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The Kadison-Kaplansky conjecture for word-hyperbolic groups

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Introduction

In this paper we prove the Kadison-Kaplansky idempotent conjecture for torsion-free word-hyperbolic groups. The conjecture asserts that the following equivalent statements hold for a torsion-free discrete group Γ:

- The reduced group C^*-algebra $C_r^*(\Gamma)$ contains no idempotents except 0 and 1.
- The spectrum of every element of the reduced group C^*-algebra is connected.
- The canonical trace on $C_r^*(\Gamma)$ takes integer values on idempotents.

The last assertion can be viewed as a statement about the pairing between the K-theory and the (local) cyclic cohomology of the group C^*-algebra. It is in this setting that we will prove the conjecture. Our proof is based on a partial analysis of the assembly maps in K-theory and local cyclic homology. We compare these assembly maps by means of an equivariant bivariant Chern-Connes character.

Before going into details, we recall some previous work on the conjecture. The first progress was achieved by Pimsner and Voiculescu [PV] who proved the Kadison-Kaplansky conjecture for free groups as a consequence of their computation of the K-theory of the group C^*-algebra. Subsequently it was realized that more generally the Kadison-Kaplansky conjecture was a consequence of the Baum-Connes conjecture which gives a geometric description of $K_*(C_r^*(\Gamma))$ for any torsion-free discrete group. In fact the Baum-Connes conjecture states that the K-theoretic assembly map

$$\mu : K_*^{top}(B\Gamma) \longrightarrow K_*(C_r^*(\Gamma))$$

is an isomorphism [BCH]. In particular, the Kadison-Kaplansky conjecture holds for all torsion-free groups, for which the Baum-Connes conjecture is known to be true [HK], [La]. Recently Mineyev-Yu [MY] and Lafforgue (unpublished) proved the Baum-Connes conjecture for word-hyperbolic groups. Their work therefore also yields a proof of the Kadison-Kaplansky conjecture for torsion-free word-hyperbolic groups.

A different approach to the idempotent conjecture, based on cyclic cohomology, was initiated by Connes in [Co]. Connes realized that to prove the Kadison-Kaplansky conjecture for free groups one could introduce a Fredholm module, implicit in the work of Pimsner and Voiculescu. Connes showed that the Chern-character of this Fredholm module, defined through his theory of cyclic cohomology, coincided with the canonical trace of $C_r^*(\Gamma)$. This immediately implied the conjecture.

The use of cyclic homology as a tool for studying the assembly map in K-theory goes back to Connes and Moscovici. In [CM] they established for any torsion-free discrete group Γ a natural commutative diagram

$$\begin{array}{rccc} \mu: & K_*^{top}(B\Gamma) & \longrightarrow & K_*(\mathfrak{a}(\Gamma)) \\ & \downarrow{\scriptstyle ch} & & \downarrow{\scriptstyle ch} \\ \mu_{coh}: & H_*(\Gamma, HP_*(\mathbb{C})) & \longrightarrow & HP_*(\mathfrak{a}(\Gamma)) \end{array}$$

where μ is the K-theoretic assembly map and $\mathfrak{a}(\Gamma)$ denotes any Banach algebra completion of the complex group ring $\mathbb{C}\Gamma$ inside the reduced group C^*-algebra $C_r^*(\Gamma)$. Connes and Moscovici use this to deduce the Novikov conjecture for word-hyperbolic groups from the injectivity of the homological assembly map μ_{coh}.

The arguments of Connes and Moscovici apply immediately so that the diagram above remains valid if periodic cyclic homology is replaced by local cyclic homology [Pu1]. Our whole effort concentrates on studying the lower line

$$\mu_{coh} : H_*\big(\Gamma, HC_*^{loc}(\mathbb{C})\big) \longrightarrow HC_*^{loc}(\mathfrak{a}(\Gamma))$$

of the diagram for appropriate $\mathfrak{a}(\Gamma)$. Properties of the assembly map are deduced from a detailed explicit calculation of $HC_*^{loc}(\mathfrak{a}(\Gamma))$ using homological methods.

We work with local cyclic cohomology for two reasons. On the one hand bivariant local cyclic cohomology of Banach algebras [Pu1] is known to be a bifunctor which is stable, satisfies excision [Pu2], and is invariant not only under smooth but even under continuous homotopies. The last properties make local cyclic cohomology much better behaved on the category of Banach-algebras than other cyclic cohomology theories. On the other hand, it is also explicitely computable in a certain sense. In fact, if a Banach algebra A satisfies the metric approximation property and is the topological direct limit of a countable directed family $(A_n)_{n\in\mathbb{N}}$ of Banach algebras with nuclear transition maps, then the local cyclic homology of A is given as the direct limit of the entire cyclic homology groups $HC_*^{\epsilon}(A_n)$ [Co2] of the algebras A_n. At first glance one might seem to have replaced one intractable problem by another, since entire cyclic groups are notoriously difficult to calculate. However the direct limit only requires information on the transition maps which in contrast are amenable to study.

In particular we can determine $HC_*^{loc}(\ell^1(\Gamma))$ by this strategy because $\ell^1(\Gamma)$ is the topological direct limit of the Banach algebras of summable functions of exponential decay (with respect to a word metric) on Γ [Bo].

The cyclic homology of group rings has been determined by Burghelea [Bu] and Nistor [Ni], and is closely related to group homology. By their works, one knows that the Hochschild complex of $\mathbb{C}\Gamma$ can be identified with the bar complex of Γ twisted by the adjoint representation. Moreover the adjoint representation decomposes canonically as a direct sum of a homogeneous and an inhomogeneous part, corresponding to the unit respectively to the nontrivial group elements. There is a corresponding canonical decomposition of the various cyclic complexes of good completions of $\mathbb{C}\Gamma$. Only the homogeneous parts are required for the proof of the Kadison-Kaplansky conjecture.

In order to construct a contracting homotopy of the bar complex of Γ in large dimensions, we compare the bar resolution of the constant Γ-module with the resolution coming from a Rips complex [Gr]. As a consequence we see that algebraic differential forms with degree higher than the dimension of the Rips complex do not contribute to $HP_*(\mathbb{C}\Gamma)$. The continuity properties of the contracting homotopy allow to establish the same assertion for $HC_*^{loc}(\ell^1(\Gamma))$. After this it is not difficult to identify the homogeneous part $HC_*^{loc}(\ell^1(\Gamma))_{hom}$ with the group homology $H_*(\Gamma, HC_*^{loc}(\mathbb{C}))$. We then use an argument of Connes and Moscovici [CM] to complete the proof of the first main result of the paper: the homological assembly map

$$\mu_{coh} : H_*\big(\Gamma, HC_*^{loc}(\mathbb{C})\big) \longrightarrow HC_*^{loc}(\mathfrak{a}(\Gamma))$$

is an isomorphism onto the homogeneous part $HC_*^{loc}(\mathfrak{a}(\Gamma))_{hom}$ of the local cyclic homology when $\mathfrak{a}(\Gamma)$ is a good completion of $\mathbb{C}\Gamma$ (Γ torsion-free and word-hyperbolic).

After this we use the diagram of Connes and Moscovici above to compare the assembly maps in K-theory and local cyclic homology. In order to carry this out we introduce equivariant bivariant Chern-Connes characters. These are natural transformations

$$ch_{biv}^{\Gamma} : KK^{\Gamma}(-,-) \longrightarrow HC_0^{loc}(- \rtimes_r \Gamma, - \rtimes_r \Gamma)$$

and

$$ch_{biv}^{\mathfrak{a}(\Gamma)} : KK^{\Gamma}(-,-) \longrightarrow HC_0^{loc}(\mathfrak{a}(\Gamma,-), \mathfrak{a}(\Gamma,-))$$

from Kasparov's equivariant KK-theory [Ka1], [Ka2] to bivariant local cyclic cohomology which are multiplicative and compatible with descent [Ka2], with ordinary Chern characters and with the canonical decomposition into homogeneous and inhomogeneous parts. (Here $\mathfrak{a}(\Gamma)$

denotes any good completion of $\mathbb{C}\Gamma$.) The existence of these Chern-Connes characters is an immediate consequence (see [Cu1]) of the characterization of equivariant KK-theory as the universal stable and split exact homotopy functor on the category of separable Γ-C^*-algebras [Th].

The second main result of this paper states that the decomposition of $K_*(\mathfrak{a}(\Gamma))$ into the direct sum of the image and the cokernel of the assembly map, given by a γ-element, is compatible under the Chern character with the canonical decomposition of $HC_*^{loc}(\mathfrak{a}(\Gamma))$. More precisely, one has

Theorem 0.1 *Let Γ be a torsion-free discrete group. Suppose that $\mathfrak{a}(\Gamma)$ is a good completion of $\mathbb{C}\Gamma$ such that the homological assembly map defines an isomorphism $\mu_{coh} : H_*(\Gamma, HC_*^{loc}(\mathbb{C})) \overset{\simeq}{\longrightarrow} HC_*^{loc}(\mathfrak{a}(\Gamma))_{hom}$. Let $\gamma \in KK^{\Gamma}(\mathbb{C}, \mathbb{C})$ be a γ-element for Γ. Then the equivariant Chern-Connes character*

$$ch_{biv}^{\mathfrak{a}(\Gamma)}(\gamma) \in HC_*^{loc}(\mathfrak{a}(\Gamma), \mathfrak{a}(\Gamma))$$

acts on the local cyclic homology $HC_*^{loc}(\mathfrak{a}(\Gamma))$ as the canonical projection onto the homogeneous part $HC_*^{loc}(\mathfrak{a}(\Gamma))_{hom}$.

In particular, such good completions contain no idempotents: the canonical trace τ is a homogeneous cocycle and therefore vanishes on $ch((1-\gamma)\, K_*(\mathfrak{a}(\Gamma)))$, since from the theorem it is inhomogeneous. Hence

$$\tau(K_*(\mathfrak{a}(\Gamma))) = \tau(ch(\gamma \cdot K_*(\mathfrak{a}(\Gamma)))) \subset \mathbb{Z}$$

by the L^2-index theorem of Atiyah and Singer [At], [Si].

In particular if Γ is torsion-free and word-hyperbolic such good $\mathfrak{a}(\Gamma)$ exist. Moreover $\mathfrak{a}(\Gamma)$ can be chosen to be closed under the holomorphic functional calculus in $C_r^*(\Gamma)$ [Jol]. Hence the Kadison-Kaplansky conjecture is true for all such Γ.

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1. Hyperbolic spaces and hyperbolic groups

We recall a few basic notions of Gromov's theory of hyperbolic metric spaces and groups.

Definition 1.1 *(Gromov) [Gr] A geodesic metric space X is called δ-hyperbolic if any four points $x, y, u, v \in X$ satisfy*

$$d(x, y) + d(u, v) \leq max\{d(x, u) + d(y, v), d(x, v) + d(y, u)\} + 2\delta$$

An important property of hyperbolic geodesic metric spaces, reminiscent of nonpositively curved manifolds is

Proposition 1.2 *(Gromov) [Gr] The distance function on a δ-hyperbolic geodesic metric space is 6δ-convex: if $z_0 := (1-t)x + ty$ and $z_1 := (1-t)u + tv$ are points on geodesic segments $[x, y]$ and $[u, v]$ dividing the segments in the ratio $t : (1-t)$, then*

$$d(z_0, z_1) \leq (1-t)d(x, u) + td(y, v) + 6\delta$$

The proof can be found in [Gr], (7.4).

Definition 1.3 *A straight k-polyhedron $\langle x_0, \ldots, x_k\rangle$ with vertices $x_0, \ldots, x_k$ in a geodesic metric space X is a subset of X which is the union $\langle x_0, \ldots, x_k\rangle = \bigcup_{z\in\langle x_1,\ldots,x_k\rangle} [x_0, z]$ of geodesic segments with origin x_0 and endpoint $z \in \langle x_1, \ldots, x_k\rangle$. A straight zero polyhedron with vertex x_0 is just the one point set $\{x_0\} \subset X$.*

Corollary 1.4 *Let $\langle x_0, \ldots, x_k\rangle$ be a straight k-polyhedron in a δ-hyperbolic geodesic metric space X and let $z', z'' \in X$. Then*

$$\sup_{z\in\langle x_0,\ldots,x_k\rangle} (d(z, z') + d(z, z'')) \leq \max_{0\leq i\leq k} (d(x_i, z') + d(x_i, z'')) + 12k\delta$$

The proof is straightforward from the 6δ-convexity of the distance function.

Definition 1.5 *(Rips-complex) [Gr]*

a) For a nonempty set X let $\Delta(X)$ be the simplicial set with n-simplices $\Delta(X)_n := X^{n+1}$ and with face and degeneracy maps $\partial_i(x_0, \ldots, x_n) := (x_0, \ldots, \widehat{x_i}, \ldots, x_n)$ and $s_i(x_0, \ldots, x_n) := (x_0, \ldots, x_i, x_i, \ldots, x_n)$. The simplicial set $\Delta(X)$ is contractible.

b) Let X be a metric space and let $R > 0$. The Rips complex $P_R(X)$ is the simplicial subset of $\Delta(X)$ with n-simplices

$$P_R(X)_n := \{(x_0, \ldots, x_n) \in \Delta(X)_n | d(x_i, x_j) < R, \forall i, j\}$$

The construction of the simplicial set $\Delta(X)$ is functorial in X. In particular any group acting on X induces a simplicial action on $\Delta(X)$. Similarly any isometric action of a group on a metric space X induces a simplicial action of the group on the Rips complexes $P_R(X)$.

The Rips complex of a hyperbolic metric space turns out to be useful because of

Proposition 1.6 *(Rips) [Gr] Let X be a δ-hyperbolic geodesic metric space and let Y be an ϵ-dense subset of X. Then $P_R(Y)$ is contractible for $R > 4\delta + 6\epsilon$.*

The proof can be found in [Gr], (1.7). It will also be recalled for the Rips complex of a hyperbolic group in 2.4.

Let now Γ be a finitely generated group and let S be a finite symmetric set of generators of Γ. Denote by $|g| := l_S(g)$ the word length with respect to S of an element $g \in \Gamma$ and let d_S be the left invariant word metric $d_S(g, g') := l_S(g^{-1}g')$.

Definition 1.7 *(Gromov) [Gr] A finitely generated group (Γ, S) is called word-hyperbolic if its Cayley graph with respect to S is δ-hyperbolic for some $\delta \geq 0$.*

If Γ is word-hyperbolic with respect to one set of generators it is so for any other set of generators so that hyperbolicity is an intrinsic notion of a group.

A finitely generated group (Γ, S) acts simplicially and properly on the Rips complex $P_R(\Gamma, d_S)$ for any $R > 0$. If Γ is torsion-free then it acts freely and the geometric realization of $P_R(\Gamma)/\Gamma$ is a finite simplicial complex.

The Rips complex of a finitely generated group has no particularly interesting properties. On the contrary, if Γ is a word-hyperbolic group in the sense of Gromov, then the Rips complex $P_R(\Gamma)$ is contractible for $R \gg 0$ (1.6) and is therefore a universal proper Γ-space. This implies for example that a torsion-free, word-hyperbolic group possesses a finite simplicial complex as classifying space.

As a group is 1-dense in any of its Cayley graphs, (1.6) shows that the Rips complex $P_R(\Gamma)$ of a δ-hyperbolic group (Γ, S) is contractible for $R > 4\delta + 6$. We describe now a simplicial homotopy operator that will be used in (2.5) to construct an explicit contracting homotopy of the singular chain complex of $P_R(\Gamma)$.

Definition and Lemma 1.8 *[Gr]*

a) Let (Γ, S) be a finitely generated group and let $R > 0$ be an even integer. Let $\sigma : \Gamma \to \Gamma$ be any map such that $\sigma(g) = e$ if $|g| \leq \frac{R}{2}$ and such that otherwise $\sigma(g)$ lies on a geodesic segment $[g, e]$ joining g and e in the Cayley graph at distance $\frac{R}{2}$ from g. Note that σ is by no means unique.

b) Let $h : \Delta(\Gamma)_* \to \Delta(\Gamma)_{*+1}$ be defined by

$$h(g_0, \ldots, g_i, \ldots, g_n) := (-1)^i(g_0, \ldots, \sigma(g_i), g_i, \ldots, g_n)$$

where g_i is characterized by the conditions $|g_i| = \max_j |g_j|$ and $|g_{i'}| < |g_i|$, $i' < i$.

c) *Suppose that* (Γ, S) *is* δ*-hyperbolic and let* $R > 4\delta + 6$ *be an even integer. Then the operator* h *of b) preserves the Rips complex* $P_R(\Gamma)$*, i.e. it defines a map* $h : P_R(\Gamma)_* \to P_R(\Gamma)_{*+1}$.

