ON THE UNIQUENESS OF THE HEISENBERG COMMUTATION RELATIONS

MARC A. RIEFFEL

The purpose of this paper is to give an elementary proof of Mackey's form [9] of the theorem of Stone and von Neumann concerning the uniqueness of the Schrödinger representation of the Heisenberg commutation relations, whose main idea is to make precise the comment which Mackey makes on page 316 that the theorem "may be regarded as an infinite dimensional generalization of a classical result on the representations of full matrix algebras". The main part of our proof involves showing that any representation of the commutation relations is equivalent to a representation of an algebra of finite rank operators on an appropriate inner-product space. The second part of our proof then consists of applying the well-known theorem (for which we indicate a proof in Lemma 4) describing the representations of such an algebra (or, equivalently, of the algebra of compact operators). Other proofs can be found in [2], [7], [10], [14], [15]. For related results see [5] and [8] and the references mentioned there as well as [1] and [9].

An elementary proof of the uniqueness of the Heisenberg commutation relations was recently given by Segal and Kunze [15; Theorem 10.6] which has many points of close contact with the proof which we give. In fact, their proof can be interpreted to a large extent as being a mixture of the two parts of our proof. We feel that by separating these two parts and, in particular, by making explicit the role played by an algebra of finite rank operators, we contribute additional motivation and clarity to the proof.

It is well-known that the theorem on the uniqueness of the Heisenberg commutation relations is a special case of the imprimitivity theorem for induced representations of locally compact groups [10], [11], [14]. In a paper now in preparation (the main results of which were announced in [13]) we will show that one can associate to induced representations also an analogue of an algebra of finite rank operators, and we will use this fact to give a proof of the imprimitivity theorem for induced representations of groups and of C^*-algebras.

Let G be a locally compact group, and let $C_\infty(G)$ denote the C^*-algebra of continuous complex-valued functions on G which vanish at infinity and with pointwise operations. By a *unitary G-module* we will mean a Hilbert space W on which G acts by means of a strongly continuous unitary representation.

Received June 12, 1972. Revisions received September 22, 1972. The research for this paper was carried out while I was visiting at the University of Pennsylvania. I would like to thank the members of the Department of Mathematics of the University of Pennsylvania for their warm hospitality during my stay there. This research was partially supported by National Science Foundation grant GP-25082.

If A is a C^*-algebra (such as $C_\infty(G)$), then by a *Hermitian A-module* we will mean a Hilbert space W on which A acts by means of a continuous non-degenerate *-representation into the algebra of bounded operators on W (Continuity is automatic [3; 1.3.7] but we will not need this fact.).

If we fix a left Haar measure for G, then we can form the Hilbert space $L^2(G)$, and $L^2(G)$ becomes a unitary G-module if we let G act by left translation, that is, if for $x \in G$ and $f \in L^2(G)$ we define $xf \in L^2(G)$ by $(xf)(y) = f(x^{-1}y)$ for all $y \in G$. Furthermore, $L^2(G)$ also becomes a Hermitian $C_\infty(G)$-module if we let $C_\infty(G)$ act by pointwise multiplication, that is, if for $F \in C_\infty(G)$ and $f \in L^2(G)$ we define $Ff \in L^2(G)$ by $(Ff)(y) = F(y)f(y)$ for all $y \in G$. Now if we also let G act on $C_\infty(G)$ by left translation so that $(xF)(y) = F(x^{-1}y)$ for all $F \in C_\infty(G)$ and $x, y \in G$, then it is easily verified that the actions of G and $C_\infty(G)$ on $L^2(G)$ are related by the formula

$$x(Ff) = (xF)(xf)$$

for all $x \in G$, $F \in C_\infty(G)$ and $f \in L^2(G)$. This formula can be considered as expressing the Heisenberg commutation relation between the x's and the F's, and the representations of G and $C_\infty(G)$ on $L^2(G)$ can be considered as the *Schrödinger representation* of these commutation relations.

More generally, by an arbitrary representation of the Heisenberg commutation relations we will mean a Hilbert space W which is simultaneously a unitary G-module and a Hermitian $C_\infty(G)$-module in such a way that

$$x(Fw) = (xF)(xw) \tag{1}$$

for all $x \in G$, $F \in C_\infty(G)$ and $w \in W$. For convenience we will call the space of such a representation simply a *Heisenberg G-module.* Thus, under the Schrödinger representation $L^2(G)$ becomes the prototype Heisenberg G-module.

