortly after a winter-school at our institute in February 1979, organized by Siegfried Grosser (see [53]). It was at this occasion that I presented for the first time "my Segal algebra"[2] $\boldsymbol{S}_0(\mathbb{R})$, as the smallest isometrically character invariant Segal algebra.

I reported that I had discovered a Fourier-invariant Banach space, which was characterized by so-called uniform partitions of unity, as they had been used (already) by Norbert Wiener [204, 205] in his work on Tauberian theorems. One of the spaces, used in Wiener's description of what I call his second Tauberian theorem, was the Segal algebra called "Wiener's algebra" in [174]. I had observed earlier that this space, described by the symbol $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1)(\mathbb{R}^d)$, was the minimal Segal algebra which at the same time was also a pointwise $\boldsymbol{C}_0(\mathbb{R}^d)$-module (see [50]). Looking back now it is easy to recognize that this was the beginning of my approach to Wiener amalgam spaces.

Classical amalgam spaces have been treated in a number of papers in the early 1980s, see, e.g., [15, 16, 27, 102]. During my semester in the spring of 1980 in Heidelberg I took a closer look into the most general approach to what I called *Wiener-type* spaces $\boldsymbol{W}(\boldsymbol{B}, \boldsymbol{C})$ (which are now called *Wiener amalgam spaces*). These spaces are described by the global behavior — decay and summability, that is expressed by the *global component* $\boldsymbol{C}$ — of the local property that is expressed by the *local component* $\boldsymbol{B}$. I remember working hard on giving a very compact description of this concept in "full generality," but also having fun with my brand-new typewriter. The outcome, and still a standard reference, was the article [60].[3] That paper showed that certain continuous descriptions are equivalent to corresponding discrete characterizations, using so-called BUPUs (*bounded uniform partitions of unity*). Moreover, basic convolution relations for Wiener amalgam spaces were derived. While Wiener amalgams have been an important tool within the majority of research topics I have worked on since that time, they are still not widely used, and some useful information appears to be hidden in publications which have received little attention by the scientific community, such as [66, 67, 68, 69, 89, 70, 92, 86] or the Master's thesis [45] of Thomas Dobler from 1989.

While my first interest in Banach spaces of functions (already during my Ph.D. thesis) was in weighted $\boldsymbol{L}^p$-spaces and their generalizations (a still informative summary of properties of weight functions is given in [50],[4] and see [121, 118] for recent accounts of the subject), soon function spaces defined by dyadic decompositions turned out to be natural in the derivation of convolution estimates. I found such spaces also in the work of Carl Herz and Raymond John-

[2] A PDF version of the extended abstract to this event is available at http://www.univie.ac.at/nuhag-php/bibtex/login/files/areyfegi79_winter79.pdf

[3] see http://www.univie.ac.at/nuhag-php/bibtex/open_files/fe83_wientyp1.pdf

[4] see http://www.univie.ac.at/nuhag-php/bibtex/open_files/fe79-2_gewfkta.pdf

son [133, 140], and above all in the work of Jaak Peetre and Hans Triebel (see [165, 14, 193, 194] for just some of the relevant books on the subject available at that time). I was familiar with (real and complex) interpolation theory for Banach spaces, and that it is desirable to have families of Banach spaces of distributions which are closed under duality (as long as the test functions are dense in the Banach space and therefore the dual can be considered as a Banach space of distributions as well), and interpolation methods. The prototypical examples of this type were the Besov–Triebel–Lizorkin spaces. The unified view on these spaces by means of their Fourier-analytic description (which in turn was built on Paley–Littlewood theory) was well-described in the work by Peetre and Triebel, long before wavelet theory arose. In fact, I consider the work on atomic decompositions of Besov spaces using the φ-transform, due to Frazier and Jawerth [103, 104, 105, 106], to be an immediate consequence of those Fourier-analytic characterizations of Besov spaces: the φ-transform allows one to apply the Shannon sampling principle to the different contributions in the dyadic frequency bands. Moreover, interpolation theory provides for the method of *retracts* with respect to vector-valued Banach spaces in order to establish interpolation results for these spaces (cf. [14]).

In the years since my Ph.D. thesis (in 1974), I had been very much interested in all kinds of Banach convolution algebras (in particular weighted L^p-spaces), but I had also done a lot of reading on Banach spaces of sequences (so-called BK-spaces), and generalizations of the family of $\boldsymbol{L}^p$-spaces, such as Lorentz and Orlicz spaces. From all that I drew the conclusion that it does not make sense to concentrate too much on the study of individual Banach function spaces, but that one should consider whole families of Banach spaces, which ideally should be (almost) closed under duality and, say, complex interpolation. The class of all weighted $\boldsymbol{L}^p$-spaces is such an "ideal family," and Besov spaces and Triebel–Lizorkin spaces provide another example. In this light the class of solid BF-spaces[5] appeared as a very natural domain for many considerations. I had learned about complex interpolation methods (cf. [14, 165, 194]), and that the complex interpolation of weighted interpolation spaces results in another weighted L^p-space, while the applications of real interpolation (according to the real K-method) results in decomposition spaces, the decomposition domains being given by the level sets of the moderate function $m = w_1/w_2$, typically dyadic coronas in the case of polynomial weight functions w_1, w_2.