We end this section with the

Remark 1.9 Let (Γ, S) be a finitely generated group and let $\sigma : \Gamma \to \Gamma$ be a map as constructed in (1.8). Denote by m the action of Γ by left translation and let $(g_0, \ldots, g_k) \in \Gamma^{k+1}$. Then the set

$$\left\{\left(m(g_0)\sigma^{j_0}m\left(g_0^{-1}g_1\right)\ldots\sigma^{j_{k-1}}m\left(g_{k-1}^{-1}g_k\right)\right)(e)\,\middle|\, j_0 \geq 0, \ldots j_{k-1} \geq 0\right\}$$

is contained in a straight k-polyhedron $\langle g_0, \ldots, g_k\rangle$ in the Cayley graph of (Γ, S) with vertices $g_0, \ldots, g_k$.

2. Controlled resolutions of hyperbolic groups

Bar resolution and Rips resolutions

Let Γ be an abstract group. We recall two well known projective resolutions of the constant Γ-module $\mathbb{C}$. The first is the bar resolution, which provides a universal resolution whereas the second resolution, obtained from the Rips complex, is defined for word-hyperbolic groups in the sense of Gromov, and reflects the geometry of the given group.

Definition and Lemma 2.1 *Let* Γ *be a discrete group and let* $\Delta(\Gamma)$ *be the simplicial set introduced in (1.5).*

a) *The bar resolution* $C_*(\Gamma)$ *of the constant* Γ*-module* $\mathbb{C}$ *is the chain complex of complex vector spaces associated to the simplicial* Γ*-set* $\Delta(\Gamma)$*. Thus* $C_n(\Gamma)$ *is the complex vector space generated by the set of n-simplices* $\Delta(\Gamma)_n$ *and the differential of* $C_*(\Gamma)$ *is given by the alternating sum of the face maps of* $\Delta(\Gamma)$.
b) *The reduced bar resolution* $\overline{C}_*(\Gamma)$ *is the quotient of the full bar resolution by the subcomplex spanned by the degenerate simplices of* $\Delta(\Gamma)$.
c) *Both resolutions are augmented by the* Γ*-map* $C_0(\Gamma) \to \mathbb{C}$ *sending a simplex* $g \in \Gamma$ *to 1 and define resolutions of the constant* Γ*-module* $\mathbb{C}$ *by free* Γ*-modules.*
d) *A natural* Γ*-basis of the (reduced) bar resolution is given by the simplices of the form* $(e, g_1, \ldots, g_n)$*. A contracting linear homotopy is given by* $s(g^0, \ldots, g^n) := (-1)^{n+1}(g_0, \ldots, g_n, e)$.
e) *The antisymmetrization operator* π *defined on simplices by* $\pi(g_0, \ldots, g_n) := \frac{1}{(n+1)!}\sum_{\sigma\in\Sigma_{n+1}}(-1)^{sig(\sigma)}(g_{\sigma(0)}, \ldots, g_{\sigma(n)})$ *is a map of (reduced) bar resolutions. It equals the identity in degree zero.*

Definition and Lemma 2.2 *Let* Γ *be a finitely generated group with finite symmetric set of generators* S *and let* $R > 0$.

a) *Denote by* $C_*^R(\Gamma)$ *the chain complex of complex vector spaces associated to the simplicial* Γ*-set* $P_R(\Gamma)$. *It is a complex of free* Γ*-modules and a subcomplex of the full bar resolution* $C_*(\Gamma)$.
b) *Denote by* $\overline{C}_*^R(\Gamma)$ *the quotient chain complex of* $C_*^R(\Gamma)$ *by the subcomplex generated by degenerate simplices. It is a complex of free* Γ*-modules and a subcomplex of the reduced bar resolution* $\overline{C}_*(\Gamma)$.
c) *The complexes* $C_*^R(\Gamma)$ *and* $\overline{C}_*^R(\Gamma)$ *are stable under antisymmetrization.*

Lemma 2.3 *Let* (Γ, S) *be a finitely generated group and let* $h : \Delta(\Gamma)_* \to \Delta(\Gamma)_{*+1}$ *be a homotopy operator as constructed in (1.8). Denote the corresponding linear operator on* $\overline{C}_*(\Gamma)$ *by the same letter.*

a) *The operator* h *maps degenerate simplices to degenerate simplices and the operators* h *and* $\varphi := Id - (h\partial + \partial h)$ *descend therefore to operators on the reduced bar resolution* $\overline{C}_*(\Gamma)$.
b) *Put*

$$\chi := \sum_{n=0}^{\infty} h \circ \varphi^n : \overline{C}_*(\Gamma) \to \overline{C}_{*+1}(\Gamma)$$

and let

$$\varphi^\infty := Id - (\chi\partial + \partial\chi) = \lim_{n\to\infty} \varphi^n$$

Then φ^∞ *defines a deformation retraction of* $\overline{C}_*(\Gamma)$ *onto the constant* Γ*-module* $\mathbb{C}$ *and* χ *is a contracting homotopy of the reduced bar resolution.*
c) *The homotopy operator* χ *satisfies* $\chi^2 = 0$.
d) *For a reduced bar simplex* $(g_0, \ldots, g_n)$ *one has*

$$\chi(g_0, \ldots, g_n) = \sum \pm(g_0', \ldots, g_{n+1}')$$

where the simplices $(g_0', \ldots, g_{n+1}')$ *satisfy* $\max_j |g_j'| \leq \max_i |g_i|$ *and the number of summands is less than or equal to* $C(\sum_{i=0}^n |g_i|)$ *for a constant* C *depending only on* (Γ, S) *and the choice of* h.

Proof: Assertion a) is clear from the definition of h. For c) note that by construction the image of h^2 consists of degenerate simplices so that $h^2 = 0$ on $\overline{C}_*(\Gamma)$. Thus $Im\, h$ is stable under φ as $\varphi \circ h = (Id - h \circ \partial - \partial \circ h) \circ h = h \circ (Id - \partial \circ h)$. This implies already that $\chi^2 = 0$.

Let $(g_0, \ldots, g_n)$ be a simplex in $\overline{C}_*^R(\Gamma)$. A straightforward calculation shows

$$\varphi(g_0, \ldots, g_i, \ldots, g_n) = (g_0, \ldots, \sigma(g_i), \ldots, g_n) - (-1)^i h(g_0, \ldots, \widehat{g_i}, \ldots, g_n)$$

if $g_{i-1} \neq \sigma(g_i)$, i.e. if $(g_0, \ldots, g_n)$ is not in the image of h and

$$\varphi(h(g_0, \ldots, g_i, \ldots, g_n)) = (-1)^{i+1} h(g_0, \ldots, \sigma(g_i), \widehat{g_i}, \ldots, g_n)$$

otherwise. These formulas show that φ^∞ is zero on $\overline{C}_*(\Gamma)$, $* > 0$ and projects $\overline{C}_0(\Gamma)$ onto $\mathbb{C}(e)$. As it is by definition chain homotopic to the identity it is a deformation retraction as claimed in b). The formulas above show moreover that $h \circ \varphi^k(g_0, \dots, g_n) = 0$ if $(k - l)\frac{R}{2} > \sum_{i=0}^n |g_i|$ where $l = \#\{i | g_i \neq e\}$. (Here R is the constant chosen in (1.8).) Moreover $\varphi^k(g_0, \dots, g_n)$ consists of a single simplex modulo $Im\, h$ so that $\pm h \circ \varphi^k(g_0, \dots, g_n)$ is actually given by a single simplex. This shows that $\chi(g_0, \dots, g_n) = \sum_{k=0}^\infty h \circ \varphi^k(g_0, \dots, g_n)$ is of the form claimed in d). □

If Γ is word-hyperbolic the simplicial homotopy operator h above is known to preserve the Rips complex $P_R(\Gamma)$ for large R (1.8). It follows that the contracting homotopy χ of the reduced bar complex of (2.3) yields also a contracting homotopy of the reduced cellular chain complex of the Rips complex.

Corollary 2.4 *[Gr] Let Γ be a word-hyperbolic group in the sense of Gromov and let S a finite symmetric set of generators of Γ. Let $R \gg 0$ be large and let h and χ be homotopy operators as in (1.8) and (2.3). Then χ defines a contracting homotopy of the reduced cellular chain complex $\overline{C}^R_*(\Gamma)$ of the Rips complex $P_R(\Gamma)$ onto its base point. In particular, $\overline{C}^R_*(\Gamma)$ defines a resolution of the trivial Γ-module $\mathbb{C}$ by free Γ-modules. Any such resolution will be called a Rips resolution.*

Lemma 2.5 *Let Γ be a word-hyperbolic group with finite symmetric set of generators S. Let $R \gg 0$ be large enough that the Rips complex $P_R(\Gamma)$ is contractible and let χ be a contracting homotopy of it as in (2.3). Fix an integer $d > \#\{g \in \Gamma,\ |g| < R\}$.*

a) There exists a Γ-equivariant chain map $\Phi' : \overline{C}_*(\Gamma) \to \overline{C}^R_*(\Gamma) \subset \overline{C}_*(\Gamma)$ which equals the identity in degree zero and is given on simplices by the formula

$$\Phi'(g_0, \dots, g_n) := \left(m(g_0) \circ \chi \circ m\left(g_0^{-1} g_1\right) \circ \dots \circ \chi \circ m\left(g_{n-1}^{-1} g_n\right)\right)(e)$$

Here $m(g)$ denotes the left translation of $g \in \Gamma$ on $\overline{C}_*(\Gamma)$.

b) Let $\Phi : \overline{C}_*(\Gamma) \to \overline{C}^R_*(\Gamma) \subset \overline{C}_*(\Gamma)$ be the composition of Φ' and the antisymmetrization operator: $\Phi := \pi \circ \Phi'$. Then Φ is a Γ-equivariant chain map which equals the identity in degree zero and vanishes in degrees larger or equal to d.

Proof: The reduced bar resolution $\overline{C}_*(\Gamma)$ of the constant Γ-module $\mathbb{C}$ consists of free Γ-modules with a natural basis described in (2.1) d). If Γ is a word-hyperbolic group there exist on the other hand the Rips resolutions $\overline{C}^R_*(\Gamma)$ of the constant Γ-module $\mathbb{C}$ which coincide with the reduced bar resolution in degree zero and possess a contracting $\mathbb{C}$-linear homotopy χ. A well known argument from homological algebra allows to extend the identity in degree zero inductively to a chain map from the reduced bar resolution to the Rips resolution. Because the homotopy operator χ satisfies $\chi^2 = 0$ (2.3)c) this chain map of complexes of Γ-modules can be given by the explicit formula in a). As the antisymmetrization operator π commutes with the group action and equals the identity in degree zero it remains only to verify that π annihilates simplices in $P_R(\Gamma)$ of degree larger or equal to d. This is clear however as the factors of such a simplex cannot be pairwise different by the choice of d. □

Operators on twisted bar resolutions

Definition 2.6 *Let Γ be a finitely generated group.*

a) Let $Ad(\Gamma)$ be the Γ-space with underlying set Γ equipped with the adjoint action. Denote by $\overline{C}_*(\Gamma) \otimes Ad(\Gamma)$ the Γ-chain complex of complex vector spaces associated to the simplicial Γ-set $\Delta(\Gamma) \times Ad(\Gamma)$ with the diagonal Γ-action.

b) Let $B' : \overline{C}_*(\Gamma) \otimes Ad(\Gamma) \to \overline{C}_{*+1}(\Gamma) \otimes Ad(\Gamma)$ be the Γ-equivariant linear operator given on simplices by the formula

$$B'(g^0, \ldots , g^n, v) := \sum_{i=0}^{n} (-1)^{n(i+1)} (g^i, g^{i+1}, \ldots , g^n, vg^0, vg^1, \ldots , vg^i, v)$$

c) Suppose that Γ is word-hyperbolic and let Φ be a Γ-equivariant chain map as in (2.5). Let $\nabla' : \overline{C}_*(\Gamma) \otimes Ad(\Gamma) \to \overline{C}_{*+1}(\Gamma) \otimes Ad(\Gamma)$ be the Γ-equivariant linear operator given on simplices by

$$\nabla'(g^0, \ldots , g^n, v) := (-1)^n \sum_{i=0}^{n} (\Phi(g^0, \ldots , g^i), g^i, \ldots , g^n, v)$$

Note that the number of summands is bounded independently of n because Φ vanishes in large degrees.

Lemma 2.7 *The operator ∇' of (2.6) is a Γ-equivariant contracting homotopy of $\overline{C}_*(\Gamma) \otimes Ad(\Gamma)$ for $* \geq d$ the constant introduced in (2.5).*

Proof: According to (2.5) the Γ-equivariant chain map $Id - \Phi : \overline{C}_*(\Gamma) \to \overline{C}_*(\Gamma)$ equals the identity in degrees larger or equal to d and vanishes in degree zero. As $\overline{C}_*(\Gamma)$ is a complex of free Γ-modules with canonical Γ-basis (2.1) and is linearly contractible (2.1)d) the map $Id - \Phi$ is canonically nullhomotopic. The same holds for the operator $(Id - \Phi) \otimes Id$ on $\overline{C}_*(\Gamma) \otimes Ad(\Gamma)$. The operator ∇' is the corresponding homotopy operator. □

Norm estimates

Now the continuity properties of the operators constructed so far will be studied. We begin by introducing a family of norms on the bar resolution of a finitely generated group.

Definition 2.8 *Let* (Γ, S) *be a finitely generated group.*

a) For a simplex $(g_0, \ldots, g_n, v) \in \Delta(\Gamma)_n \times Ad(\Gamma)$ *put*

$$|(g_0, \ldots, g_n, v)| := d_S(g_0, g_1) + \ldots + d_S(g_{n-1}, g_n) + d_S(g_n, vg_0)$$

b) For a monotone increasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ *let* $\| - \|_f$ *be the largest seminorm on* $\overline{C}_*(\Gamma) \otimes Ad(\Gamma)$ *satisfying*

$$\| (g_0, \ldots, g_n, v) \|_f \leq f(|(g_0, \ldots, g_n, v)|)$$

c) The seminorms associated to the functions $f(t) := \lambda^t$, $\lambda > 1$ *will be denoted by* $\| - \|_\lambda$.

The action of Γ on $\overline{C}_*(\Gamma) \otimes Ad(\Gamma)$ is isometric with respect to the norms $\| \ \|_f$.

Proposition 2.9 *Let* (Γ, S) *be a word-hyperbolic group. Let* Φ *be a chain map as in (2.5) and let* ∇', B' *be the operators introduced in (2.6). Fix a number* d *such that the chain map* Φ *vanishes in degrees larger than* d. *Then there exist constants* C_0, C_1 *such that for every monotone increasing function* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *and every* $\alpha_n \in \overline{C}_n(\Gamma) \otimes Ad(\Gamma), n \in \mathbb{N}$, *the estimates*

$$\| \nabla'(\alpha_n) \|_f \leq C_0 \| \alpha_n \|_{f'}$$

and

$$\| \nabla' \circ B'(\alpha_n) \|_f \leq C_0 (n+1)^{1-d} \| \alpha_n \|_{f'}$$

hold for f' *the function* $f'(t) := t^d f(t + C_1)$.

Proof: The norms $\| - \|_f$ are weighted ℓ^1-norms so that it suffices to verify the estimates on simplices $(g_0, \ldots, g_n, v) \in \Delta(\Gamma)_n \times Ad(\Gamma)$. The explicit formulas in (2.3) and (2.5) show that

$$(\Phi'(g_0, \ldots, g_k), g_k, \ldots, g_n, v) = \sum \pm (g'_0, \ldots, g'_k, g_k, \ldots, g_n, v)$$

is an alternating sum of at most

$$C^k \cdot k! \cdot (d_S(g_0, g_1) + \ldots + d_S(g_{k-1}, g_k))^k$$

simplices $(g'_0, \ldots, g'_k, g_k, \ldots, g_n, v)$ of the following form:

- $k \leq d$
- $(g'_0, \ldots, g'_k)$ is a Rips simplex, i.e. $d_S(g'_i, g'_j) < R$, $\forall i, j$

- The factor g'_k is of the form
$$g'_k = m(g_0)\sigma^{j_0} m\big(g_0^{-1} g_1\big) \dots \sigma^{j_{k-1}} m\big(g_{k-1}^{-1} g_k\big)(e)$$
and is therefore by (1.9) contained in a straight k-polyhedron with vertices $(g_0, \dots, g_k)$ in the Cayley-graph of (Γ, S).

In particular

$$\begin{aligned} &|(g'_0, \dots, g'_k, g_k, \dots, g_n, v)| \\ &= \sum_{i=0}^{k-1} d_S(g'_i, g'_{i+1}) + \sum_{j=k}^{n-1} d_S(g_j, g_{j+1}) + d_S(g'_k, g_k) + d_S(g_n, vg'_0) \\ &\leq \sum_{j=k}^{n-1} d_S(g_j, g_{j+1}) + d_S(g_n, vg'_k) + d_S(g'_k, g_k) + (k+1)R \end{aligned}$$

which can be estimated after (1.4) by

$$\begin{aligned} &\leq \sum_{j=k}^{n-1} d_S(g_j, g_{j+1}) + \max_{0 \leq i \leq k}(d_S(g_n, vg_i) + d_S(g_i, g_k)) + (k+1)R + 12k\delta \\ &\leq |(g_0, \dots, g_n, v)| + (k+1)R + 12k\delta \end{aligned}$$

A similar estimate holds after antisymmetrization in the first k variables for $(\Phi(g_0, \dots, g_k), g_k, \dots, g_n, v)$ so that altogether

$$\begin{aligned} &\| \, (\Phi(g_0, \dots, g_k), g_k, \dots, g_n, v) \, \|_f \leq \\ &\leq C(k)(d_S(g_0, g_1) + \dots + d_S(g_{k-1}, g_k))^k f(|(g_0, \dots, g_n, v)| + C'(k)) \end{aligned}$$

From this one obtains

$$\begin{aligned} &\|\nabla'(g_0, \dots, g_n, v) \, \|_f \\ &\quad \leq C'(d)(d_S(g_0, g_1) + \dots + d_S(g_{n-1}, g_n))^d f(|(g_0, \dots, g_n, v)| + C''(d)) \end{aligned}$$

which establishes the first inequality.