If W and W' are two Heisenberg G-modules, then by $\mathrm{Heis}_G\,(W, W')$ we will denote the Banach space of all bounded linear transformations from W to W' which commute with the actions of both G and $C_\infty(G)$. It is clear that the collection of all Heisenberg G-modules together with the $\mathrm{Heis}_G\,(W, W')$ as spaces of homomorphisms forms a category, which we will call the category of Heisenberg G-modules. Two Heisenberg G-modules W and W' will be said to be unitarily equivalent if there is a unitary operator in $\mathrm{Heis}_G\,(W, W')$. A Heisenberg G-module is said to be irreducible if it has no proper subspace which is invariant under the actions of both G and $C_\infty(G)$. It is clear that the Hilbert space direct sum of a (possibly infinite) family of Heisenberg G-modules is again a Heisenberg G-module in the obvious way.

THEOREM (Mackey). *Every Heisenberg G-module is unitarily equivalent to a direct sum of (possibly infinitely many) copies of the Schrödinger Heisenberg G-module $L^2(G)$. The Schrödinger Heisenberg G-module is irreducible, and thus every irreducible Heisenberg G-module is unitarily equivalent to the Schrödinger Heisenberg G-module.*

Proof. Let $C_c(G)$ denote the space of continuous complex-valued functions on G of compact support. Then $C_c(G)$ can be viewed as a dense subspace of $L^2(G)$ and so as an inner-product space in its own right. It is clear that the actions of G and $C_\infty(G)$ on $L^2(G)$ carry $C_c(G)$ into itself.

Let E denote the algebra of continuous finite rank operators on $C_c(G)$ which is spanned by the rank one operators $T_{(f,g)}$ for $f, g \in C_c(G)$, where $T_{(f,g)}h = \langle h, g\rangle f$ for $h \in C_c(G)$. (Here $\langle\ ,\ \rangle$ denotes the inner product on $L^2(G)$.) Note that E is not the algebra of *all* continuous finite rank operators on $C_c(G)$ since a $T_{(f,g)}$ with $f \in C_c(G)$, $g \in L^2(G)$ need not be in E if $g \notin C_c(G)$. It is easily seen that the adjoint of $T_{(f,g)}$ is $T_{(g,f)}$; so E is in fact a self-adjoint algebra of operators on $C_c(G)$. We will find it convenient to write $\langle f, g\rangle_E$ instead of $T_{(f,g)}$ so that $\langle\ ,\ \rangle_E$ can be viewed as an E-valued inner-product on $C_c(G)$. Thus

$$\langle f, g\rangle_E h = \langle h, g\rangle f, \tag{2}$$

and $\langle f, g\rangle_E^* = \langle g, f\rangle_E$ for $f, g, h \in C_c(G)$.

Of course, E can also be viewed in the obvious way as acting on $L^2(G)$. In general, by a Hermitian E-module we will mean a Hilbert space on which E acts by a non-degenerate $*$-representation by bounded operators. (We make no continuity assumption since E is not considered to carry any topology, but it is not difficult to verify that if E is equipped with the C^*-norm coming from its action on $L^2(G)$, then any $*$-representation of E will automatically be continuous and so will extend to a $*$-representation of the algebra of all compact operators on $L^2(G)$.) Then it is clear that $L^2(G)$ is in fact an irreducible Hermitian E-module.

We would like to show that every Heisenberg G-module is in a natural way a Hermitian E-module. To this end we write the operators $\langle f, g\rangle_E$ in the form of integral operators. Specifically, a routine calculation shows that

$$(\langle f, g\rangle_E h)(x) = \int f(x)g^*(x^{-1}y)h(y^{-1}x)\, dy \tag{3}$$

for all $x \in G$, where $g^*(x) = \bar{g}(x^{-1})\Delta(x^{-1})$ as usual, Δ is the modular function of G and $^-$ denotes complex conjugation. Being sure to write the integral operator in exactly this way is perhaps the only somewhat tricky part of the whole proof. The function $(y, x) \mapsto f(x)g^*(x^{-1}y)$ is in $C_c(G \times G)$, and whenever we find it convenient we will identify this function with $\langle f, g\rangle_E$. Now the form of (3) suggests that for any element ϕ of $C_c(G \times G)$ we define an action of ϕ on $C_c(G)$ or $L^2(G)$ by

$$(\phi f)(x) = \int \phi(y, x)f(y^{-1}x)\, dy. \tag{4}$$

Routine calculations show that the composition of the actions of two elements ϕ and ψ of $C_c(G \times G)$ is given by the action of $\phi * \psi$, where

$$(\phi * \psi)(z, x) = \int \phi(y, x)\psi(y^{-1}z, y^{-1}x)\, dy,$$

and that the adjoint of the action of ϕ is given by ϕ^*, where

$$\phi^*(y, x) = \bar{\phi}(y^{-1}, y^{-1}x)\Delta(y^{-1}).$$

These are exactly the definitions of the product and involution on a transformation group algebra [4; 3.3, 3.5]. Furthermore, routine calculations show that with these definitions of product and involution $C_c(G \times G)$ is a $*$-algebra. Of course, it contains E as a $*$-subalgebra (actually as a two-sided ideal, but we will not need this fact).