3 Basic Properties of Modulation Spaces

It is not necessary to report here on all of the important properties of modulation spaces, for at least two reasons. First, most of the important facts with

[5]Such spaces appear under the name of *Banach function spaces* in the work of Luxemburg and Zaanen [209].

respect to TF-analysis are well-presented in Gröchenig's book [115], and second, it would take far too much space within this note. So let me reveal some hidden connections: Because the original approach to modulation spaces was based on the understanding that one is "just doing" a form of Wiener amalgam spaces on the Fourier transform side, typically spaces of the form $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \boldsymbol{L}^q)(\mathbb{R}^d)$, it was clear to me from the very beginning (and indeed a motivation to provide a description of Wiener amalgam spaces in full generality) that one can invoke all the results from the theory of amalgam spaces already presented at two conferences in 1980 (see [56] and [60]). This concerns facts about the independence of the use of the specific choice of the partition of unity (the famous BUPUs), the equivalence of the "discrete and continuous norms" (for suitable "windows"), the density of test functions, or the translation and modulation invariance of these spaces. Results concerning duality and pointwise multipliers were even given in a more general form by describing pointwise multipliers of decomposition spaces [73, 63], see also [11, 87]. In fact, Wiener amalgam spaces are exactly the decomposition spaces obtained by coverings using balls of uniform size. It also can be shown that some of the modulation spaces are Banach algebras with respect to pointwise multiplication by verifying the corresponding convolution properties of Wiener amalgams on the Fourier transform side (cf. [186, 71, 60]). A similar statement can be made with respect to the interpolation methods that were discussed for Wiener amalgam spaces in [56].

The original report [59] described most of those properties in the context of locally compact Abelian groups, because they are the natural context for these kind of function spaces (much in the same way that LCA groups are the natural context for harmonic analysis as such, according to André Weil, see [202]). On the other hand, the conference report [60] put its emphasis on the family of (by now classical) modulation spaces $\boldsymbol{M}^s_{p,q}(\mathbb{R}^d)$. In fact, this program worked out well, and it was even possible to verify trace theorems (restrictions to subgroups for sufficiently smooth functions), showing exactly the same loss of smoothness as for the corresponding Besov classes. Those details were finally published in [71]. The important atomic characterizations were first reported at an approximation theory conference in Edmonton in the summer of 1986 (the publication [65] resulted from this event), during my first trip to North America. It started with a talk on Wiener amalgam spaces at the Canadian Summer Meeting in Newfoundland[6] (June 1986), and ended with my participation at the ICM in Berkeley where I gave a summary of the properties of $\boldsymbol{S}_0(\mathbb{R}^d)$. Unfortunately, plans to collect the various facts concerning $\boldsymbol{S}_0(\mathbb{R}^d)$ into a larger publication have not been realized even yet, but hopefully this will change in the near future with an ongoing book project, carried out jointly with Georg Zimmermann.

[6] I specifically remember enjoying drinking fresh milk there, for it was soon after the Chernobyl catastrophe, which contaminated the grass all over Austria with the result that there "was no milk" for some time.

From my personal point of view, a lot of the general results about the modulation spaces, i.e., at least for the classical spaces $\boldsymbol{M}^s_{p,q}(\mathbb{R}^d)$, were known from the beginning, even in the context of general locally compact Abelian groups, and described in the report [60].[7] This is true in particular for embeddings of spaces with different parameters into each other, or the invariance properties of the spaces with respect to translation, modulation, dilation or the Fourier transform.

4 The Most Important Modulation Space: $\boldsymbol{M}^1(\mathbb{R}^d)$

It is fair to say that (aside from $\boldsymbol{L}^2$ and the $\boldsymbol{L}^2$-Sobolev spaces, which are modulation spaces as well as Besov spaces) the first "true" modulation space was $(\boldsymbol{M}^1(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^1})$, introduced under the name $\boldsymbol{S}_0(\mathbb{R}^d)$ in [57], see below. By pointing out some of the important properties of this space, we will emphasize that it may still be considered as the prototypical model of a modulation space and is probably the most important in the entire family of modulation spaces. In fact, not only is it itself a suitable tool for research, but it also provides a not-so-technical introduction to the theory of generalized functions and their Fourier transforms (see [64], and hopefully soon a complete summary in an ongoing book project on the subject). It allows us to express basic facts about the Fourier transform and about the Kohn–Nirenberg and spreading symbols of operators (cf. [85]) based on the concept of the Banach Gelfand triple $(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{L}^2(\mathbb{R}^d), \boldsymbol{S}_0'(\mathbb{R}^d))$, avoiding both Lebesgue integration and the theory of topological vector spaces. Therefore I recommend such an approach to engineers who obviously have to go beyond "ordinary functions" for modeling but wish to avoid the heavy abstraction of the theory of topological vector spaces.[8]

When the space $\boldsymbol{M}^1(\mathbb{R}^d)$ was introduced in 1979, the name $\boldsymbol{S}_0(\mathbb{R}^d)$ was chosen in order to indicate that this space is the smallest among all the *Segal algebras* that are isometrically invariant under modulations. Nowadays I would describe it simply as the smallest (non-trivial) Banach space of functions which has the extra property of being isometrically invariant under all TF-shifts $\pi(\lambda)$, $\lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. This space was also discovered by Jean-Paul Bertrandias independently. In a technical report (around 1982) they described the space and its Fourier invariance. However, he and his coauthors did not recognize the large potential of this space (based on a number of further properties) and its potential for a number of applications in analysis.