Concerning the second note that

$$|(g_i, g_{i+1}, \dots, g_n, vg_0, vg_1, \dots, vg_i, v)| \; = \; |(g_0, \dots, g_n, v)|$$

which implies

$$\begin{aligned} &\| \, \nabla' \circ B'(g_0, \dots, g_n, v) \, \|_f \leq \\ &\leq C' f(|(g_0, \dots, g_n, v)| + C'') \sum_{i=0}^{n} (d_S(g_i, g_{i+1})' + \dots + d_S(g_{i+d}, g_{i+d+1})')^d \end{aligned}$$

where the indices $i, \dots, i+d+1 \in \mathbb{Z}/(n+1)\mathbb{Z}$ are counted in cyclic order and $d_S(g_i, g_{i+1})' := d_S(g_i, g_{i+1})$ for $0 \leq i < n$ and $d_S(g_n, g_0)' := d_S(g_n, vg_0)$.

If $x_0, \dots, x_n$ are positive real numbers, then

$$\sum_{i=0}^{n}(x_i + x_{i+1} + \dots + x_{i+d})^d \leq C''(d) \cdot (n+1)^{1-d} \cdot (x_0 + \dots + x_n)^d$$

where the indices are counted in cyclic order. To see this one can assume by homogeneity that $x_0+\dots+x_n = n+1$. A simple analysis shows that the left hand side attains its maximum under these conditions for $x_0 = \dots = x_n = 1$ and in this case the claimed inequality is obvious. Applying it one finds

$$\| \nabla' \circ B'(g_0, \dots, g_n, v) \|_f \leq$$
$$\leq C_0(n+1)^{1-d} \cdot |(g_0, \dots, g_n, v)|^d \cdot f(|(g_0, \dots, g_n, v)| + C_1)$$

which proves the second inequality. □

The aim of this section is to establish

Corollary 2.10 *Let* (Γ, S) *be a word-hyperbolic group and let* ∇', B' *be the operators on* $\overline{C}_*(\Gamma) \otimes Ad(\Gamma)$ *introduced in (2.6). Let* λ_0, λ_1 *be a pair of real numbers with* $1 < \lambda_0 < \lambda_1$ *and let* $\| - \|_{\lambda_0}, \| - \|_{\lambda_1}$ *be the corresponding seminorms on* $\overline{C}_*(\Gamma) \otimes Ad(\Gamma)$ *introduced in (2.8) c). Then there exist constants* C_2, C_3 *such that the estimates*

$$\| \nabla'(\alpha_n) \|_{\lambda_0} \leq C_2 \| \alpha_n \|_{\lambda_1}$$

and

$$\| (\nabla' \circ B')^k (\alpha_n) \|_{\lambda_0} \leq (n+1)(n+3) \cdot \dots \cdot (n+2k-1) \cdot C_3^k \cdot \| \alpha_n \|_{\lambda_1}$$

hold for all $\alpha_n \in \overline{C}_n(\Gamma) \otimes Ad(\Gamma)$ *and all* $n, k \in \mathbb{N}$.

Proof: The first assertion is an immediate consequence of (2.9) by noting that for $f_0(t) := \lambda_0^t$, $f_1(t) := \lambda_1^t$ the function $\frac{f_0'}{f_1}$ is bounded on $\mathbb{R}_+$. The second inequality of (2.9) yields for powers of the degree two operator $\nabla' \circ B'$ the estimate

$$\left\|(\nabla' \circ B')^k(\alpha_n)\right\|_f \leq C_0(n+2k-1)^{1-d}\left\|(\nabla' \circ B')^{k-1}(\alpha_n)\right\|_{f'}$$
$$\leq C_0^k(n+2k-1)^{1-d} \cdot (n+2k-3)^{1-d} \cdot \dots \cdot (n+1)^{1-d} \| \alpha_n \|_{f^{(k)}}$$

so that

$$\left\|(\nabla' \circ B')^k(\alpha_n)\right\|_{\lambda_0} \leq$$
$$\leq C_0^k((n+2k-1) \cdot (n+2k-3) \cdot \dots$$
$$\cdot (n+1))^{1-d} \sup_{t \geq 0}\left(\frac{f_0^{(k)}}{f_1}(t)\right) \| \alpha_n \|_{\lambda_1}$$

Here d, C_0, C_1 are the constants introduced in (2.9). Choose now a constant C such that $exp\left(\frac{dC_0}{C}\right) < \frac{\lambda_1}{\lambda_0}$ and let $C_3 > exp\left(\frac{dC_0C_1}{C}\right) \cdot C^d \cdot \lambda_0^{C_1}$. Then

$$C_0^k \cdot ((n+2k-1)\cdot(n+2k-3)\cdot\ldots\cdot(n+1))^{-d}\cdot\left(\frac{f_0^{(k)}}{f_1}(t)\right)$$
$$\leq C_0^k \cdot \left(\frac{1}{k!}\right)^d \cdot (t+kC_1)^{kd}\cdot\lambda_0^{(t+kC_1)}\cdot\lambda_1^{-t}$$
$$\leq \left(\frac{(C_0\cdot C^{-1}\cdot(t+kC_1))^k}{k!}\right)^d\cdot C^{kd}\cdot\lambda_0^{(t+kC_1)}\cdot\lambda_1^{-t}$$
$$\leq \left(exp\left(\frac{C_0}{C}\cdot(t+kC_1)\right)\right)^d\cdot C^{kd}\cdot\lambda_0^{(t+kC_1)}\cdot\lambda_1^{-t}$$
$$= exp\left(d\cdot\frac{C_0}{C}\cdot(t+kC_1)\right)\cdot\left(C^d\cdot\lambda_0^{C_1}\right)^k\cdot\left(\frac{\lambda_0}{\lambda_1}\right)^t$$
$$= \left(exp\left(\frac{dC_0C_1}{C}\right)\cdot C^d\cdot\lambda_0^{C_1}\right)^k\cdot\left(exp\left(\frac{dC_0}{C}\right)\cdot\frac{\lambda_0}{\lambda_1}\right)^t \leq C_3^k$$

which establishes the second inequality. □

3. Local cyclic homology of group Banach algebras

Hochschild and cyclic homology

To introduce notation a few well known facts about Hochschild and cyclic homology of algebras are collected in

Definition 3.1 *[Co], [Co1]*

a) For a complex algebra A define the A-bimodule of algebraic differential forms by

$$\Omega^n A := \widetilde{A}\otimes A^{\otimes n} \qquad \Omega A := \bigoplus_n \Omega^n A$$

with $\widetilde{A} := A\oplus\mathbb{C}1$ the algebra obtained from A by adjoining a unit. The A-bimodule structure on ΩA is the obvious one.

b) The Hochschild complex of a complex algebra is given by

$$C_*(A) := (\Omega^* A,\ b)$$

with Hochschild differential

$$b(a_0\otimes\ldots\otimes a_n) := \sum_{i=0}^{n-1}(-1)^i a_0\otimes\ldots\otimes a_i a_{i+1}\otimes\ldots\otimes a_n + (-1)^n a_n a_0\otimes\ldots\otimes a_{n-1}$$

Its homology $HH_*(A, A) := H_*(C_*(A))$ is called the Hochschild homology of A. There is a canonical isomorphism

$$HH_*(A, A) \simeq Tor_*^{\widetilde{A}\otimes\widetilde{A}^{op}}(A, A)$$

c) *The cyclic bicomplex of a complex algebra is the $\mathbb{Z}/2$-graded chain complex*

$$CC_*(A) := \left(\bigoplus_n \Omega^{*+2n}A,\ b+B\right)$$

where the Connes differential B is given by

$$B(a_0 \otimes \ldots \otimes a_n) := \sum_{j=0}^{n}(-1)^j\, 1 \otimes a_j \otimes \ldots \otimes a_n \otimes a_0 \otimes \ldots \otimes a_{j-1}$$

d) *The Hodge-filtration of the cyclic bicomplex is the descending filtration defined by the subcomplexes*

$$\mathrm{Fil}^k_{Hodge}\, CC_*(A) := \left(b\Omega^k A \bigoplus \Omega^{\geq k}A, b+B\right)$$

generated by algebraic differential forms of degree at least k.

e) *The periodic cyclic bicomplex $\widehat{CC}_*(A)$ of a complex algebra is the completion of the cyclic bicomplex with respect to the Hodge filtration:*

$$\widehat{CC}_*(A) := \lim_{\substack{\leftarrow \\ n}} CC_*/\mathrm{Fil}^n_{Hodge}CC_*(A)$$

Its homology $HP_*(A) := H_*(\widehat{CC}_*(A))$ is called the periodic cyclic homology of A.

f) *The reduced Hochschild- and cyclic complexes of a unital algebra A are defined similarly by using the A-bimodule $\overline{\Omega}A$ of reduced algebraic differential forms. It is obtained from ΩA by adding the relation $d(1_A) = 0$. The reduced Hochschild and cyclic complexes are naturally chain homotopic to the corresponding full complexes.*

g) *The bimodule of continuous differential forms over a complex Fréchet algebra A is the A-bimodule*

$$\Omega^n A_{cont} := \widetilde{A} \otimes_\pi A^{\otimes_\pi^n} \qquad \Omega A_{cont} := \bigoplus_n \Omega^n A_{cont}$$

The continuous Hochschild, continuous cyclic and continuous periodic cyclic complexes of a Fréchet algebra are defined similarly to the corresponding algebraic complexes by using continuous instead of algebraic differential forms.

Periodic cyclic (co)homology is a smooth homotopy functor and satisfies excision [Co], [CQ]. There exists a Chern character on the topological K-theory of Banach algebras with values in continuous periodic cyclic homology [Co].

Cyclic homology of group rings

We collect some well known facts about the cyclic homology of group rings which can be found for example in [Bu] and [Ni].

For every group Γ there exists a commutative diagram of functors

$$\begin{array}{ccc} \Gamma - Bimodules & \xrightarrow{Ad} & \Gamma - Modules \\ {\scriptstyle -/[\mathbb{C}\Gamma,-]}\big\downarrow & & \big\downarrow{\scriptstyle (-)_\Gamma} \\ \mathbb{C} - Vect & = & \mathbb{C} - Vect \end{array}$$

where the upper horizontal arrow associates to a Γ-bimodule M the same vector space M with the adjoint action and the vertical arrows are given by taking the commutator quotient $M \to M/[\mathbb{C}\Gamma, M] = M \otimes_{\mathbb{C}\Gamma\otimes\mathbb{C}\Gamma^{op}} \mathbb{C}\Gamma$ respectively by taking coinvariants $N \to N_\Gamma := N/\langle n - gng^{-1} | n \in N, g \in \Gamma\rangle$. The functor Ad is obviously exact and turns projective objects into projective objects. From this one obtains a canonical isomorphism of functors

$$HH_*(-, \mathbb{C}\Gamma) := Tor_*^{\mathbb{C}\Gamma\otimes\mathbb{C}\Gamma^{op}}(-, \mathbb{C}\Gamma) \simeq H_*(\Gamma, Ad(-))$$

which identifies the Hochschild-homology of $\mathbb{C}\Gamma$ with coefficients in a $\mathbb{C}\Gamma$-bimodule M with the group homology with coefficients in $Ad\, M$. Taking $M = \mathbb{C}\Gamma$ one obtains a canonical isomorphism

$$HH_*(\mathbb{C}\Gamma, \mathbb{C}\Gamma) \simeq H_*(\Gamma, Ad(\Gamma))$$

This isomorphism can be described on the level of the projective standard resolutions, the bar resolutions. Moreover, Connes' cyclic differential B corresponds to an explicit operator on the bar complex $(C_*(\Gamma) \otimes Ad(\Gamma))_\Gamma$. This allows to describe the cyclic homology of group rings in terms of group homology. The formulas necessary for our purpose are summarized in

Lemma 3.2 *[Ni] Let Γ be a group and let $C_*(\mathbb{C}\Gamma, \mathbb{C}\Gamma)$ be the Hochschild complex of its complex group ring $\mathbb{C}\Gamma$. Let $C_*(\Gamma) \otimes Ad(\Gamma)$ be the bar resolution of Γ twisted by the adjoint representation and let $(C_*(\Gamma) \otimes Ad(\Gamma))_\Gamma$ be its complex of Γ-coinvariants.*

a) There exist canonical isomorphisms

$$(C_*(\Gamma) \otimes Ad(\Gamma))_\Gamma \underset{\nu}{\overset{\mu}{\rightleftarrows}} C_*(\mathbb{C}\Gamma, \mathbb{C}\Gamma) \cong \Omega^*(\mathbb{C}\Gamma) .$$

inverse to each other. They are given on generators by the formulas

$$\mu(g_0, \ldots, g_n, v) = \big(g_n^{-1} v g_0\big) d\big(g_0^{-1} g_1\big) d\big(g_1^{-1} g_2\big) \cdots d\big(g_{n-1}^{-1} g_n\big)$$
$$\nu(g_0 dg_1 \ldots dg_n) = (1, g_1, g_1 g_2, \ldots, g_1 \ldots g_n, g_1 \ldots g_n g_0) .$$

These isomorphisms descend to isomorphisms

$$(\overline{C}_*(\Gamma) \otimes Ad(\Gamma))_\Gamma \underset{\nu}{\overset{\mu}{\rightleftarrows}} \overline{C}_*(\mathbb{C}\Gamma, \mathbb{C}\Gamma) \cong \overline{\Omega}^*(\mathbb{C}\Gamma)$$

of the corresponding reduced complexes.

b) *The operator* $B' : C_*(\Gamma) \otimes Ad(\Gamma) \to C_{*+1}(\Gamma) \otimes Ad(\Gamma)$ *of (2.6) is* Γ*-equivariant and descends therefore to an operator on the complex of* Γ*-coinvariants. It corresponds under the isomorphisms in a) to Connes' differential* B. *A similar statement holds for the reduced complexes.*

c) *The operator* $\nabla' : \overline{C}_*(\Gamma) \otimes Ad(\Gamma) \to \overline{C}_{*+1}(\Gamma) \otimes Ad(\Gamma)$ *of (2.6) is* Γ*-equivariant and descends therefore to an operator on the complex of* Γ*-coinvariants. It corresponds under the isomorphisms in a) to a connection* ∇ *on* $\overline{\Omega}^n(\mathbb{C}\Gamma)$ *for* $n \geq d$ *in the sense of Cuntz-Quillen [CQ1].*

Proof: Statements a) and b) are straightforward and c) is a tautology because connections in the sense of Cuntz and Quillen are by definition contracting homotopies of the Hochschild complex. □

A characteristic feature of the cyclic homology of group rings and crossed products is described in

Lemma 3.3 *[Ni] (Homogeneous decomposition)*

a) *The* Γ*-module* $Ad(\Gamma)$ *is the direct sum of irreducible* Γ*-modules*

$$Ad(\Gamma) = \bigoplus_{\langle\gamma\rangle} Ad(\Gamma)_{\langle\gamma\rangle}$$

labeled by the conjugacy classes $\langle\gamma\rangle$ *of* Γ *where* $Ad(\Gamma)_{\langle\gamma\rangle}$ *is the linear span of the elements of* $\langle\gamma\rangle$.

b) *The decomposition of* $Ad(\Gamma)$ *into irreducible submodules induces a corresponding decomposition of* $(C_*(\Gamma) \otimes Ad(\Gamma))_\Gamma$ *and therefore of the Hochschild-complex* $C_*(\mathbb{C}\Gamma, \mathbb{C}\Gamma)$. *It is compatible with Connes' differential* B *and provides thus a decomposition of the cyclic bicomplex* $CC_*(\mathbb{C}\Gamma)$. *A similar statement holds for the reduced complexes.*

c) *The decomposition of b) is given explicitely by*

$$CC_*(\mathbb{C}\Gamma) = \bigoplus_{\langle\gamma\rangle} CC_*(\mathbb{C}\Gamma)_{\langle\gamma\rangle}$$

where $CC_*(\mathbb{C}\Gamma)_{\langle\gamma\rangle}$ *is the span of all algebraic differential forms* $g_0 dg_1 \ldots dg_n$ *with* $g_0 \cdot \ldots \cdot g_n \in \langle\gamma\rangle$. *The direct summand*

$$CC_*(\mathbb{C}\Gamma)_{hom} = CC_*(\mathbb{C}\Gamma)_{\langle e\rangle}$$

corresponding to the conjugacy class of the unit is called the homogeneous part of $CC_*(\mathbb{C}\Gamma)$ *whereas*

$$CC_*(\mathbb{C}\Gamma)_{inhom} = \bigoplus_{\langle\gamma\rangle\neq\langle e\rangle} CC_*(\mathbb{C}\Gamma)_{\langle\gamma\rangle}$$

is called the inhomogeneous part.

d) *A similar decomposition holds for the Hochschild- and cyclic complexes of crossed product algebras.*

e) *The homogeneous decomposition of the Hochschild- and cyclic complexes induces a homogeneous decomposition of Hochschild- and cyclic (co)homology. In particular*

$$HH_*(\mathbb{C}\Gamma, \mathbb{C}\Gamma)_{hom} \simeq H_*(\Gamma, \mathbb{C}).$$

Local cyclic homology

Local cyclic homology is a cyclic homology theory for topological algebras, for example Banach algebras. It is the target of a Chern character from topological K-theory and possesses properties similar to the K-functor, in particular continuous homotopy invariance, excision property and stability [Pu1], [Pu2]. In addition it commutes with topological direct limits and is stable under passage to smooth dense subalgebras for Banach algebras with metric approximation property [Pu1].