Given the other properties of general Segal algebras (e.g., they are dense subspaces of $\boldsymbol{L}^1(\mathbb{R}^d)$ and are also homogeneous Banach spaces in the sense

[7] This report was only recently published (in 2003), in the proceedings of the ICWA [149]. Consequently, the modulation spaces went widely unnoticed for a long time.

[8] PDF-versions of two talks delivered at a conference (Bremen: January 2006) are found at: `http://www.univie.ac.at/nuhag-php/bibtex/open_files/fe03-1_modspa03.pdf`

of Katznelson [144]), this extra property is equivalent to being a pointwise $\mathcal{F}L^1(\mathbb{R}^d)$-module. Indeed, this was the key for introducing the description of $\boldsymbol{M}^1(\mathbb{R}^d)$ as $\boldsymbol{W}(\mathcal{F}L^1, \ell^1)(\mathbb{R}^d)$. I certainly owe a lot to Hans Reiter, who helped me to recognize the usefulness of $\boldsymbol{S}_0(\mathbb{R}^d)$ by asking me whether I could prove a variety of properties (essentially those known to hold for the Schwartz–Bruhat spaces). He was interested in these properties because he wanted to give a detailed account of André Weil's work [203] related to the metaplectic group (published in [175, 176]). It was Viktor Losert who showed in [154, 155] that $\boldsymbol{S}_0(G)$ is the only Segal algebra defined for general LCA groups that satisfies a number of functorial properties. More recently, Franz Luef was able to show that $\boldsymbol{S}_0(\mathbb{R}^d)$ is not only a good replacement for the Schwartz–Bruhat space in the work of Weil, but also in the work of Marc Rieffel on non-commutative tori [156, 157].

Obviously $\boldsymbol{S}_0(\mathbb{R}^d)$ and other modulation spaces of the form $\boldsymbol{M}^1_w(\mathbb{R}^d)$ are very good tools for describing the perturbation invariance of Gabor frames, be it in the context of jitter error (implicitly, such results appear in [75]), or even more interestingly for the case of "varying the TF-lattice," as described in the paper with Norbert Kaiblinger [83]. It is one of the corollaries in this paper that for any $g \in \boldsymbol{S}_0(\mathbb{R}^d)$, the set of all parameters (a, b) with $a > 0$, $b > 0$ such that the Gabor family with lattice constants (a, b) is a Gabor frame is an open set, and that the dual Gabor atom depends continuously in the $\boldsymbol{S}_0$-norm on the lattice constants within this open sets. Hence the calculation of a dual Gabor window $\tilde{g}$ for (g, a, b) can be replaced by the calculation of the dual $\tilde{g}_1$ for (g, a', b'), where (a', b') could be — just for the sake of illustration — a sufficiently close rational approximation to (a, b), at the cost of a small error in the $\boldsymbol{S}_0$-norm. This in turn implies that deviation of operator, which uses $\tilde{g}_1$ instead of $\tilde{g}$ for the reconstruction of signals f from their STFT with window g over the lattice $a\mathbb{Z}^d \times b\mathbb{Z}^d$, will be small in operator norm on $\boldsymbol{L}^2(\mathbb{R}^d)$. In the proof many of the interesting properties of $\boldsymbol{S}_0(\mathbb{R}^d)$ are used, and it is quite clear that without the use of modulation spaces fairly involved sufficient conditions (most likely some form of sufficient condition for g to belong to $\boldsymbol{S}_0(\mathbb{R}^d)$) would have to be formulated in order to obtain results of this kind.

One of the deep results used in the course of this paper is the insight that a regular Gabor frame generated from a Gabor window in $\boldsymbol{S}_0(\mathbb{R}^d)$ has its dual Gabor window $\widetilde{g} = S^{-1}(g)$ in $\boldsymbol{S}_0(\mathbb{R}^d)$ as well, because the Gabor frame operator S, which is invertible on $\boldsymbol{L}^2(\mathbb{R}^d)$ and bounded on $\boldsymbol{S}_0(\mathbb{R}^d)$ is automatically invertible as an operator on $\boldsymbol{S}_0(\mathbb{R}^d)$ (cf. [122], the rational case was done by different methods in [78]). This is in sharp contrast to the situation for general $\boldsymbol{L}^2(\mathbb{R}^d)$-windows, where even the Bessel property of the corresponding Weyl–Heisenberg families may vary wildly depending on the underlying lattice [81].

In the past years it has also been shown that $\boldsymbol{S}_0(\mathbb{R}^d)$ and its dual are convenient tools for the treatment of generalized stochastic processes, in particular for stationary ones. Some key results in this direction had been obtained already in the early 80s, but not published. The most relevant visible contribution in this direction is the 1989 Ph.D. thesis [137] of Wolfgang Hörmann, and the 2003 Ph.D. thesis of Bernard Keville [145]. In their work the various key properties of $\boldsymbol{S}_0(\mathbb{R}^d)$, including Fourier invariance and the existence of a kernel theorem, show that $\boldsymbol{S}_0(\mathbb{R}^d)$ is a very convenient and technically relatively simple, yet flexible, tool for this context. As in the deterministic case, the distributional setup of $\boldsymbol{S}_0'(\mathbb{R}^d)$ makes the Fourier transform and its inverse much more symmetric. One does not need different arguments in order to describe the Fourier transform and the inverse Fourier transform (the spectral representation of stochastic processes). More recently there are the papers [199, 197, 198] by Patrik Wahlberg which make use of these modulation spaces in a stochastic context. He also introduced vector-valued modulation spaces very recently in [200].

We also want to emphasize the role of $\boldsymbol{S}_0(G)$ as a natural domain for the Poisson formula (in standard or symplectic form). It had been observed by Tolimieri and Orr in [190] that Poisson's formula can be used to derive the fundamental identity for Gabor analysis. In full generality this principle is described in the recent paper [86]. The validity of Poisson's formula over LCA groups is also one of the cases where the use of $\boldsymbol{S}_0(G)$ instead of the Schwartz–Bruhat space $\boldsymbol{\mathcal{S}}(G)$ provides more general statements with less technical effort.

The restriction theorem for $\boldsymbol{S}_0(\mathbb{R}^d)$ also allows us to apply ordinary pointwise sampling (and, conversely, quasi-interpolation operators [84]) to functions in $\boldsymbol{S}_0(\mathbb{R}^d)$. In particular, the sufficient conditions described in the paper by A. J. E. M. Janssen [139] are satisfied for $f \in \boldsymbol{S}_0(\mathbb{R}^d)$. Consequently, one can calculate the samples of a dual Gabor window $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ by calculating the dual Gabor window of the sampled version of the given Gabor window. This has far-reaching consequences and leads in particular to the possibility of approximating dual Gabor atoms for (g, a, b) numerically by using finite models: If the samples of a Gabor atom $g \in \boldsymbol{S}_0(\mathbb{R})$ are choosen in a suitable manner, as well as appropriate discrete parameters (a_n, b_n) the dual atom within the finite context can be used to obtain a good approximation of the dual Gabor atom $\widetilde{g}$ (in the $\boldsymbol{S}_0(\mathbb{R})$-sense) by means of quasi-interpolation methods (see [143] or [182]).

Finally let us mention that the definition of $\boldsymbol{S}_0(\mathbb{R}^d)$ also captures features relevant to the context of classical summability theory. Results of this kind have been worked out in a recent series of joint papers with Ferenc Weisz (among them [93, 94]), providing a long list of classical kernels within $\boldsymbol{S}_0(\mathbb{R}^d)$.

5 Thinking in Terms of Families: Banach Frames

The Segal algebra $\boldsymbol{S}_0(\mathbb{R}^d)$, together with $\boldsymbol{L}^2(\mathbb{R}^d)$ and the dual space $\boldsymbol{S}_0'(\mathbb{R}^d)$, allows us to prove many useful properties and to discuss a number of issues that arise in a natural way in the context of harmonic analysis in general, and in particular within Gabor analysis (cf. the results provided by [95, 88]). For example, one can formulate the fact that Wilson bases are a suitable family of orthonormal bases for modulation spaces by stating that the mapping from f to its Wilson coefficients establishes a unitary Gelfand triple isomorphism between $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$ and $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)$. In other words, the usual orthonormal expansion of elements from the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$ with respect to a Wilson orthonormal basis allows us to characterize the elements of $\boldsymbol{S}_0(\mathbb{R}^d)$ as exactly those functions in $\boldsymbol{L}^2(\mathbb{R}^d)$ which have an absolutely convergent Wilson expansion. Moreover, if one chooses a Wilson basis generated by a Schwartz function, then one can identify $\boldsymbol{S}_0'(\mathbb{R}^d)$ with the subspace of tempered distributions which have bounded Wilson coefficients. Since the first appearance of Wilson bases in [44], a number of constructions have been obtained by different people. More or less as an immediate reaction to a discussion with Ingrid Daubechies in 1989, when she disclosed the construction described in [44] to me, it was clear that these bases should be well-suited to characterize the modulation spaces, very much as wavelet bases are well-suited to characterize the Besov–Triebel–Lizorkin spaces. The details are published in [79, 80]. More recent papers on Wilson bases are [192, 168, 17, 150, 206]. An alternative way to characterize modulation spaces via weighted mixed-norm conditions on their coefficients are the local Fourier bases, such as the ones constructed by Coifman and Meyer in [4, 29]. Therefore it may not be surprising that modulation spaces arise in the characterization of spaces of tempered distributions which show a certain decay rate for the non-linear least n-term approximations with respect to such expansions (cf. [123]). (Besov spaces played a similar role for wavelet expansions in the early work of Popov and DeVore). In this context the case $p \leq 1$ appears to be most relevant, see [109]. It is worthwhile noting that modulation spaces with these parameters have already been treated in Triebel's very early paper [195], while [173] is a very recent contribution to the subject.