The local cyclic homology of a Banach algebra is given by an inductive limit of cyclic type homology groups of dense subalgebras which makes it to some extent computable. We describe now its definition.

Let A be a Banach algebra and let U be its open unit ball. For each compact subset $K \subset U$ denote by A_K the completion of the subalgebra of A generated by K in the largest submultiplicative seminorm for which $\| K \| \leq 1$. In particular A_K is a Banach algebra. For an auxiliary Banach algebra A' denote by $\| - \|_{N,m}$, $N \geq 1$, $m \in \mathbb{N}$ the largest seminorm on $\Omega A'$ satisfying

$$\| a^0 da^1 \dots da^n \|_{N,m} \leq \frac{1}{c(n)!}(2 + 2c(n))^m N^{-c(n)} \| a^0 \|_{A'} \cdot \dots \cdot \| a^n \|_{A'}$$

with $c(2n) = c(2n+1) = n$. Let $\Omega A'_{(N)}$ be the completion of $\Omega A'$ with respect to the seminorms $\| - \|_{N,m}$, $m \in \mathbb{N}$. In fact $\Omega^n A'_{(N)}$ is then just a projective tensor power of A' and $\Omega A'_{(N)}$ becomes a weighted topological direct sum of the subspaces $\Omega^n A'_{(N)}$. The cyclic differentials b and B extend to bounded operators on $\Omega A'_{(N)}$. (This would not hold after completion with respect to only one of the seminorms introduced above.) Denote by $CC_*(A')_{(N)}$ the cyclic bicomplex $CC_*(A')_{(N)} := (\Omega A'_{(N)}, b + B)$.

Definition 3.4 *[Pu1] Let A be a Banach algebra. Then in the notations introduced above the local cyclic homology of A is defined as the direct limit*

$$HC_*^{loc}(A) := \varinjlim_{\substack{K \subset U \\ N \to \infty}} H_*(CC_*(A_K)_{(N)})$$

Recall that a Banach space E possesses the metric approximation property if the finite rank operators are dense in $\mathcal{L}(E)$ with respect to the compact open topology and that a Banach algebra possesses this property if its underlying Banach space does.

For Banach algebras with metric approximation property the local cyclic homology can be calculated as a countable direct limit. Instead of running over all compact subsets of the open unit ball it suffices to take the limit over the countable family of compact subsets $K_j := V_j \cap \overline{B}\left(\frac{j}{j+1}\right)$ where $0 \subset V_0 \subset \ldots \subset V_j \subset \ldots \subset A$ is an increasing sequence of finite dimensional subspaces such that $\bigcup_j V_j$ is a dense subalgebra of A and $\overline{B}\left(\frac{j}{j+1}\right)$ is the closed $\frac{j}{j+1}$-ball in A.

Proposition 3.5 *[Pu1] Let A be a Banach algebra with metric approximation property. Then in the notations above*

$$HC_*^{loc}(A) := \lim_{\substack{j\to\infty \\ N\to\infty}} H_*(CC_*(A_{K_j})_{(N)})$$

Proof: This is [Pu1], (4.2). □

Local cyclic homology of group Banach algebras

In the sequel the notion of a formal inductive limit or Ind-object will be used. Recall that for a category $\mathcal{C}$ the category Ind $\mathcal{C}$ of Ind-objects or formal inductive limits over $\mathcal{C}$ is defined as follows.

The objects of Ind $\mathcal{C}$ are small directed diagrams over $\mathcal{C}$:

$$\begin{aligned} \text{ob Ind}\,\mathcal{C} &= \left\{ \text{“}\varinjlim_{I}\text{”} A_i \;\middle|\; I \text{ a partially ordered directed set} \right\} \\ &= \left\{ A_i,\, f_{ij} : A_i \to A_j,\, i \le j \in I \;\middle|\; f_{jk} \circ f_{ij} = f_{ik} \right\} \end{aligned}$$

The morphisms between two Ind-objects are given by

$$\text{mor}_{\text{Ind}\,\mathcal{C}}(\text{“}\varinjlim_{I}\text{”}A_i,\ \text{“}\varinjlim_{J}\text{”}B_j) := \varprojlim_{I}\varinjlim_{J} \text{mor}_{\mathcal{C}}(A_i, B_j)$$

where the limits on the right hand side are taken in the category of sets.

Let Γ be a discrete group and let $\ell^1(\Gamma)$ be the convolution algebra of summable functions on Γ. The Banach algebra $\ell^1(\Gamma)$ possesses the metric approximation property. Suppose that Γ is finitely generated and let S be a finite symmetric set of generators. A natural choice for an increasing chain $(V_j),\ j \in \mathbb{N}$ of finite dimensional subspaces of $\ell^1(\Gamma)$ with dense union is to take V_j as the linear span of all elements $g \in \Gamma$ with word length $l_S(g) \le j$. Let K_j be the corresponding family of compact subsets of $\ell^1(\Gamma)$ as defined above. We describe the inductive family of Banach algebras $\ell^1(\Gamma)_{K_j},\ j \in \mathbb{N}$, explicitely. In order to do this we recall

Definition 3.6 *(Bost) [Bo] Let* (Γ, S) *be a finitely generated group and denote for* $\lambda \geq 1$ *by* $\ell^1_\lambda(\Gamma)$ *the completion of* $\mathbb{C}\Gamma$ *with respect to the largest seminorm satisfying*

$$\| g \| \leq \lambda^{l_S(g)}$$

The family $\ell^1_\lambda(\Gamma),\ \lambda \geq 1$ is the inductive system of the Banach algebras of summable functions on Γ of exponential decay in the word metric and $\ell^1_1(\Gamma) = \ell^1(\Gamma)$.

Lemma 3.7 *For a finitely generated group* (Γ, S) *there is a natural isomorphism*

$$\text{“}\lim_{j\to\infty}\text{”}\, \ell^1(\Gamma)_{K_j} \simeq \text{“}\lim_{\substack{\lambda\to 1\\ \lambda>1}}\text{”}\, \ell^1_\lambda(\Gamma)$$

of Ind-Banach algebras. Moreover these are independent of the choice of the set of generators S.

Corollary 3.8 *For a finitely generated group* (Γ, S)

$$HC^{loc}_*(\ell^1(\Gamma)) \simeq \lim_{\substack{\lambda\to 1,\lambda>1\\ N\to\infty}} H_*\big(CC_*\big(\ell^1_\lambda(\Gamma)\big)_{(N)}\big)$$

Proof: This follows from (3.5) and (3.7). □

The norms on $CC_*(\ell^1_\lambda(\Gamma))_{(N)}$ are given by weighted ℓ^1-norms on unions of powers of Γ. An immediate consequence of this is

Lemma 3.9 *Let* $\ell^1(\Gamma)$ *be the Banach algebra of a finitely generated group* Γ*. Then* $HC^{loc}_*(\ell^1(\Gamma))$ *possesses a homogeneous decomposition similar to the one described in (3.3).*

It should be observed that there is no reason for the existence of a homogeneous decomposition of $HC^{loc}_*(\overline{\mathbb{C}\Gamma})$ for completions $\overline{\mathbb{C}\Gamma}$ of $\mathbb{C}\Gamma$ with respect to other norms than weighted ℓ^1-norms.

The calculation of the local cyclic homology of $\ell^1(\Gamma)$ for a word hyperbolic group will now be done in two steps.

In the first it is shown that the subcomplexes $Fil^n_{Hodge}CC^{loc}_*(\ell^1(\Gamma))$ given by the Hodge filtration of the cyclic bicomplex do not contribute to $HC^{loc}_*(\ell^1(\Gamma))$ for large n. This is equivalent to the statement that the local cyclic homology of $\ell^1(\Gamma)$ is isomorphic to the direct limit of the continuous periodic cyclic homology groups of the Banach algebras $\ell^1_\lambda(\Gamma)$.

Proposition 3.10 *Let* Γ *be a word hyperbolic group. Let d be an integer as introduced in (2.5). Then the following holds.*

a) The natural map of Ind-*complexes*

$$\text{“}\lim_{\substack{\lambda\to 1,\lambda>1\\ N\to\infty}}\text{”}\overline{CC}_*\big(\ell^1_\lambda(\Gamma)\big)_{(N)} \longrightarrow \text{“}\lim_{\substack{\lambda\to 1\\ \lambda>1}}\text{”}\overline{CC}_*/\mathrm{Fil}^d_{\mathrm{Hodge}}\overline{CC}_*\big(\ell^1_\lambda(\Gamma)\big)$$

is an isomorphism in the homotopy category of Ind-*complexes of complete, locally convex vector spaces. Here* $\overline{CC}_*(\ell^1_\lambda(\Gamma))_{(N)}$ *is the completion of the reduced cyclic bicomplex of the Banach algebra* $\ell^1_\lambda(\Gamma)$ *with respect to the seminorms* $\| - \|_{N,m}$ *introduced in (3.4) and* $\overline{CC}_*/\mathrm{Fil}^d_{\mathrm{Hodge}}\overline{CC}$ *denotes the quotient of the reduced continuous cyclic bicomplex by the subcomplex given by the closure of the d-th step of the Hodge filtration (3.1).*

b) *In particular*

$$HC^{loc}_*(\ell^1(\Gamma)) \simeq \lim_{\substack{\lambda\to 1\\ \lambda>1}} H_*\big(\overline{CC}_*/\mathrm{Fil}^d_{\mathrm{Hodge}}\overline{CC}_*\big(\ell^1_\lambda(\Gamma)\big)\big)$$

c) *The isomorphisms of a) and b) are compatible with the harmonic decomposition (3.3).*

Proof: Let (Γ, S) be a word hyperbolic group. Let ∇' be the contracting Γ-equivariant homotopy operator of (2.6) on the twisted reduced bar resolution $\overline{C}_*(\Gamma) \otimes Ad(\Gamma)$ for $* \geq d$ and let $\nabla : \overline{\Omega}^*\mathbb{C}\Gamma \to \overline{\Omega}^{*+1}\mathbb{C}\Gamma$ be the connection on $\overline{\Omega}^*(\mathbb{C}\Gamma)$ corresponding to ∇' under the isomorphism of (3.2) a).

Denote by $\| - \|_{\lambda,N,m}$ the seminorms $\| - \|_{N,m}$ (3.4) on $\overline{CC}_*(\ell^1_\lambda(\Gamma))_{(N)}$. They satisfy $\| - \|_{\lambda,N',m} \leq C(m, N, N') \| - \|_{\lambda,N,0}$ for every pair $1 \leq N < N'$ so that one can restrict to the norms labeled by $(\lambda, N, 0)$ in order to check the continuity of a morphism of the Ind-complexes under consideration.

The results of (2.10) provide, via the identifications (3.2), the following estimates:

If $\lambda > \lambda' > 1$ are real numbers and if C_2, C_3 are the constants introduced in (2.10), then for every pair of real numbers $1 \leq N \leq N'$ and for all $\omega \in \overline{\Omega}(\mathbb{C}\Gamma)$ the estimates

$$\| \nabla(\omega) \|_{\lambda',N',0} \leq C_2 \| \omega \|_{\lambda,N,0}$$

and

$$\| (\nabla \circ B)^k(\omega) \|_{\lambda',N',0} \leq \left(\frac{2C_3}{N'}\right)^k \| \omega \|_{\lambda,N,0}$$

hold.

The estimates above show that for fixed (λ, N) and any λ' satisfying $1 < \lambda' < \lambda$ there exists N' such that the linear maps ∇ and $\eta := \sum_{k=0}^{\infty}(-\nabla \circ B)^k \circ \nabla$ extend to bounded operators from $\overline{CC}_*(\ell^1_\lambda(\Gamma))_{(N)}$ into $\overline{CC}_*(\ell^1_{\lambda'}(\Gamma))_{(N')}$.

In particular the operators ∇ and η define bounded endomorphisms of the Ind-complex

$$\text{“}\lim_{\substack{\lambda\to 1,\lambda>1\\ N\to\infty}}\text{”}\mathrm{Fil}^d_{Hodge}\overline{CC}_*\big(\ell^1_\lambda(\Gamma)\big)_{(N)}$$

and in fact the operator η defines a contracting homotopy of this Ind-complex because ∇ is a contracting homotopy of the Hochschild complex in degrees larger or equal to d. Let

$$p : \text{“}\lim_{\substack{\lambda\to 1,\lambda>1\\ N\to\infty}}\text{”}\overline{CC}_*\big(\ell^1_\lambda(\Gamma)\big)_{(N)} \longrightarrow \text{“}\lim_{\substack{\lambda\to 1,\lambda>1\\ N\to\infty}}\text{”}\overline{CC}_*/\mathrm{Fil}^d_{\mathrm{Hodge}}\overline{CC}_*\big(\ell^1_\lambda(\Gamma)\big)_{(N)}$$

be the natural quotient map. There exists a bounded linear map

$$s : \text{“}\lim_{\substack{\lambda\to 1,\lambda>1\\ N\to\infty}}\text{”}\overline{CC}_*/\mathrm{Fil}^d_{\mathrm{Hodge}}\overline{CC}_*\big(\ell^1_\lambda(\Gamma)\big)_{(N)} \longrightarrow \text{“}\lim_{\substack{\lambda\to 1,\lambda>1\\ N\to\infty}}\text{”}\overline{CC}_*\big(\ell^1_\lambda(\Gamma)\big)_{(N)}$$

of Ind-complexes which splits the projection p. It is defined by $s \circ p := Id - b \circ \nabla$ on differential forms of degree $d - 1$ and by zero respectively the identity in degrees above respectively below $d - 1$. The operator $s' := s + \eta(s\partial - \partial s)$, with $\partial := b + B$ the differential of the cyclic bicomplex, is then a bounded chain map of Ind-complexes. In fact it is a homotopy inverse of p. To see this note that $p \circ s' = Id$ and that $Id - s' \circ p = \partial\eta' + \eta'\partial$ with $\eta' := \eta(Id - s'p)$. Finally

$$\text{“}\lim_{\substack{\lambda\to 1,\lambda>1\\ N\to\infty}}\text{”}\overline{CC}_*/\mathrm{Fil}^d_{\mathrm{Hodge}}\overline{CC}_*\big(\ell^1_\lambda(\Gamma)\big)_{(N)} \simeq \text{“}\lim_{\substack{\lambda\to 1\\ \lambda>1}}\text{”}\overline{CC}_*/\mathrm{Fil}^d_{\mathrm{Hodge}}\overline{CC}_*\big(\ell^1_\lambda(\Gamma)\big)$$

as the weight factor N plays no role if only differential forms of uniformly bounded degree are considered. The proof of a) is thus complete and b) is an immediate consequence of (3.8) and a).

For c) note that by definition the contracting homotopy operator ∇' of the reduced twisted bar resolution of (2.6) is natural with respect to the chosen coefficient module and therefore compatible with the decomposition of $Ad(\Gamma)$ into irreducible subspaces. By (3.2) this is equivalent to the compatibility of the connection ∇ with the homogeneous decomposition of $\overline{CC}_*(\mathbb{C}\Gamma)$ from which c) follows along the lines of the proof of a). □

The homology of the inductive system of truncated cyclic complexes of the Ind-Banach algebra $\text{“}\lim_{\substack{\lambda\to 1\\ \lambda>1}}\text{”}\ell^1_\lambda(\Gamma)$ is studied further by comparing it with the twisted bar complexes and Rips chain complexes of the previous section.

We calculate only the homogeneous part of $HC^{loc}_*(\ell^1(\Gamma))$ as this is what we need for the purpose of this paper. The inhomogeneous part will be considered elsewhere.

Theorem 3.11 *Let Γ be a word-hyperbolic group. Then there exists a natural isomorphism*

$$HC^{loc}_*(\ell^1(\Gamma))_{hom} \simeq H_*\big(\Gamma, HC^{loc}_*(\mathbb{C})\big) := \bigoplus_n H_{*+2n}(\Gamma, \mathbb{C})$$

between the homogeneous part of the local cyclic homology of $\ell^1(\Gamma)$ and the homology of Γ with two-periodic complex coefficients.