Since Wilson bases are not true Weyl–Heisenberg families, but are systems typically obtained from tight Gabor frames of redundancy 2 by suitable pairings (see [44]), one may naturally ask whether a similar characterization is also possible using proper Gabor frames. Recall that according to the Balian–Low principle the existence of Gabor Riesz bases (and in particular orthonormal Gabor bases) for $\boldsymbol{L}^2(\mathbb{R}^d)$ with generator $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ or even $g \in \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1)(\mathbb{R}^d)$, is excluded. Again, we will explain the situation for the Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$ only. Assume that we have a Gabor atom $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ which defines a Gabor frame $(g_\lambda)_{\lambda\in\Lambda}$, i.e., the frame operator S is invertible on $\boldsymbol{L}^2(\mathbb{R}^d)$. In this situation a deep result on the automatic invertibility of S over $\boldsymbol{S}_0(\mathbb{R}^d)$ (cf. [122, 78]) implies that the dual atom $\tilde{g} = S^{-1}(g)$ belongs to $\boldsymbol{S}_0(\mathbb{R}^d)$ as well. Hence the coefficient operator $C\colon f \mapsto c_\lambda = \langle f, \pi(\lambda)g\rangle, \lambda \in \Lambda$, which is first of all defined on $\boldsymbol{L}^2(\mathbb{R}^d)$, extends to a bounded linear mapping from the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$ into

$(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)$. It has closed range because there exists a right inverse R, which is simply of the form $\mathbf{c} = (c_\lambda) \mapsto \sum_{\lambda\in\Lambda} c_\lambda \pi(\tilde{g})$, that maps $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)$ back into $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$, with $R \circ C = Id$ at all three levels. It is also possible to change the order of $\tilde{g}$ and g above, so that one comes in a linear way to the minimal norm coefficients necessary to represent a distribution f as an unconditionally convergent double series using the Gabor building blocks $(\pi(\lambda)g)_{\lambda\in\Lambda}$, with coefficients $c_\lambda = \langle f, \pi(\lambda)\tilde{g}\rangle$.

In fact, we see here a simple example of the much more general concept of a *Banach frame for Gelfand triples*, which was introduced by Karlheinz Gröchenig in [113] for the context of individual Banach spaces. So in fact, the above statement[9] describes a retract of Gelfand triples, which is more than just having three Banach frames. Indeed, the concept of Gelfand triples puts emphasis on the fact that the mappings at the different levels are natural continuations of each other, very much as the Fourier transform at the $\boldsymbol{L}^2$-level is the unique extension to all of $\boldsymbol{L}^2(\mathbb{R}^d)$ of the ordinary Fourier transform defined for $\boldsymbol{S}_0(\mathbb{R}^d)$ (given by Riemann integrals) or $\boldsymbol{L}^1(\mathbb{R}^d)$ (using the Lebesgue integral).

Quite similar statements hold true for regular and irregular Gabor families, and are often immediate consequences of the general coorbit theory developed in [75, 76]. The results of these papers can be applied to the (irreducible) Schrödinger representation of the reduced Heisenberg group and yield results for the family of all $\boldsymbol{M}^s_{p,q}(\mathbb{R}^d)$-spaces (see [77] for the translation process between group representation concepts and signal processing terminology). For sufficiently nice elements of $\boldsymbol{M}^1_w$ or the Schwartz space, such as the Gauss-function or finite linear combinations of Hermite functions, one can guarantee that any sufficiently dense, uniformly separated set of points $(\lambda_i)_{i\in I}$ in the time-frequency plane (or phase space) $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ can be used to represent any f as a series of the form $\sum_{i\in I} c_i \pi(\lambda_i) g$, with coefficients belonging to some weighted mixed-norm space sequence space resembling the weighted mixed-norm space used to define $\boldsymbol{M}^s_{p,q}(\mathbb{R}^d)$.

Another insight gained by the study of modulation spaces and more general function spaces is the idea of *equivalence* of orthonormal bases (or frames), which are typically bases (or Banach frames) for a whole family of Banach spaces "around" the original Hilbert space. For example, one of the fine properties of all the "good" wavelet transforms, based on admissible wavelets with sufficient decay, smoothness, and vanishing moments, is the fact that these systems are equally well-suited to describe any of the Besov–Triebel–Lizorkin spaces. In particular, operators such as the localization operators in [42], or wavelet thresholding operators, preserve the membership of a function in one of those spaces (see e.g. [47]).

In a completely similar way, good Gabor frames allow us to characterize the modulation spaces, and Wilson basis provide unconditional bases for this family

[9] In interpolation theory one uses the notion of a *retract*, see [14].

of spaces. Wilson bases were rigorously developed in the paper [44], of which I learned already during a first discussion with Ingrid Daubechies in late 1989 (the paper [79] grew out of the observation that they should do more or less the same for the modulation spaces as what the wavelet bases do for the classical smoothness spaces.)

With further generalizations going on — e.g., the alpha-modulation spaces, and coorbit spaces with respect to sections, as developed in a series of papers by Dahlke and his coauthors, see [99, 40, 41, 24, 171] — it has became clear that also in this more general context, there are equivalence classes among the various orthonormal bases for $\boldsymbol{L}^2(\mathbb{R}^d)$ that have particular interesting properties. For example, one can call two such bases ℓ^1-equivalent if the set of all absolutely convergent series are the same. More generally, one obtains narrower equivalence classes if one poses similar requirements for certain families of sequence spaces instead of the single space ℓ^1. In wavelet theory this phenomenon is well-known: if one wants to characterize distributions in Besov spaces, one has to use wavelet bases with atoms satisfying decay and moment conditions. There is also a benefit from these restrictions: the bases in those smaller equivalence classes exhibit a lot of robustness (e.g., to jitter error). In contrast, from the point of view of Hilbert space theory, all the orthonormal bases are "equivalent," although they are not equally well-suited for practical purposes. For example, the orthonormal Gabor system at critical density, i.e., with $g = \mathbf{1}_{[0,1]}$ and $a = 1 = b$, shows very little robustness against jitter perturbations. Furthermore one cannot characterize $\boldsymbol{S}_0(\mathbb{R})$ with this basis.

As there are now different families of Banach spaces, the modulation spaces and Besov spaces being typical cases, the question of mutual embeddings arises. Already the Ph.D. thesis of Peter Gröbner [112] contained optimal embedding results between Besov and (alpha-)modulation spaces, while the embeddings among the classical modulation spaces are covered by the corresponding results for Wiener amalgams. More recent results are obtained by Kasso Okoudjou [160] and Joachim Toft ([186] and many other papers ([188, 187, 189] by this author), and we should also mention [119] and [135] in this context.

6 Usefulness of Modulation Spaces

It is a true (but maybe trivial) statement that *any* new family of Banach spaces of distributions will lead to new mathematical results, some of which are just contributing to clarification of concepts or to a comparison between new families and established ones. New results which help to answer questions arising from applications typically serve much better as explanations of *why* a new family of spaces should be studied more closely. For a demonstration of the relevance of modulation spaces, the theory of pseudo-differential operators comes to mind. There is already a long list of publications related to the use of modulation spaces in pseudo-differential operator theory, e.g., [184, 119, 185, 151, 152, 169, 39, 129, 120, 2, 13, 19, 22, 186, 188, 187, 170, 12, 23].

The development started with the work of Kazuya Tachizawa who described the behavior of classical pseudo-differential operators on modulation spaces in [184, 178], while at the same time it became more and more clear that it is natural to describe pseudo-differential operators themselves (resp. their Kohn–Nirenberg or spreading symbols) also in terms of the modulation spaces. One of the most interesting results in this direction goes back to Christopher Heil and Karlheinz Gröchenig, improving on the classical Calderón–Vaillancourt theorem [120]. Modulation spaces are also implicitly used in [131], but also completely independently in the work of Johannes Sjöstrand [181], and N. Lerner [153], cf. also [116, 125].

There are also other connections to the theory of pseudo-differential operators. Only recently it has been discovered (see [20]) that the so-called *Shubin classes* $\mathcal{Q}_s$ are in fact typical modulation spaces ($\boldsymbol{M}^2_{v_s}$ in the notation of [115]), with radially symmetric weights on the time-frequency plane. Various interesting properties (aside from the obvious invariance of these spaces under metaplectic transformations, including the Fourier transform) can be derived from such a characterization. For example, they are pointwise algebras (as well as convolution algebras) for $s > d/2$. Shubin classes have been discussed in some detail from the viewpoint of time-frequency analysis in papers such as [20, 37, 21]. In an alternative description, they are of the form $\boldsymbol{L}^2_{w_s} \cap \mathcal{H}^2_s(\mathbb{R}^d)$, the intersection of a weighted $\boldsymbol{L}^2$-space with a classical Sobolev space.

The description of time-varying channels for wireless communications is another area where the use of the spreading function is quite useful and very helpful. Due to the limited distance between the sending station and the receiver, as well as a maximal Doppler shift due to the maximal possible speed of movement of the receiver, the communication channels may be modeled as slowly time-varying channels, or, respectively, as "underspread linear systems". Such systems are the subject of an active area of research, where time-frequency methods play a significant role (see [85, 158, 128, 124]). Beam-forming can be interpreted as a task of looking for approximate eigenvectors for these systems under the side constraint of keeping their norms in the modulation spaces $\boldsymbol{M}^1_{v_s}(\mathbb{R}^d)$ small in order to maximize TF-concentration as well as robustness of the system. It is this area of application which shows mathematical similarities with the theory of Calderón–Zygmund operators. While Calderón–Zygmund operators are almost diagonalized by "good wavelet systems," underspread operators appear to be almost diagonalized by the Banach frames of Gabor type which are built from well-concentrated Gabor atoms. New results from the theory of Banach algebras (generalizing Wiener's inversion theorem [174]), imply that the inverse, resp. pseudo-inverse, of the corresponding matrices share various types of off-diagonal decay properties with the doubly-infinite matrices arising in this context.