Proof: The homogeneous decomposition takes the form

$$HC_*^{loc}(\ell^1(\Gamma)) = HC_*^{loc}(\ell^1(\Gamma))_{\langle e\rangle} \bigoplus HC_*^{loc}(\ell^1(\Gamma))_{inhom}$$

so that by the results of (3.10)

$$HC_*^{loc}(\ell^1(\Gamma))_{hom} \simeq \lim_{\substack{\lambda\to 1\\ \lambda>1}} H_*\big(\overline{CC}_*/Fil^d_{Hodge}\overline{CC}_*\big(\ell^1_\lambda(\Gamma)\big)_{\langle e\rangle}\big)$$

Define a bicomplex $\overline{C}_{**}(\Gamma)^{(d)}_\Gamma$ by

$$\overline{C}_{pq}(\Gamma)^{(d)}_\Gamma := \begin{cases} \overline{C}_{q-p}(\Gamma)_\Gamma & 0 \leq q-p < d-1 \\ \overline{C}_{d-1}/\partial_{bar}\overline{C}_d(\Gamma)_\Gamma & q-p = d-1 \\ 0 & \text{otherwise} \end{cases}$$

with differentials $d_0 := \partial_{bar} : \overline{C}_{pq}(\Gamma)^{(d)}_\Gamma \to \overline{C}_{p(q-1)}(\Gamma)^{(d)}_\Gamma$ the differential of the bar complex and $d_1 := B' : \overline{C}_{pq}(\Gamma)^{(d)}_\Gamma \to \overline{C}_{(p-1)q}(\Gamma)^{(d)}_\Gamma$ (2.6).

Denote by $\overline{C}^R_{**}(\Gamma)^{(d)}_\Gamma$ the corresponding bicomplex associated to the Rips chain complex $\overline{C}^R_*(\Gamma)_\Gamma \subset \overline{C}_*(\Gamma)_\Gamma$ (2.2). It is a subbicomplex of $\overline{C}_{**}(\Gamma)^{(d)}_\Gamma$. Note that $\overline{C}^R_*(\Gamma)_\Gamma$ and therefore also $\overline{C}^R_{**}(\Gamma)^{(d)}_\Gamma$ are finite dimensional in each degree.

The natural morphisms of (3.2) induce an isomorphism of complexes

$$\nu : \overline{CC}_*/Fil^d_{Hodge}\overline{CC}_*(\mathbb{C}\Gamma)_{\langle e\rangle} \xrightarrow{\simeq} Tot\big(\overline{C}_{**}(\Gamma)^{(d)}_\Gamma\big)$$

between the quotient of the homogeneous part of the cyclic bicomplex of $\mathbb{C}\Gamma$ by the d-th step of the Hodge filtration and the total complex of the bicomplex $\overline{C}_{**}(\Gamma)^{(d)}_\Gamma$. After completion of $\overline{C}_{**}(\Gamma)^{(d)}_\Gamma$ with respect to the seminorms $\| - \|_\lambda$ of (2.8) one obtains a similar isomorphism

$$\nu : \text{“}\lim_{\substack{\lambda\to 1\\ \lambda>1}}\text{”}\overline{CC}_*/Fil^d_{Hodge}\overline{CC}_*\big(\ell^1_\lambda(\Gamma)\big)_{\langle e\rangle} \xrightarrow{\simeq} \text{“}\lim_{\substack{\lambda\to 1\\ \lambda>1}}\text{”}Tot\big(\big(\overline{C}_{**}(\Gamma)^{(d)}_\Gamma\big)_\lambda\big)$$

of Ind-complexes.

We claim that the inclusion

$$\iota : Tot\big(\overline{C}^R_{**}(\Gamma)^{(d)}_\Gamma\big) \longrightarrow \text{“}\lim_{\substack{\lambda\to 1\\ \lambda>1}}\text{”}Tot\big(\big(\overline{C}_{**}(\Gamma)^{(d)}_\Gamma\big)_\lambda\big)$$

is a chain homotopy equivalence of Ind-complexes. Note that it suffices to prove that the inclusion is a chain homotopy equivalence on the columns of the bicomplexes because the bicomplexes are concentrated in a strip of finite width. In fact the chain map $\Phi : \overline{C}_*(\Gamma)_\Gamma \to \overline{C}_*(\Gamma)_\Gamma$ of (2.5) factors through $\overline{C}^R_*(\Gamma)_\Gamma$ and by (2.9) gives rise to a morphism

$$\Phi : \text{“}\lim_{\substack{\lambda\to 1\\ \lambda>1}}\text{”}\big((\overline{C}_*(\Gamma)^{(d)}_\Gamma\big)_\lambda\big) \to \text{“}\lim_{\substack{\lambda\to 1\\ \lambda>1}}\text{”}\big((\overline{C}^R_*(\Gamma)^{(d)}_\Gamma\big)_\lambda\big) = \overline{C}^R_*(\Gamma)^{(d)}_\Gamma$$

because $\text{“}\lim_{\substack{\lambda\to 1\\ \lambda>1}}\text{”}((\overline{C}^R_*(\Gamma)^{(d)}_\Gamma)_\lambda)$ is a constant, finite dimensional Ind-complex. The endomorphism $\iota\circ\Phi$ of $\text{“}\lim_{\substack{\lambda\to 1\\ \lambda>1}}\text{”}((\overline{C}_*(\Gamma)^{(d)}_\Gamma)_\lambda)$ is chain homotopic to the identity because $Id - \iota\circ\Phi = \nabla'\partial + \partial\nabla'$ and ∇' is a bounded operator on the considered Ind-complex by (2.10). On the other hand the chain endomorphism $\Phi\circ\iota$ of the finite dimensional complex $\overline{C}^R_*(\Gamma)^{(d)}_\Gamma$ is chain homotopic to the identity because $Id - \Phi\circ\iota$ is a chain map of the Rips resolution which vanishes in degree zero and any such chain map is nullhomotopic. Altogether this implies that

$$HC^{loc}_*(\ell^1(\Gamma))_{hom} \simeq H_*\big(Tot\big(\overline{C}^R_{**}(\Gamma)^{(d)}_\Gamma\big)\big)$$

Let π be the antisymmetrization operator on $\overline{C}^R_*(\Gamma)$ (2.1) and denote by $\overline{C}^R_{**}(\Gamma)'_\Gamma$ the bicomplex with $\overline{C}^R_{pq}(\Gamma)'_\Gamma := \pi(\overline{C}^R_{q-p}(\Gamma)_\Gamma)$ and differentials $d_0 = \partial_{bar}$ and $d_1 = 0$. Note that $\overline{C}^R_{pq}(\Gamma)'_\Gamma = 0$ for $q - p \geq d$ as π annihilates $\overline{C}^R_*(\Gamma)$ for $* \geq d$. The antisymmetrization map defines then a map

$$\pi' : \overline{C}^R_{**}(\Gamma)^{(d)}_\Gamma \to \overline{C}^R_{**}(\Gamma)'_\Gamma$$

of bicomplexes because $\pi\circ B' = 0$ on $\overline{C}_*(\Gamma) \subset \overline{C}_*(\Gamma)\otimes Ad(\Gamma)$.

The map π' is a chain homotopy equivalence on columns as π is a map of Rips-resolutions equal to the identity in degree zero. Therefore π' is a chain homotopy equivalence of total complexes, too, and

$$HC^{loc}_*(\ell^1(\Gamma))_{hom} \simeq \bigoplus_n H_{*+2n}\big(\pi\big(\overline{C}^R_*(\Gamma)\big)_\Gamma\big)$$

The Rips resolution $\overline{C}^R_*(\Gamma)$ is a free resolution of the constant Γ-module $\mathbb{C}$. Its complex of Γ-coinvariants calculates therefore the homology of Γ with complex coefficients and the same holds for the image of the complex under antisymmetrization. This finally yields the isomorphism

$$HC^{loc}_*(\ell^1(\Gamma))_{hom} \simeq \bigoplus_n H_{*+2n}(\Gamma,\mathbb{C})$$

claimed by the theorem. The notation $H_*(\Gamma, HC^{loc}_*(\mathbb{C}))$ has been introduced to underline the analogy of the map above with the assembly map in K-theory. □

Remark 3.12 In [Pu3] it was shown that Theorem (3.11) holds also for fundamental groups of compact nonpositively curved manifolds. In fact the homogeneous part of $HC_*^{loc}(\ell^1(\Gamma))$ can be identified with certain homology groups of Γ for a quite large class of groups. This together with an analysis of the inhomogeneous part will be presented elsewhere.

4. Auxiliary results about crossed products and their local cyclic homology

The rapid decay property of discrete groups

Let Γ be a finitely generated discrete group with finite symmetric set of generators S. Denote the associated word length function by l_S.

Let $\ell^2(\Gamma)$ be the Hilbert space of square intergrable functions on Γ. The group algebra $\mathbb{C}\Gamma$ acts as $*$-algebra of operators on $\ell^2(\Gamma)$ and both $\mathbb{C}\Gamma$ and its closure $C_r^*(\Gamma)$ under the operator norm can be identified with linear subspaces of $\ell^2(\Gamma)$ by associating to an operator the image of the cyclic and separating vector $e_1 \in \ell^2(\Gamma)$.

Definition 4.1 *[Jol] Denote by $\mathcal{A}(\Gamma)$ the completion of $\mathbb{C}\Gamma$ with respect to the family of seminorms*

$$\| \sum a_\gamma u_\gamma \|_k^2 := \sum_\gamma (1 + l_s(\gamma))^{2k} |a_\gamma|^2,\ k \geq 0$$

It is a linear subspace of $\ell^2(\Gamma)$ containing $\mathbb{C}\Gamma$ and is independent of the choice of the finite generating set S. A finitely generated group Γ is said to possess the property of rapid decay (RD) if

$$\mathcal{A}(\Gamma) \subset C_r^*(\Gamma)$$

as subspaces of $\ell^2(\Gamma)$.

Proposition 4.2 *[Jol] Let Γ be a finitely generated group which possesses the property (RD) of rapid decay. Then $\mathcal{A}(\Gamma)$ is an admissible Fréchet algebra which is closed under holomorphic functional calculus in the reduced group C^*-algebra $C_r^*\Gamma$. It is called the Jolissaint algebra of Γ.*

Proof: Let $\ell^2(\Gamma)$ be the Hilbert space with standard orthonormal basis $\{\xi_g,\ g \in \Gamma\}$. Let F be the unbounded selfadjoint operator on $\ell^2(\Gamma)$ given by $F(\xi_g) := l_S(g)\xi_g$ and let D be the unbounded derivation $D := [F, -]$ on $\mathcal{B}(\ell^2(\Gamma))$. The group ring $\mathbb{C}\Gamma$ acts by convolution as involutive algebra of bounded operators on $\ell^2(\Gamma)$. It satisfies $\mathbb{C}\Gamma \subset \bigcap_n dom(D^n)$. Moreover $\| D^n(a)(e_1) \|_{\mathcal{H}}^2 = \sum_{g\in\Gamma} l_S(g)^{2n} |a_g|^2$ for $a \in \mathbb{C}\Gamma$.

If Γ satisfies the property of rapid decay then by the closed graph theorem $\| a \|_{\mathcal{B}(\mathcal{H})} \leq C \| a \|_k$ for $a \in \mathcal{A}(\Gamma)$ and some constants C and k.

A straightforward calculation shows then $\| D^n(a) \|_{\mathcal{B}(\mathcal{H})} \leq C \| a \|_{n+k}$ for $a \in \mathcal{A}(\Gamma)$ and $n \in \mathbb{N}$. Therefore $\mathcal{A}(\Gamma)$ coincides with the closure of $\mathbb{C}\Gamma$ in $\bigcap\limits_n dom(D^n) \subset \mathcal{B}(\ell^2(\Gamma))$, i.e. with the completion of $\mathbb{C}\Gamma$ with respect to the seminorms $\| a \|'_k := \| D^k(a) \|_{\mathcal{B}(\ell^2(\Gamma))}$. This implies that $\mathcal{A}(\Gamma)$ is a Fréchet algebra and shows also that it is closed under holomorphic functional calculus in $C^*_r(\Gamma)$.

We claim that $\mathcal{A}(\Gamma)$ is admissible. This means that it possesses an "open unit ball" U such that the multiplicative closure of any compact subset of U is relatively compact in the ambient algebra [Pu]. Our choice for an open unit ball is $U := \{a \in \mathcal{A}(\Gamma), \| \sum\limits_\gamma |a_\gamma| u_\gamma \|_{C^*_r(\Gamma)} < 1\}$. In fact let $K \subset U$ be compact, $\| K \|_{C^*_r(\Gamma)} \leq \epsilon < 1$. Then

$$\begin{aligned} &\| D^k(K^n) \|_{\mathcal{B}(\mathcal{H})} \\ \leq &\sum_{\substack{j_1+\ldots+j_m=k \\ i_1+\ldots+i_{m+1}=n-k}} \| K^{i_1} \| \| D^{j_1}(K) \| \| K^{i_2} \| \cdot \ldots \cdot \| D^{j_m}(K) \| \| K^{i_{m+1}} \| \\ &\leq n^k \epsilon^{n-k} \max_{0 \leq j \leq k} \| D^j(K) \|_{\mathcal{B}(\mathcal{H})} \leq C'(k, K) \end{aligned}$$

which shows that the multiplicative closure of K is bounded with respect to all the defining seminorms of the Jolissaint algebra and thus a relatively compact subset of $\mathcal{A}(\Gamma)$. This proves our claim. □

The homogeneous decomposition for good completions of crossed products

Following Lafforgue [La] we introduce the notion of a good completion of a group algebra inside its reduced group C^*-algebra. We use an argument due to Connes and Moscovici [CM] to show that the local cyclic homology groups of any good completion of $\mathbb{C}\Gamma$ possess a homogeneous decomposition and that the homogeneous parts coincide for any two sufficiently large good completions.

Definition 4.3 *[La] Let Γ be a discrete group. A good completion of the group ring $\mathbb{C}\Gamma$ is an admissible Fréchet subalgebra $\mathfrak{a}(\Gamma)$ of $C^*_r(\Gamma)$ which can be defined by seminorms satisfying the conditions*

$$\| \sum_\gamma a_\gamma u_\gamma \| = \| \sum_\gamma |a_\gamma| u_\gamma \|$$

and

$$|a_\gamma| \leq |b_\gamma|, \ \forall \gamma \in \Gamma \Rightarrow \| \sum_\gamma a_\gamma u_\gamma \| \leq \| \sum_\gamma b_\gamma u_\gamma \|$$

A good completion of $\mathbb{C}\Gamma$ is called sufficiently large if it contains all the Banach algebras $\ell^1_\lambda(\Gamma)$, $\lambda > 1$, (3.6) of summable functions of exponential decay.

Examples of good completions are the group Banach algebra $\ell^1(\Gamma)$ and the Jolissaint algebra of a group with the property of rapid decay. Still following Lafforgue one has

Lemma 4.4 *[La] Let* Γ *be a discrete group and let* $\mathfrak{a}(\Gamma)$ *be a good completion of* $\mathbb{C}\Gamma$ *with defining seminorms* $\| - \|_k, k \in \mathbb{N}$. *Let A be a Banach-algebra with isometric* Γ*-action.*

a) The completion $\mathfrak{a}(\Gamma, A)$ *of the algebraic crossed product* $A \rtimes \Gamma$ *with respect to the seminorms*

$$\| \sum_{\gamma} a_\gamma u_\gamma \|'_k := \| \sum_{\gamma} \|a_\gamma\|_A \, u_\gamma \|_k$$

is an admissible Fréchet algebra.

b) If A is a Γ*-*C^**-algebra then* $\mathfrak{a}(\Gamma, A)$ *is a dense Fréchet subalgebra of the reduced crossed product* C^**-algebra* $A \rtimes_r \Gamma$.

Proof: Let A be a Banach algebra with isometric Γ-action.

For $a = \sum a_\gamma u_\gamma \in \mathfrak{a}(\Gamma, A)$ put $|a| := \sum \|a_\gamma\|_A \, u_\gamma \in \mathfrak{a}(\Gamma)$. It follows easily from the definitions and the identity $\|a\|_k^{\mathfrak{a}(\Gamma,A)} = \| \, |a| \, \|_k^{\mathfrak{a}(\Gamma)}$ that $\mathfrak{a}(\Gamma, A)$ is a Fréchet algebra and is admissible [Pu], an open unit ball being given by $U := \{a \in \mathfrak{a}(\Gamma, A), \ |a| \in U'\}$ where U' is an open unit ball of $\mathfrak{a}(\Gamma)$.

Suppose now that A is a Γ-C^*-algebra which is represented faithfully on the Hilbert space $\mathcal{H}_A$ and consider the associated representation of $A \rtimes_{red} \Gamma$ on $\mathcal{H}_A \widehat{\otimes} \ell^2(\Gamma)$. A simple calculation shows that $\| \, a \, \|_{\mathcal{B}(\mathcal{H}_A \widehat{\otimes} \ell^2(\Gamma))} \leq \| \, |a| \, \|_{C^*_r(\Gamma)}$ which implies $\mathfrak{a}(\Gamma, A) \subset A \rtimes_{red} \Gamma$. □

In particular the local cyclic homology groups of good completions of crossed products are well defined as these are admissible Fréchet algebras.