7 What Came After Modulation Spaces?

Seeing the modulation spaces and the Besov spaces side by side (e.g. as in [43, 132]) as rather parallel theories with a lot of similarities — but also differences — it is natural to ask in which sense they might be made "comparable" or connected. I suggest two possible answers.

The first is provided by the *theory of coorbit spaces and their atomic characterizations*, which was developed together with Karlheinz Gröchenig in the late 1980s. An important impetus for this work came during a summer school organized by DMV in 1985 (with courses provided by Roger Howe, Detlev Poguntke and Elias Stein), which I attended together with Gröchenig. Here we learned (mostly from Howe) a lot about the (reduced) Heisenberg group and its relevance for many branches of analysis [**?**]. In the same year the paper [127] appeared (I had seen a preprint some time earlier) on the decomposition of functions into building blocks of "constant shape". Then, in late 1986, I received notice of Yves Meyer's report [159], describing his first construction of an orthonormal wavelet system. From the beginning it was clear that these new wavelet systems are unconditional bases for a large class of Banach spaces which had already played an important role in analysis, for example in connection with the theory of Calderón–Zygmund operators (such as the real Hardy space $H^1(\mathbb{R}^d)$ or BMO, and of course $\boldsymbol{L}^p$-spaces for $1 < p < \infty$). A short visit to Yves Meyer in Paris in February 1987 confirmed to me that a fascinating development within mathematical analysis had started. Having in front of us both the modulation spaces and the Besov–Triebel–Lizorkin spaces, it was natural to look for a joint background, which we found in the theory of integrable group representations. This theory was developed jointly with Gröchenig in a series of papers ([74, 75, 76, 77, 113]). In the context of these papers one starts (based on work of Duflo and Moore [49]) with a general integrable and irreducible representation π of some locally compact group $\mathcal{G}$ on a certain Hilbert space $\mathcal{H}$. Depending on whether one uses the Schrödinger representation of the Heisenberg group or the standard representation of the affine group (the "$ax+b$"-group), one would characterize different Banach spaces of distributions via the so-called generalized wavelet or voice transform. Indeed, one obtains the modulation and the Besov spaces in this way, by using these two representations.

The second option would be to interpolate the geometries which are used in the atomic characterization of these two families of spaces (namely, uniform versus dyadic partitions of unity). This idea leads to the so-called alpha-modulation spaces, discussed below.

Since one has both Besov and modulation spaces over $\mathbb{R}^d$ sharing a number of joint properties, it was natural from the very beginning to look for ways of "interpolating" between the two families. The obvious way — the use of standard interpolation methods — did not work out, but a relatively natural approach using (geo)metric ideas did turn out to be feasible. The foundations of this theory were described in the Ph.D. thesis of Peter Gröbner (worked out between 1980 and 1992).[10] In fact, the definition of the so-called alpha-modulation spaces was motivated by the general theory of decomposition spaces (cf. [73, 63]).

In the last few years the alpha-modulation spaces have received a lot of attention through the work of Massimo Fornasier, Lasse Borup and Morten Nielsen [73, 63, 161, 22, 99, 101, 23, 24, 25, 41]. Among other results, the existence of Banach frames for these spaces has been established.

The developments around modulation spaces and their generalizations — i.e., the general coorbit theory in its various forms — has also led to the introduction of the concept of *localization* for frames. Especially in the time-frequency context, localization of frames has a natural meaning: a regular or irregular Gabor frame resp. the elements of its dual frame (which are thought to have a "natural center within the TF-plane") can be considered uniformly concentrated around their respective centers if their STFTs show a uniform decay (same order, joint constants). Technically, the concept of localization appeared in two different and independent ways. The team of Radu Balan, Peter Casazza, Christopher Heil, and Zeph Landau came up with this notion in their work on "excess and redundancy" of frames, see [6, 7], following their earlier work [5] on excess of Gabor frames. The other approach is based on concepts from the theory of Banach algebras, especially the inverse-closedness of certain algebras of infinite matrices which are dominated by well-concentrated convolution-type matrices. The key publication in this direction is due to Karlheinz Gröchenig [117], where *localization of a frame with respect to some given orthonormal system* is defined. Subsequently, a concept of *intrinsic localization* and related concepts of mutual localization of a (dual) pair of Banach frames, expressed in terms of their (cross)-Gramian matrices, have been developed, see e.g., the paper [100] by Fornasier and Gröchenig, and this also was developed in [6]. Further interesting results in this direction can be expected in the near future, expanding the range of situations where such arguments can be employed. In view of the recent paper [101], I believe that many cases will be found where such principles can be applied to Banach frames that are obtained by applying suitable discretization procedures to continuous frames (that are indexed by some metric space, so that elements with parameters "far away from each other" naturally show little correlation).