A crucial step in the proof of the Novikov conjecture for word-hyperbolic groups by Connes and Moscovici [CM] is the observation that every continuous homogeneous cyclic cocycle on the group Banach algebra $\ell^1(\Gamma)$ of a finitely generated discrete group extends to a continuous cyclic cocycle on any good completion of $\mathbb{C}\Gamma$. This is somewhat surprising because there is a priori no relation between the cyclic groups of different good completions. Repeating the argument of Connes and Moscovici in our context we find

Proposition 4.5 *(after Connes and Moscovici) Let* Γ *be a finitely generated discrete group and let* $\mathfrak{a}(\Gamma)$ *be a sufficiently large good completion of* $\mathbb{C}\Gamma$ *(4.3). Let A be a Banach algebra with isometric* Γ*-action.*

a) The local cyclic homology groups of $\mathfrak{a}(\Gamma, A)$ *possess a homogeneous decomposition*

$$HC_*^{loc}\big(\mathfrak{a}(\Gamma, A)\big) \simeq HC_*^{loc}\big(\mathfrak{a}(\Gamma, A)\big)_{hom} \oplus HC_*^{loc}\big(\mathfrak{a}(\Gamma, A)\big)_{inhom}$$

similar to the one in (3.2).

b) There exists a natural isomorphism

$$HC_*^{loc}\big(\ell^1(\Gamma, A)\big)_{hom} \xrightarrow{\simeq} HC_*^{loc}(\mathfrak{a}(\Gamma, A)\big)_{hom}$$

of the homogeneous parts of local cyclic homology, i.e. the homogeneous part is independent of the choice of a sufficiently large good completion.

Proof: The proof is that of [CM], (6.5). We begin with some preliminaries. The notations of (3.4) and (3.5) are permanently used. The inclusions $\ell^1_\lambda(\Gamma) \to \ell^1_{\lambda'}(\Gamma)$, $\lambda > \lambda' > 1$ are compact so that by assumption the same holds for the inclusions $\ell^1_\lambda(\Gamma) \to \mathfrak{a}(\Gamma)$. In particular one obtains a natural bounded morphism η : $\text{“}\lim\limits_{\substack{\lambda\to 1\\ \lambda>1}}\text{”}\,\ell^1_\lambda(\Gamma) \longrightarrow \text{“}\lim\limits_{K\subset U}\text{”}\,\mathfrak{a}(\Gamma)_K$ of Ind-Banach-algebras. This induces a morphism of cyclic Ind-complexes which yields the canonical homomorphism $\eta_* : HC_*^{loc}(\ell^1(\Gamma)) \longrightarrow HC_*^{loc}(\mathfrak{a}(\Gamma))$ of local cyclic homology groups.

Let U be an open unit ball for $\mathfrak{a}(\Gamma)$ given by an open ball with respect to some seminorm satisfying the conditions of (4.3) and let $K_j \subset \mathfrak{a}(\Gamma)$ be the compact set $K_j := V_j \cap \frac{j}{j+1}U$ where V_j is the linear span of the set of elements of Γ of word length at most j. We claim that there exists $\lambda > 1$ (depending on j) such that the linear endomorphism of $\mathbb{C}\Gamma$ given by $u_g \to \lambda^{l_S(g)}u_g$ extends to a bounded linear map $\iota_\lambda : \mathfrak{a}(\Gamma)_{K_j} \to \mathfrak{a}(\Gamma)$. In fact this is true provided that $\lambda > 1$ is such that $\lambda^j \cdot K_j \subset U$ as a simple calculation shows.

Let π_{hom} be the canonical projection onto the homogeneous part of $CC_*(\mathbb{C}\Gamma)$. Its restriction to $(\mathbb{C}\Gamma)^{\otimes^{n+1}}$ is given by the formula

$$\begin{array}{rccc} \pi_{hom}: & (\mathbb{C}\Gamma)^{\otimes^{n+1}} & \longrightarrow & (\mathbb{C}\Gamma)^{\otimes^{n+1}} \\ & a^0\otimes\ldots\otimes a^n & \longrightarrow & \sum\limits_{\gamma_0\ldots\gamma_n=e}(a_{\gamma_0}u_{\gamma_0})^0\otimes\ldots\otimes(a_{\gamma_n}u_{\gamma_n})^n \end{array}$$

Let $j \in \mathbb{N}$ and suppose that $\lambda > 1$ is chosen so that $\iota = \iota_\lambda$ extends to a bounded operator as described above. Then

$$\|\,\pi_{hom}(a^0\otimes\ldots\otimes a^n)\,\|_{\ell^1_\lambda(\Gamma)^{\otimes^{n+1}_\pi}} \leq \sum_{\gamma_0\ldots\gamma_n=e}(\lambda^{l_S(g_0)}|a_{\gamma_0}|)\ldots(\lambda^{l_S(g_n)}|a_{\gamma_n}|)$$

$$= |\,\langle\xi_e, |\iota(a^0)|\cdot\ldots\cdot|\iota(a^n)|\xi_e\rangle\,| \;\leq \|\,|\iota(a^0)|\,\|_{C_r^*(\Gamma)}\cdots\|\,|\iota(a^n)|\,\|_{C_r^*(\Gamma)}$$
$$\leq\; C^{n+1}\,\|\,\iota(a^0)\,\|_k\ldots\|\,\iota(a^n)\,\|_k$$

for some constant C and some seminorm $\|-\|_k$ by the closed graph theorem

$$\leq\; C(j,k)^{n+1}\,\|\,a^0\,\|_{\mathfrak{a}(\Gamma)_{K_j}}\cdots\|\,a^n\,\|_{\mathfrak{a}(\Gamma)_{K_j}}$$

This suffices to verify that π_{hom} defines a bounded chain map of Ind-complexes

$$\pi'_{hom}:\text{“}\lim_{\substack{j\to\infty\\ N\to\infty}}\text{”}\,CC_*(\mathfrak{a}(\Gamma)_{K_j})_{(N)} \longrightarrow \text{“}\lim_{\substack{\lambda\to 1,\lambda>1\\ N\to\infty}}\text{”}\,CC_*(\ell^1_\lambda(\Gamma))_{(N)}$$

Good completions of $\mathbb{C}\Gamma$ possess the metric approximation property. So by the approximation theorem (3.5) their local cyclic homology can be calculated by the Ind-complex considered above. Together with (3.8) it follows that this chain map considered above induces a natural homomorphism

$$\pi'_{hom_*} : HC_*^{loc}(\mathfrak{a}(\Gamma)) \longrightarrow HC_*^{loc}(\ell^1(\Gamma))_{hom}$$

of local cyclic homology groups. It is clear that the endomorphism

$$\pi_{hom_*} := \eta_* \circ \pi'_{hom_*} : HC_*^{loc}(\mathfrak{a}(\Gamma)) \longrightarrow HC_*^{loc}(\mathfrak{a}(\Gamma))$$

is idempotent and defines a homogeneous decomposition of $HC_*^{loc}(\mathfrak{a}(\Gamma))$. Moreover it follows from the continuity of π'_{hom} that the restriction of η_* to the homogeneous part

$$\eta_* : HC_*^{loc}(\ell^1(\Gamma))_{hom} \longrightarrow HC_*^{loc}(\mathfrak{a}(\Gamma))_{hom}$$

is an isomorphism. The extension of this to the case of crossed products is straightforward. □

Before we proceed a lemma about the compatibility of homogeneous decompositions with boundary maps in long exact homology sequences is needed.

Lemma 4.6 *Let*

$$0 \to I \to A \xrightarrow{f} B \to 0$$

be a Γ*-equivariant extension of* Γ*-algebras with (not necessarily equivariant) linear section. Then the long exact periodic cyclic homology sequence associated to the extension*

$$0 \to I \rtimes \Gamma \to A \rtimes \Gamma \xrightarrow{f \rtimes \Gamma} B \rtimes \Gamma \to 0$$

of crossed products decomposes into the direct sum of long exact sequences of the homogeneous parts

$$\ldots \to HP_*(A \rtimes \Gamma)_{hom} \xrightarrow{f \rtimes \Gamma} HP_*(B \rtimes \Gamma)_{hom} \xrightarrow{\delta} HP_{*-1}(I \rtimes \Gamma)_{hom} \to \ldots$$

and the inhomogeneous parts

$$\ldots \to HP_*(A \rtimes \Gamma)_{inhom} \xrightarrow{f \rtimes \Gamma} HP_*(B \rtimes \Gamma)_{inhom} \xrightarrow{\delta} HP_{*-1}(I \rtimes \Gamma)_{inhom} \to \ldots$$

Proof: The inclusions

$$\widehat{CC}_*(I \rtimes \Gamma) \subset \widehat{CC}_*(A \rtimes \Gamma, B \rtimes \Gamma) \subset Cone(f \rtimes \Gamma)_*$$

of bicomplexes are compatible with the homogeneous decomposition. The excision theorem in periodic cyclic homology [CQ] states that the inclusions above induce isomorphisms on homology. The same has to be true then for the inclusions of the homogeneous respectively inhomogeneous parts of the bicomplexes which is equivalent to the assertion of the lemma. □

Lemma 4.7 *Let* Γ *be a finitely generated group and let* $\mathfrak{a}(\Gamma)$ *be a sufficiently large good completion of* $\mathbb{C}\Gamma$. *Let*

$$0 \to I \to A \to B \to 0$$

be an equivariant extension of Banach-algebras with isometric Γ*-action which possesses a (not necessarily* Γ*-equivariant) bounded linear section.*

Then the long exact sequence of local cyclic homology groups of the extension

$$0 \to \mathfrak{a}(\Gamma, I) \to \mathfrak{a}(\Gamma, A) \to \mathfrak{a}(\Gamma, B) \to 0$$

decomposes into the direct sum of long exact sequences of the homogeneous respectively inhomogeneous parts.

Proof: The Ind-complexes which define the local cyclic homology (3.4) of $\mathfrak{a}(\Gamma, -)$ possess a homogeneous decomposition by (4.5). The extensions of crossed products above have an obvious bounded linear section derived from the section of the original extension. The assertion follows then by the same arguments as in the proof of (4.6) from the excision theorem in local cyclic homology for extensions of admissible Fréchet algebras [Pu2]. □

The homogeneous decomposition for the local cyclic homology of some crossed products is determined now. We are interested in crossed products of Γ-C^*-algebras over proper and free Γ-spaces.

Proposition 4.8 *Let* Γ *be a torsion-free finitely generated group and let* $\mathfrak{a}(\Gamma)$ *be a sufficiently large good completion of* $\mathbb{C}\Gamma$. *Let* $\widetilde{X}$ *be a simplicial complex on which* Γ *acts freely and simplicially and suppose that* $X := \widetilde{X}/\Gamma$ *is a finite complex. Let* A *be a* Γ-$C_0(\widetilde{X})$-C^**-algebra [Ka2], (1.5).*

Then the canonical homomorphisms

$$HC_*^{loc}(\mathfrak{a}(\Gamma, A)) \xrightarrow{\simeq} HC_*^{loc}(C_r^*(\Gamma, A))$$

and

$$HC_*^{loc}(\mathfrak{a}(\Gamma, A))_{hom} \xrightarrow{\simeq} HC_*^{loc}(\mathfrak{a}(\Gamma, A))$$

are isomorphisms of local cyclic homology groups.

Proof: Recall that a Γ-$C_0(\widetilde{X})$-C^*-algebra is a Γ-C^*-algebra with a Γ-equivariant structure homomorphism $C_0(\widetilde{X}) \to Z(\mathcal{M}(A))$ of $C_0(\widetilde{X})$ to the center of the multiplier algebra of A which satisfies $\varinjlim u_n a = a, \ \forall a \in A$, for some bounded approximate unit (u_n) of $C_0(\widetilde{X})$.

Let Δ be a topdimensional simplex of X and put $Y := X - \overset{\circ}{\Delta}$. Y is a simplicial complex with one simplex less than X. Denote by $\pi : \widetilde{X} \to X$ the canonical projection and put $\widetilde{Y} := \pi^{-1}Y, \ \widetilde{\Delta} := \pi^{-1}(\overset{\circ}{\Delta})$. Then

$$0 \to C_0(\widetilde{\Delta}) \to C_0(\widetilde{X}) \to C_0(\widetilde{Y}) \to 0$$

is a Γ-equivariant extension of Γ-$C_0(\widetilde{X})$-C^*-algebras with equivariant bounded linear section. The sequence

$$0 \to C_0(\widetilde{\Delta}) \otimes_{C_0(\widetilde{X})} A \to C_0(\widetilde{X}) \otimes_{C_0(\widetilde{X})} A \to C_0(\widetilde{Y}) \otimes_{C_0(\widetilde{X})} A \to 0$$

(see [Ka2] (1.6)) is again an extension of Γ-C^*-algebras with bounded linear section. Note that $C_0(\widetilde{X}) \otimes_{C_0(\widetilde{X})} A = A$ and that $I := C_0(\widetilde{\Delta}) \otimes_{C_0(\widetilde{X})} A$ respectively $B := C_0(\widetilde{Y}) \otimes_{C_0(\widetilde{X})} A$ is a Γ-$C_0(\widetilde{\Delta})$- respectively a Γ-$C_0(\widetilde{Y})$-algebra.

The action of Γ on $\widetilde{X}$ being free the maximal and reduced crossed products of any Γ-$C_0(\widetilde{X})$-C^*-algebra with Γ agree. It follows that the sequence of reduced crossed products associated to the extension $0 \to I \to A \to B \to 0$ is exact because the corresponding sequence of maximal crossed products is always exact. In particular there exists a natural map of long exact sequences

$$\begin{array}{ccccccc}
\overset{\delta}{\to} HC_*^{loc}(C_r^*(\Gamma, I)) & \to & HC_*^{loc}(C_r^*(\Gamma, A)) & \to & HC_*^{loc}(C_r^*(\Gamma, B)) \overset{\delta}{\to} \\
\uparrow & & \uparrow & & \uparrow \\
\overset{\delta}{\to} HC_*^{loc}(\mathfrak{a}(\Gamma, I)) & \to & HC_*^{loc}(\mathfrak{a}(\Gamma, A)) & \to & HC_*^{loc}(\mathfrak{a}(\Gamma, B)) \overset{\delta}{\to}
\end{array}$$

By the results of (4.7) there exists also a natural map of long exact sequences

$$\begin{array}{ccccccc}
\overset{\delta}{\to} HC_*^{loc}(\mathfrak{a}(\Gamma, I)) & \to & HC_*^{loc}(\mathfrak{a}(\Gamma, A)) & \to & HC_*^{loc}(\mathfrak{a}(\Gamma, B)) \overset{\delta}{\to} \\
\uparrow & & \uparrow & & \uparrow \\
\overset{\delta}{\to} HC_*^{loc}(\mathfrak{a}(\Gamma, I))_{hom} & \to & HC_*^{loc}(\mathfrak{a}(\Gamma, A))_{hom} & \to & HC_*^{loc}(\mathfrak{a}(\Gamma, B))_{hom} \overset{\delta}{\to}
\end{array}$$

We argue now by induction over the number of simplices of X. The two diagrams above allow with the help of the five lemma to reduce to the case that X is a single (open or closed) simplex.

Under this condition a Γ-$C_0(\widetilde{X})$-algebra is necessarily of the form $A \simeq C_0(\Gamma) \otimes A'$ for a C^*-algebra A' with trivial Γ-action.

Let $S \subset \Gamma$ be a finite subset, let $\chi_S \in C_c(\Gamma)$ be its characteristic function and denote by $P_S \in C_c(\Gamma) \rtimes \Gamma$ the image of χ_S under the canonical inclusion. One verifies easily that P_S is a multiplier of $(C_c(\Gamma) \otimes A') \rtimes \Gamma$ and that the maps $\pi_S : (C_c(\Gamma) \otimes A') \rtimes \Gamma \to (C_c(\Gamma) \otimes A') \rtimes \Gamma,\ \pi_S(\alpha) := P_S \cdot \alpha \cdot P_S$ extend to idempotent, norm decreasing linear endomorphisms of $\mathfrak{a}(\Gamma, A)$ and $C_r^*(\Gamma, A)$ satisfying $\lim\limits_{\overrightarrow{S\subset\Gamma}} \pi_S = Id$ pointwise. Moreover the images of the maps π_S are closed subalgebras. It is well known that $\pi_S((C_c(\Gamma) \otimes A') \rtimes \Gamma) \simeq M_n(A')$ for $n = \#S$. In particular $\pi_{\{e\}}((C_c(\Gamma) \otimes A') \rtimes \Gamma) \simeq A'$. The stability under passage to matrix algebras and the approximation theorem for local cyclic homology [Pu1], (4.2) imply then that the maps

$$HC_*^{loc}(A') \to \lim_{\overrightarrow{S\subset\Gamma}} HC_*^{loc}(\pi_S(\mathfrak{a}(\Gamma, A))) \to HC_*^{loc}(\mathfrak{a}(\Gamma, A))$$

and

$$HC_*^{loc}(A') \to \lim_{\overrightarrow{S\subset\Gamma}} HC_*^{loc}(\pi_S(C_r^*(\Gamma, A))) \to HC_*^{loc}(C_r^*(\Gamma, A))$$

are isomorphisms from which the first assertion follows. It is also clear by construction that the first isomorphism above factors as

$$HC_*^{loc}(A') \to HC_*^{loc}(\mathfrak{a}(\Gamma, A))_{hom} \to HC_*^{loc}(\mathfrak{a}(\Gamma, A))$$

which shows the second assertion.