[10] According to Peter's diary the first suggestions for this topic of the thesis had been made during my visiting professorship in Heidelberg in the spring of 1980. His thesis took a very long time because he was pursuing research as an "amateur," while working full time as a high school teacher.

8 What should we call a Modulation Space?

Since there now exist various generalizations of the classical modulation spaces $\boldsymbol{M}^s_{p,q}(\mathbb{R}^d)$, it has become an issue as to how far one can or should stretch the terminology, and in particular which features need to be satisfied by a space in order to call it a *modulation space* (at least in the wider sense). Let us jump right to our proposed answer to this question: Modulation spaces should be those Banach spaces of distributions on a LCA group which are described by the behavior of their STFT. But let us make this statement more precise and for this purpose review various options, again including historical aspects.

The modulation spaces $\boldsymbol{M}^s_{p,q}$, which one might now call the "classical modulation spaces," have a number of specific features, including the following. On the Fourier transform side they are Wiener amalgam spaces of the form $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \boldsymbol{\ell}^q_{w_s})$, or equivalently: they can be characterized using a mixed $\boldsymbol{L}^p$-$\boldsymbol{L}^q$-norm conditions on the STFT, with integration in the time-direction first [115, Chap. 11], and with weights of polynomial type which depend only on the frequency variable. In other words, the classical modulation spaces are a specific family of coorbit spaces (as described in [75, 77])arising via the Schrödinger representation of the reduced Heisenberg group through time-frequency shift operators (the resulting "generalized wavelet transform" is then just the STFT). We think that this is the most important feature of modulation spaces, and should be taken as characteristic of this terminology, rather than the specific form of solid BF-spaces which are used in the definition.

Let us recall the different aspects of the components which are used to build the classical (and more recent variants of) modulation spaces:

- If one wants to start within the space of tempered distributions (for simplicity or because no more is needed), then of course it is important to admit only weights of at most polynomial growth if one is interested in a family of Banach spaces that will be closed under duality and (complex) interpolation.

- One can still keep the approach via BUPUs on the Fourier transforms as long as one uses only subexponential weights, such as weights of the form $\xi \mapsto e^{\beta|x|^\alpha}, 0 \leq \alpha < 1, \beta > 0$, or, more generally, weight functions which satisfy the so-called Beurling–Domar non-quasi-analyticity condition [174, 177]. Such weights occur in the early report [59, 71], and weights of this kind are the subject of the work of Nenad Teofanov and Stevan Pilipovic [168, 170]. In the latter papers, the notion of ultra-modulation spaces is introduced, which naturally reminds us of the need to make use of ultra-distribution spaces as a reservoir to select from if one wants to establish a family of spaces which is closed under duality. On the other hand, certain aspects of those spaces are covered in a different way by the general coorbit theory developed in [74], where the reservoir is the dual space of a coorbit space associated to some $\boldsymbol{L}^1_w(G)$-space over $G = \mathbb{R}^d \times \widehat{\mathbb{R}}^d =$ phase space.

- The dependence of the weight on the frequency variable is another aspect of the original approach. It has already been removed in Gröchenig's description of modulation spaces in [115], e.g. the spaces $\boldsymbol{M}^p_{v_s}$. In fact, radial weight functions of polynomial growth over phase space are particularly attractive, because they give rise to a new family of modulation spaces which are invariant with respect to metaplectic transformations, including partial Fourier transforms. As already mentioned above, one obtains the Shubin classes for the choice $p = 2$. In turn, according to a recent result by Hogan and Lakey, the Shubin classes can be characterized via weighted $\boldsymbol{\ell}^2$-conditions on their Hermite coefficients.
- There is no reason why one should not allow subexponential radial weights on phase space, such as $w(x, y) = \exp(\sqrt{(x^2 + y^2)})$.

Summarizing, we can say that the terminology of modulation spaces should be used for Banach spaces of (ultra-)distributions which are characterized by the membership of STFTs (with respect to the Gaussian window g_0) in some solid and translation invariant Banach function space over phase space. Only recently it has been shown [31] that there is a joint dense subspace of $\boldsymbol{S}_0(\mathbb{R}^d)$ resp. $\boldsymbol{L}^2(\mathbb{R}^d)$ which is contained in all of these spaces.

Moreover, as a consequence of the atomic theory developed in the context of coorbit spaces in [75, 76], one has immediate characterizations of those general modulation spaces using sufficiently dense regular or irregular Gabor families generated from sufficiently good atoms, i.e., from one of the modulation spaces $\boldsymbol{M}^1_w$. Since such atomic characterizations (cf. [65]) are also quite typical for the case of the classical modulation spaces, we have another argument for our suggestion above.

As a final remark let us mention that the name *"modulation space"* refers to the fact that the STFT describes the behavior of f under convolution with a *modulated* version of g , due to the following identity ([115], p. 39)

$$|V_g f(x, \omega)| = |(M_\omega g) * f(x)|.$$

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