□

5. Equivariant Chern-Connes characters and the Kadison-Kaplansky conjecture

Equivariant Chern-Connes characters

The description of Kasparov's bivariant K-theory [Ka] in terms of universal algebras by Cuntz [Cu] allows to characterize KK-theory as the universal stable and split exact homotopy functor on the category of (separable) C^*-algebras [Hi]. Recently Thomsen gave a similar characterization of equivariant KK-theory.

Theorem 5.1 *[Th] Let Γ be a locally compact second countable group. Let $\mathcal{KK}^\Gamma$ be the additive category with separable Γ-C^*-algebras as objects and morphisms defined by $mor_{\mathcal{KK}^\Gamma}(A, B) := KK^\Gamma(A, B)$. The composition of morphisms is given by the Kasparov product. Let ι be the tautological functor from the category of separable Γ-C^*-algebras to $\mathcal{KK}^\Gamma$ which assigns to a Γ-homomorphism $f : A \to B$ the element $f_* \in KK^\Gamma(A, B)$. Then every stable and split exact homotopy functor F from the category of separable Γ-C^*-algebras to an additive category $\mathfrak{C}$ factors uniquely through $\mathcal{KK}^\Gamma$, i.e. there exists a unique functor $\widetilde{F} : \mathcal{KK}^\Gamma \longrightarrow \mathfrak{C}$ such that $F = \widetilde{F} \circ \iota$.*

We derive, similar to [Cu1], a number of consequences from this result.

Theorem 5.2 *Let Γ be a finitely generated discrete group and let $\mathfrak{a}(\Gamma)$ be a sufficiently large good completion (4.3) of the group ring $\mathbb{C}\Gamma$.*

There exist natural transformations of bifunctors,

$$ch_{biv}^{\Gamma} : KK^{\Gamma}(-,-) \longrightarrow HC_0^{loc}(- \rtimes_r \Gamma, - \rtimes_r \Gamma)$$

and

$$ch_{biv}^{\mathfrak{a}(\Gamma)} : KK^{\Gamma}(-,-) \longrightarrow HC_0^{loc}(\mathfrak{a}(\Gamma,-), \mathfrak{a}(\Gamma,-))$$

called the **equivariant Chern-Connes characters***, on the category of separable* Γ*-*C^**-algebras. They satisfy the following conditions:*

- *If* $\alpha \in KK^{\Gamma}(A,B)$ *is given by a* Γ*-equivariant homomorphism* $f : A \to B$ *of* C^**-algebras then*

$$ch_{biv}^{\Gamma}(\alpha) = (f \rtimes \Gamma)_* \in HC_0^{loc}(A \rtimes_r \Gamma, B \rtimes_r \Gamma)$$

  and

$$ch_{biv}^{\mathfrak{a}(\Gamma)}(\alpha) = (f \rtimes \Gamma)_* \in HC_0^{loc}(\mathfrak{a}(\Gamma, A), \mathfrak{a}(\Gamma, B))$$

- *The equivariant Chern-Connes characters are multiplicative, i.e. for any separable* Γ*-*C^**-algebras* A, B, C *and for any* $\alpha \in KK^{\Gamma}(A, B)$ *and* $\beta \in KK^{\Gamma}(B, C)$

$$ch_{biv}^{\Gamma}(\alpha \circ \beta) = ch_{biv}^{\Gamma}(\alpha) \circ ch_{biv}^{\Gamma}(\beta)$$

  and

$$ch_{biv}^{\mathfrak{a}(\Gamma)}(\alpha \circ \beta) = ch_{biv}^{\mathfrak{a}(\Gamma)}(\alpha) \circ ch_{biv}^{\mathfrak{a}(\Gamma)}(\beta)$$

  In fact these properties characterize the equivariant Chern-Connes characters uniquely.
- *There exist canonical natural transformations of bifunctors*

$$j_r : KK^{\Gamma}(-,-) \longrightarrow KK(- \rtimes_r \Gamma, - \rtimes_r \Gamma)$$

  and

$$j_{\mathfrak{a}} : KK^{\Gamma}(-,-) \longrightarrow Hom(K_*(\mathfrak{a}(\Gamma,-)), K_*(\mathfrak{a}(\Gamma,-)))$$

  which are compatible with products. For any $\alpha \in KK^{\Gamma}(A, B)$ *the diagram*

$$\begin{array}{ccc} K_*(\mathfrak{a}(\Gamma, A)) & \xrightarrow{j_{\mathfrak{a}}(\alpha)} & K_*(\mathfrak{a}(\Gamma, B)) \\ \downarrow & & \downarrow \\ K_*(A \rtimes_r \Gamma) & \xrightarrow{j_r(\alpha)} & K_*(B \rtimes_r \Gamma) \end{array}$$

  commutes.

- *The equivariant Chern-Connes character is compatible with the Chern character: for any $\alpha \in KK^{\Gamma}(A, B)$ the diagrams*

$$\begin{array}{ccc} K_*(A \rtimes_r \Gamma) & \xrightarrow{j_r(\alpha)} & K_*(B \rtimes_r \Gamma) \\ \downarrow ch & & \downarrow ch \\ HC_*^{loc}(A \rtimes_r \Gamma) & \xrightarrow{ch_{biv}^{\Gamma}(\alpha)} & HC_*^{loc}(B \rtimes_r \Gamma) \end{array}$$

and

$$\begin{array}{ccc} K_*(\mathfrak{a}(\Gamma, A)) & \xrightarrow{j_{\mathfrak{a}}(\alpha)} & K_*(\mathfrak{a}(\Gamma, B)) \\ \downarrow ch & & \downarrow ch \\ HC_*^{loc}(\mathfrak{a}(\Gamma, A)) & \xrightarrow{ch_{biv}^{\mathfrak{a}(\Gamma)}(\alpha)} & HC_*^{loc}(\mathfrak{a}(\Gamma, B)) \end{array}$$

commute.
- *The equivariant Chern-Connes character is compatible with the homogeneous decomposition of local cyclic homology: for any $\alpha \in KK^{\Gamma}(A,B)$ the morphism*

$$HC_*^{loc}(\mathfrak{a}(\Gamma, A)) \xrightarrow{ch_{biv}^{\mathfrak{a}(\Gamma)}(\alpha)} HC_*^{loc}(\mathfrak{a}(\Gamma, B))$$

preserves the homogeneous decomposition, i.e. it decomposes into the direct sum of the morphisms

$$HC_*^{loc}(\mathfrak{a}(\Gamma, A))_{hom} \xrightarrow{ch_{biv}^{\mathfrak{a}(\Gamma)}(\alpha)} HC_*^{loc}(\mathfrak{a}(\Gamma, B))_{hom}$$

of the homogeneous and

$$HC_*^{loc}(\mathfrak{a}(\Gamma, A))_{inhom} \xrightarrow{ch_{biv}^{\mathfrak{a}(\Gamma)}(\alpha)} HC_*^{loc}(\mathfrak{a}(\Gamma, B))_{inhom}$$

of the inhomogeneous parts.

Proof: The assertions of the theorem are immediate consequences of the characterization of equivariant KK-theory in (5.1).

Introduce a functor F on the category of separable Γ-C^*-algebras by $F(A) := HC_*^{loc}(\mathfrak{a}(\Gamma, A))$ and $F(f) := (f \rtimes \Gamma)_* \in HC_0^{loc}(\mathfrak{a}(\Gamma, A), \mathfrak{a}(\Gamma, B))$ for $f : A \to B$ a Γ-homomorphism. F is viewed as functor with values in the category $\mathfrak{C}$ with objects given by the image of F and with morphisms given by

$$mor_{\mathfrak{C}}(HC_*^{loc}(\mathfrak{a}(\Gamma, A)), HC_*^{loc}(\mathfrak{a}(\Gamma, B))) := HC_0^{loc}(\mathfrak{a}(\Gamma, A), \mathfrak{a}(\Gamma, B))$$

The composition of morphism is given by the composition product. Local cyclic homology is a stable and split exact homotopy functor [Pu1], [Pu2]. It follows therefore from Thomsen's theorem (5.1) that F factors through

$\mathcal{KK}^\Gamma$. The corresponding functor $\widetilde{F} : \mathcal{KK}^\Gamma \to \mathfrak{C}$ induces then the equivariant Chern-Connes character $ch_{biv}^{\mathfrak{a}(\Gamma)}$ on morphisms. By construction $ch_{biv}^{\mathfrak{a}(\Gamma)}(f_*) = \widetilde{F}(f_*) = F(f) = (f \rtimes \Gamma)_*$ and the multiplicativity of $ch_{biv}^{\mathfrak{a}(\Gamma)}$ is equivalent to the statement that $\widetilde{F}$ is a functor. The construction of ch_{biv}^{Γ} is similar. The uniqueness is obvious.

Introduce a functor F' on the category of separable Γ-C^*-algebras with values in the category $\mathcal{KK}$ (defined similarly to the equivariant categories in (5.1)) by $F'(A) := K_*(A \rtimes_r \Gamma)$ and $F'(f) := (f \rtimes \Gamma)_* \in KK(A \rtimes_r \Gamma, B \rtimes_r \Gamma)$ for $f : A \to B$ a Γ-homomorphism. Introduce still another functor F'' on the category of separable Γ-C^*-algebras with values in a full subcategory of the category of $\mathbb{Z}/2\mathbb{Z}$-graded abelian groups by $F''(A) := K_*(\mathfrak{a}(\Gamma, A))$ and $\widetilde{F}(f) := (f \rtimes \Gamma)_*$. As K-theory is a stable and split exact homotopy functor the functors F' and F'' factor again by Thomsens theorem through $\mathcal{KK}^\Gamma$ giving rise to the transformations j_r and $j_{\mathfrak{a}}$. Their compatibility with products is again equivalent to the fact that $\widetilde{F'}$ and $\widetilde{F''}$ are functors. The natural homomorphism $K_*(\mathfrak{a}(\Gamma, A)) \to K_*(A \rtimes_r \Gamma)$ induced by the inclusion of algebras $\mathfrak{a}(\Gamma, A) \to A \rtimes_r \Gamma$ defines a natural transformation $F' \to F''$ of functors. It gives rise to a natural transformation $\widetilde{F'} \to \widetilde{F''}$ which proves the desired commutativity of the diagram relating j_r and $j_{\mathfrak{a}}$.

The Chern character $ch : K_* \to HC_*^{loc}$ in local cyclic homology defines natural transformations between the stable and split exact homotopy functors F' and F respectively F'' and F introduced above. The associated natural transformations $ch : \widetilde{F'} \to \widetilde{F}$ respectively $ch : \widetilde{F''} \to \widetilde{F}$ of functors from $\mathcal{KK}^\Gamma$ to Ab yield the compatibility of the ordinary Chern character with the equivariant Chern-Connes character claimed in the theorem.

Consider the functors F_{hom} and F_{inhom} on the category of separable Γ-C^*-algebras given by $F_{hom}(A) := HC_0^{loc}(\mathfrak{a}(\Gamma, A))_{hom}$ respectively $F_{inhom}(A) := HC_0^{loc}(\mathfrak{a}(\Gamma, A))_{inhom}$ and $F_{hom}(f) = F_{inhom}(f) = (f \rtimes \Gamma)_*$. They are well defined because the homogeneous decomposition of $HC^{loc}(\mathfrak{a}(\Gamma, A))$ exists by (4.5) and is preserved by the elements of $HC_0^{loc}(\mathfrak{a}(\Gamma, A), \mathfrak{a}(\Gamma, B))$ of the form $(f \rtimes \Gamma)_*$. As both functors are stable, split exact and homotopy invariant by [Pu1] and (4.7), the direct sum decomposition $F \simeq F_{hom} \oplus F_{inhom}$ gives rise to a direct sum decomposition $\widetilde{F} \simeq \widetilde{F}_{hom} \oplus \widetilde{F}_{inhom}$ of functors from $\mathcal{KK}^\Gamma$ to Ab which proves the last assertion of the theorem. □

Assembly maps and the γ-element

Let Γ be a torsion-free discrete group. Let M be a smooth compact manifold without boundary and let D be an elliptic differential operator acting on the sections of a vector bundle over M. Let $\widetilde{M}$ be a Γ-covering of M and let $\widetilde{D}$ be the lift of D to a Γ-invariant elliptic differential operator acting on the sections of the appropriate Γ-equivariant vector bundle on $\widetilde{M}$. Kasparov [Ka1] associates to the data (Γ, M, D) on the one hand a topological index $Ind_t(\widetilde{D}) \in K_0^{top}(B\Gamma)$ in the topological K-homology group of the classifying space of Γ. In fact the topological indices exhaust the group $K_0(B\Gamma)$.

On the other hand he defines an analytic index $Ind_a(\widetilde{D}) \in K_0(C_r^*(\Gamma))$ in the K-group of the reduced group C^*-algebra of Γ. For the trivial group these quantities coincide with the topological and analytic indices of an elliptic operator on a compact manifold introduced by Atiyah and Singer [AS]. There exists a natural assembly homomorphism

$$\begin{array}{rccc} \mu : & K_*^{top}(B\Gamma) & \longrightarrow & K_*(C_r^*(\Gamma)) \\ & \Uparrow & & \Uparrow \\ & Ind_t(\widetilde{D}) & \longrightarrow & Ind_a(\widetilde{D}) \end{array}$$

[BC] which maps the topological index of an elliptic operator $\widetilde{D}$ to its analytic index.

The index theorem of Atiyah-Singer states that the topological and analytic index of an elliptic operator are identical. In particular the groups in which topological and analytic indices live are canonically isomorphic. The most optimistic hope for a generalization in the context of Γ-index theory would be that the assembly map μ is an isomorphism. This is the content of the famous Baum-Connes Conjecture [BC], [BCH].

All the approaches to this conjecture known up to now use in one or another form a strategy formulated by Kasparov [Ka1], [Ka2]. It is based on the construction of so called γ-elements and important for us because these elements provide a canonical decomposition of $K_*(C_r^*(\Gamma))$ into the direct sum of the image and the cokernel of the assembly map.

Definition 5.3 *[Ka1], [Ka2] Let Γ be a torsion-free discrete group and suppose that the classifying space $B\Gamma$ has the homotopy type of a finite dimensional, locally finite simplicial complex X. A Dirac-element α and a dual-Dirac element β for the group Γ are elements $\alpha \in KK^\Gamma(\mathcal{E}, \mathbb{C})$ and $\beta \in KK^\Gamma(\mathbb{C}, \mathcal{E})$ which satisfy the following conditions*

- *$\mathcal{E}$ is a Γ-$C_0(\widetilde{X})$-C^*-algebra.*
- *The reduced crossed product $\mathcal{E} \rtimes \Gamma$ is KK-equivalent to a commutative C^*-algebra $C_0(Y)$ where Y is an (even) Spanier-Whitehead dual of X. In particular $K_*^{top}(B\Gamma) \overset{\simeq}{\longrightarrow} K_*(C_0(Y)) \overset{\simeq}{\longrightarrow} K_*(\mathcal{E} \rtimes \Gamma)$ under this equivalence.*
- *The assembly map factors as*

$$\mu : \; K_*^{top}(B\Gamma) \overset{\simeq}{\longrightarrow} K_*(\mathcal{E} \rtimes \Gamma) \overset{j_r(\alpha)}{\longrightarrow} K_*(C_r^*(\Gamma))$$

  where j_r is the descent transformation defined as in (5.2).
- *$\beta \circ \alpha \,=\, 1 \in KK^\Gamma(\mathcal{E}, \mathcal{E})$*

If a Dirac element and a dual Dirac element exist the Kasparov product

$$\gamma := \alpha \circ \beta \,\in\, KK^\Gamma(\mathbb{C}, \mathbb{C})$$

is called a γ-element for the group Γ.

The element γ is an idempotent in the ring $KK^{\Gamma}(\mathbb{C},\mathbb{C})$ and $j_r(\gamma) \in KK(C_r^*(\Gamma), C_r^*(\Gamma))$ acts on $K_*(C_r^*(\Gamma))$ as projector onto the image of the assembly map μ. Its kernel $j_r(1-\gamma)\cdot K_*(C_r^*(\Gamma))$ is a canonical complement of this image. If $j_r(\gamma)$ is equal to one, then the Baum-Connes conjecture holds for Γ and the assembly map μ is an isomorphism. In general however $j_r(\gamma)$ is different from one [Sk].

Let $\mathcal{N}(\Gamma)$ be the enveloping von Neumann algebra of $C_r^*(\Gamma)$ and let $\tau : \mathcal{N}(\Gamma) \to \mathbb{C}$, $\tau(\sum a_g u_g) := a_e$ be the canonical faithful and positive trace. The value of the canonical trace on the analytic index of a Γ-invariant elliptic operator $\widetilde{D}$, called the von Neumann index of $\widetilde{D}$, can be interpreted like a classical index as

$$\tau(Ind_a(\widetilde{D})) = dim_{\mathcal{N}(\Gamma)} Ker(\widetilde{D}) - dim_{\mathcal{N}(\Gamma)} Ker(\widetilde{D}^*)$$

where ordinary dimensions have to be replaced by the real valued von Neumann dimensions. The L^2-index theorem of Atiyah and Singer [At], [Si] states that the von Neumann index of $\widetilde{D}$ on $\widetilde{M}$ is equal to the ordinary index of D on the compact manifold M

$$dim_{\mathcal{N}(\Gamma)} Ker(\widetilde{D}) - dim_{\mathcal{N}(\Gamma)} Ker(\widetilde{D}^*) = dim\, Ker(D) - dim\, Ker(D^*)$$

In particular, the value of the von Neumann index of a Γ-invariant elliptic operator, which is a priori a real number, turns out to be an integer.

The Kadison-Kaplansky conjecture

Conjecture 5.4 (Kadison-Kaplansky) Let Γ be a torsion-free group and let $C_r^*(\Gamma)$ be its reduced group C^*-algebra, i.e. the closure in operator norm of the group ring $\mathbb{C}\Gamma$ acting by convolution on the Hilbert space $\ell^2(\Gamma)$. Then the following equivalent assertions hold:

- The spectrum of any element of $C_r^*(\Gamma)$ is connected.
- $C_r^*(\Gamma)$ contains no idempotents except 0 and 1.
- The value of the canonical trace on any idempotent $e = e^2 \in C_r^*(\Gamma)$ is integral: $\tau(e) \in \mathbb{Z}$.

The equivalence of the two first assertions is a consequence of holomorphic functional calculus and the equivalence of the two last ones follows from an elementary argument given for example in [Co].

The Kadison-Kaplansky conjecture is usually deduced as a corollary of the Baum-Connes conjecture. Suppose that Γ is a torsion-free group for which the assembly map $\mu : K_*^{top}(B\Gamma) \to K_*(C_r^*(\Gamma))$ is an isomorphism and let $e = e^2 \in C_r^*(\Gamma)$ be an idempotent. Then $[e] = \mu(Ind_t(\widetilde{D})) = Ind_a(\widetilde{D})$ for some Γ-invariant elliptic operator $\widetilde{D}$ on a manifold $\widetilde{M}$ with free and cocompact Γ-action because every element of $K_0(B\Gamma)$ can be realized as a topological index. Atiyah's L^2-index theorem shows then

$\tau(e) = \tau(Ind_a(\widetilde{D})) = Ind(D) \in \mathbb{Z}$ as asserted by the third formulation of the Kadison-Kaplansky conjecture.

In particular, the Kadison-Kaplansky conjecture holds for all torsion-free discrete groups for which the Baum-Connes conjecture is known [HK], [La], [MY].

We interpret the integrality of the trace on idempotents as a statement about the pairing between K-theory and local cyclic cohomology. In our approach both of these theories play an equally important role. The existence of a γ-element for a torsion-free discrete group provides a canonical decomposition of the K-theory of $C_r^*(\Gamma)$ and of any good completion of $\mathbb{C}\Gamma$ into the direct sum of the image and the (possibly vanishing) cokernel of the assembly map. The local cyclic homology of good completions of $\mathbb{C}\Gamma$ possesses another canonical and natural decomposition, the homogeneous decomposition (4.5). The central point is to show that these a priori unrelated decompositions correspond to each other under the Chern character. Once this is done one argues as follows. Let $\mathfrak{a}(\Gamma)$ be any good completion of $\mathbb{C}\Gamma$. The canonical trace is a homogeneous cyclic cocycle and vanishes therefore on the canonical complement of the assembly map in $K_*(\mathfrak{a}(\Gamma))$ because this complement is mapped to the inhomogeneous part of local cyclic homology by the Chern character. From this the integrality of the trace on idempotents in $\mathfrak{a}(\Gamma)$ follows as before by Atiyah's L^2-index theorem. This shows that there are no nontrivial idempotents in good completions of $\mathbb{C}\Gamma$ and if one finds a good completion which is closed under holomorphic functional calculus in $C_r^*(\Gamma)$ the Kadison-Kaplansky conjecture for the considered group results.

The details of this argument are given in the next two theorems.

Theorem 5.5 *Let Γ be a torsion-free discrete group with finite classifying space and such that $HC_*^{loc}(\ell^1(\Gamma))_{hom} \simeq H_*(\Gamma, HC_*^{loc}(\mathbb{C}))$. Let $\mathfrak{a}(\Gamma)$ be a sufficiently large good completion of $\mathbb{C}\Gamma$ and let $\gamma \in KK^\Gamma(\mathbb{C}, \mathbb{C})$ be a γ-element for Γ. Then the equivariant Chern-Connes character*

$$ch_{biv}^{\mathfrak{a}(\Gamma)}(\gamma) \in HC_*^{loc}(\mathfrak{a}(\Gamma), \mathfrak{a}(\Gamma))$$

acts on $HC_*^{loc}(\mathfrak{a}(\Gamma))$ as the canonical projection onto the homogeneous part $HC_*^{loc}(\mathfrak{a}(\Gamma))_{hom}$.

Proof: We proceed in several steps.

- Let $\mathfrak{a}(\Gamma)$ be a sufficiently large good completion of $\mathbb{C}\Gamma$ and let $ch_{biv}^{\mathfrak{a}(\Gamma)}$ be the equivariant Chern-Connes character (5.2).
- Let γ be a γ-element for Γ and let $\alpha \in KK^\Gamma(\mathcal{E}, \mathbb{C})$ and $\beta \in KK^\Gamma(\mathbb{C}, \mathcal{E})$ be the Dirac- and dual Dirac elements satisfying $\beta \circ \alpha = 1$ and $\alpha \circ \beta = \gamma$.
- By (5.2), (5.3) the elements $ch_{biv}^{\mathfrak{a}(\Gamma)}(\alpha)$ and $ch_{biv}^{\mathfrak{a}(\Gamma)}(\beta)$ satisfy $ch_{biv}^{\mathfrak{a}(\Gamma)}(\beta) \circ ch_{biv}^{\mathfrak{a}(\Gamma)}(\alpha) = 1$ and $ch_{biv}^{\mathfrak{a}(\Gamma)}(\alpha) \circ ch_{biv}^{\mathfrak{a}(\Gamma)}(\beta) = ch_{biv}^{\mathfrak{a}(\Gamma)}(\gamma)$.

- The compatibility of the equivariant Chern-Connes character and the homogeneous decomposition (5.2) gives rise to homomorphisms

$$\begin{array}{ccccc}
HC_*^{loc}(\mathfrak{a}(\Gamma,\mathcal{E}))_{hom} & \xrightarrow{ch_{biv}^{\mathfrak{a}(\Gamma)}(\alpha)} & HC_*^{loc}(\mathfrak{a}(\Gamma))_{hom} & \xrightarrow{ch_{biv}^{\mathfrak{a}(\Gamma)}(\beta)} & HC_*^{loc}(\mathfrak{a}(\Gamma,\mathcal{E}))_{hom} \\
\bigoplus & & \bigoplus & & \bigoplus \\
HC_*^{loc}(\mathfrak{a}(\Gamma,\mathcal{E}))_{inhom} & \xrightarrow{ch_{biv}^{\mathfrak{a}(\Gamma)}(\alpha)} & HC_*^{loc}(\mathfrak{a}(\Gamma))_{inhom} & \xrightarrow{ch_{biv}^{\mathfrak{a}(\Gamma)}(\beta)} & HC_*^{loc}(\mathfrak{a}(\Gamma,\mathcal{E}))_{inhom}
\end{array}$$

  The composition of these maps equals the identity.
- One finds for the homogeneous part of the local cyclic homology of the crossed product $\mathfrak{a}(\Gamma, \mathcal{E})$

$$HC_*^{loc}(\mathfrak{a}(\Gamma,\mathcal{E}))_{hom} \simeq HC_*^{loc}(\mathfrak{a}(\Gamma,\mathcal{E})) \simeq HC_*^{loc}(\mathcal{E} \rtimes_r \Gamma)$$

  by (4.8)

$$\simeq HC_*^{loc}(C_0(Y))$$

  because KK-equivalent C^*-algebras are local cyclic homology equivalent by the multiplicativity of the bivariant Chern-Connes character [Pu2]

$$\simeq H_*(\Gamma, HC_*^{loc}(\mathbb{C}))$$

  because Y is an even Spanier-Whitehead dual of $X = B\Gamma$.
- For the homogeneous part of the local cyclic homology of $\mathfrak{a}(\Gamma)$ one finds

$$HC_*^{loc}(\mathfrak{a}(\Gamma))_{hom} \simeq HC_*^{loc}(\ell^1(\Gamma))_{hom} \simeq H_*(\Gamma, HC_*^{loc}(\mathbb{C}))$$

  by (4.5) and by the assumption of the theorem.
- In particular one obtains a commutative diagram

$$\begin{array}{ccccc}
HC_*^{loc}(\mathfrak{a}(\Gamma,\mathcal{E}))_{hom} & \xrightarrow{ch_{biv}^{\mathfrak{a}(\Gamma)}(\alpha)} & HC_*^{loc}(\mathfrak{a}(\Gamma))_{hom} & \xrightarrow{ch_{biv}^{\mathfrak{a}(\Gamma)}(\beta)} & HC_*^{loc}(\mathfrak{a}(\Gamma,\mathcal{E}))_{hom} \\
\uparrow & & \uparrow & & \uparrow \\
H_*(\Gamma, HC_*^{loc}(\mathbb{C})) & = & H_*(\Gamma, HC_*^{loc}(\mathbb{C})) & = & H_*(\Gamma, HC_*^{loc}(\mathbb{C}))
\end{array}$$

  with vertical arrows given by isomorphisms. This allows to conclude that

$$ch_{biv}^{\mathfrak{a}(\Gamma)}(\alpha) : HC_*^{loc}(\mathfrak{a}(\Gamma,\mathcal{E}))_{hom} \to HC_*^{loc}(\mathfrak{a}(\Gamma))_{hom}$$

  and

$$ch_{biv}^{\mathfrak{a}(\Gamma)}(\beta) : HC_*^{loc}(\mathfrak{a}(\Gamma))_{hom} \to HC_*^{loc}(\mathfrak{a}(\Gamma,\mathcal{E}))_{hom}$$

  are isomorphisms, too. If one is not willing to check the commutativity of the diagram one can deduce the assertion from the fact that all vector spaces in sight are of the same finite dimension and from the fact that $ch(\beta) \circ ch(\alpha) = 1$.

- Consequently $ch_{biv}^{\mathfrak{a}(\Gamma)}(\gamma)_{hom} = ch_{biv}^{\mathfrak{a}(\Gamma)}(\alpha)_{hom} \circ ch_{biv}^{\mathfrak{a}(\Gamma)}(\beta)_{hom}$ is the identity whereas $ch_{biv}^{\mathfrak{a}(\Gamma)}(\gamma)_{inhom} = ch_{biv}^{\mathfrak{a}(\Gamma)}(\alpha)_{inhom} \circ ch_{biv}^{\mathfrak{a}(\Gamma)}(\beta)_{inhom}$ vanishes because it factors through $HC_*^{loc}(\mathfrak{a}(\Gamma, \mathcal{E}))_{inhom}$ which is zero by (4.8). In particular $ch_{biv}^{\mathfrak{a}(\Gamma)}(\gamma)$ acts on $HC_*^{loc}(\mathfrak{a}(\Gamma))$ as the canonical projection onto its homogeneous part. □

Theorem 5.6 *Let Γ be a torsion-free discrete group which satisfies the following conditions:*

- *The classifying space $B\Gamma$ has the homotopy type of a finite simplicial complex.*
- *The group ring $\mathbb{C}\Gamma$ possesses a sufficiently large good completion which is closed under holomorphic functional calculus in $C_r^*(\Gamma)$.*
- *There exists a γ-element $\gamma \in KK^\Gamma(\mathbb{C}, \mathbb{C})$.*
- $HC_*^{loc}(\ell^1(\Gamma))_{hom} \simeq H_*(\Gamma, HC_*^{loc}(\mathbb{C}))$

Then the Kadison-Kaplansky conjecture holds for Γ, i.e. $C_r^*(\Gamma)$ contains no idempotents except 0 and 1.

Proof: We proceed again in several steps.

- Choose a sufficiently large completion $\mathfrak{a}(\Gamma)$ of $\mathbb{C}\Gamma$ which is closed under holomorphic functional calculus in $C_r^*(\Gamma)$.
- Let τ be the canonical positive faithful trace on $C_r^*(\Gamma)$. It is concentrated on the conjugacy class of the unit. The restriction of τ to $\mathfrak{a}(\Gamma)$ defines therefore a homogeneous local cyclic cocycle $[\tau] \in HC^0_{loc}(\mathfrak{a}(\Gamma))_{hom}$.
- For the pairing between local cyclic homology and cohomology we deduce

  $$\langle [\tau]\,,\, ch(j_{\mathfrak{a}}(1-\gamma)\cdot K_*(\mathfrak{a}(\Gamma)))\rangle = \left\langle [\tau]\,,\, ch_{biv}^{\mathfrak{a}(\Gamma)}(1-\gamma)\cdot ch(K_*(\mathfrak{a}(\Gamma)))\right\rangle$$

  by (5.2)

  $$= \langle \pi_{hom}[\tau]\,, \pi_{inhom} ch(K_*(\mathfrak{a}(\Gamma)))\rangle = 0$$

  by the previous theorem. In particular the canonical trace vanishes on the image under the Chern character of

  $$\begin{aligned} i_*\, j_{\mathfrak{a}}(1-\gamma)\cdot K_*(\mathfrak{a}(\Gamma)) &= j_r(1-\gamma)\cdot(i_* K_*(\mathfrak{a}(\Gamma)) \\ &= j_r(1-\gamma)\cdot K_*(C_r^*(\Gamma) \end{aligned}$$

  where $i : \mathfrak{a}(\Gamma) \to C_r^*(\Gamma)$ is the inclusion because $\mathfrak{a}(\Gamma)$ is closed under holomorphic functional calculus in $C_r^*(\Gamma)$.
- Let finally $e = e^2$ be an idempotent in $C_r^*(\Gamma)$ and let $[e]$ be its class in K-theory. Then

  $$\begin{aligned} \tau(e) &= \langle [\tau], ch(j_r(\gamma)\cdot[e])\rangle + \langle [\tau], ch(j_r(1-\gamma)\cdot[e])\rangle \\ &= \langle [\tau], ch(j_r(\gamma)\cdot[e])\rangle \in \langle [\tau], ch(Im(\mu))\rangle \subset \mathbb{Z} \end{aligned}$$

  by Atiyah's L^2-index theorem as explained at the beginning of this section. This proves the Kadison-Kaplansky conjecture for Γ. □

Our main application of this theorem is

Theorem 5.7 *Let* Γ *be a torsion-free word-hyperbolic group. Then the Kadison-Kaplansky conjecture holds for* Γ*, i.e.* $C_r^*(\Gamma)$ *contains no idempotents except 0 and 1.*

Proof: It has to be shown that hyperbolic groups verify the conditions of the previous theorem.

- A torsion-free word-hyperbolic group acts freely on its Rips complexes $P_d(\Gamma)$ which are contractible for $d \gg 0$ (2.4), (2.5). The geometric realization $|P_d(\Gamma)/\Gamma|$ of the quotient of $P_d(\Gamma)$ by the Γ-action is a finite simplicial complex which has the homotopy type of the classifying space $B\Gamma$.
- The results of Jolissaint [Jol] show that the Jolissaint algebra $\mathcal{A}(\Gamma)$ of a word-hyperbolic group is a sufficiently large completion of the group ring which is closed under holomorphic functional calculus in $C_r^*(\Gamma)$.
- A γ-element for bolic and in particular word-hyperbolic groups has been constructed by Kasparov and Skandalis in [KS].
- The homogeneous part of the local cyclic homology of the group Banach algebra $\ell^1(\Gamma)$ of a word-hyperbolic group has been calculated in (3.11) and coincides with the group homology of Γ with $\mathbb{Z}/2\mathbb{Z}$-periodic complex coefficients. □

Remark 5.8 By the results of [Ka1] and [Pu3] the previous theorem (5.6) applies also to fundamental groups Γ of compact Riemannian manifolds of nonpositive sectional curvature for which the group ring possesses a sufficiently large good completion that is closed under holomorphic functional calculus in $C_r^*(\Gamma)$. Therefore the Kadison-Kaplansky conjecture holds also for this class of groups. This result was previously known however by Lafforgue's proof of the Baum-Connes conjecture [La] for such groups.

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