Now any element ϕ of $C_c(G \times G)$ can be viewed as an element of $C_c(G, C_c(G))$, the space of continuous $C_c(G)$-valued functions on G of compact support, by setting $\phi(y)(x) = \phi(y, x)$. Here $C_c(G)$ is viewed as a dense subalgebra of $C_\infty(G)$ with the uniform norm $\|\ \|_\infty$. Then a glance at the definition (4) of the action of $C_c(G \times G)$ on the Schrödinger Heisenberg G-module suggests that for any Heisenberg G-module W and any $\phi \in C_c(G, C_c(G))$ we define an action of ϕ on W by

$$\phi w = \int \phi(y) y w \, dy \tag{5}$$

for all $w \in W$. It is easily seen that the integrand is in $C_c(G, W)$, and so this integral (and those which appear later) can be defined either as a Bochner integral or as a weak integral, that is, as is done in the proof of [15; Theorem 10.6] by everywhere taking the inner-product with elements of W and by applying the ordinary Lebesgue integral with respect to Haar measure. The following lemma is then just a special case of [4; Lemma 3.21] or [6; Theorem 1.5].

LEMMA 1. *Let W be a Heisenberg G-module. The action of $C_c(G \times G)$ on W defined in* (5) *gives a non-degenerate $*$-representation of $C_c(G \times G)$ by bounded operators on W. In fact, as a linear operator on W the norm of $\phi \in C_c(G, C_c(G))$ is no greater than $\int \|\phi(y)\|_\infty \, dy$.*

Proof. For $\phi \in C_c(G \times G)$ and $w \in W$ we have

$$\|\phi w\| = \left\| \int \phi(y) y w \, dy \right\| \leq \int \|\phi(y)\|_\infty \, \|yw\| \, dy = \|w\| \int \|\phi(y)\|_\infty \, dy;$$

so the last statement is verified. Routine calculations show that the action defined by (5) gives a $*$-representation. (If weak integrals are used, these calculations are facilitated by noting that for $w, w' \in W$ the function $F \to \langle Fw, w' \rangle$ defines a finite complex measure on G to which Fubini's theorem can be applied.)

To see that the representation is non-degenerate, let $w \in W$ and $\epsilon > 0$ be given. Since W is a non-degenerate $C_\infty(G)$-module and $C_c(G)$ is dense in $C_\infty(G)$, we can find $g \in C_c(G)$ such that $\|w - gw\| < \epsilon/2$. Let f be a non-negative function in $C_c(G)$ chosen so that $\int f(y) \, dy = 1$ and so that if y is in the support of f, then $\|w - yw\| < \epsilon/(2 \, \|g\|_\infty)$. Define $\phi \in C_c(G, C_c(G))$ by $\phi(y) = f(y)g$. Then a routine calculation shows that $\|w - \phi w\| < \epsilon$. Q.E.D.

LEMMA 2. *Let W be a Heisenberg G-module, which we view as a $C_c(G \times G)$-module according to Lemma 1. If we view W as a module over the *-subalgebra E of $C_c(G \times G)$, then W is non-degenerate as an E-module and so becomes a Hermitian E-module.*

Proof. We have seen that E can be identified with the linear span in $C_c(G \times G)$ of the functions of the form $(y, x) \to f(x)g^*(x^{-1}y)$ where $f, g \in C_c(G)$. From this it is easily seen that E, viewed as a subspace of $C_c(G \times G)$, is closed under pointwise multiplication and complex conjugation. Furthermore, if U is any precompact open subset of $G \times G$, then it is easily seen that the functions in E which are supported in U separate the points of U and so by the Stone-Weierstrass theorem [15] are uniformly dense in $C_c(U)$. (In other words, E is dense in $C_c(G \times G)$ in the usual inductive limit topology.)

Let $w \in W$ and $\epsilon > 0$ be given. By Lemma 1 we can find $\phi \in C_c(G \times G)$ such that $\|w - \phi w\| < \epsilon/2$. Let U be a precompact open subset of $G \times G$ which contains the support of ϕ, and let k be the Haar measure of the projection of U on the first component of $G \times G$. From what we saw in the previous paragraph we can find $\psi \in E$ with ψ supported in U and such that $\|\phi - \psi\|_\infty < \epsilon/(2k\,\|w\|)$. Then a routine calculation using the norm estimate of Lemma 1 shows that $\|w - \psi w\| < \epsilon$. Thus as an E-module W is non-degenerate. Q.E.D.

If W and W' are two Hermitian E-modules, then by $\mathrm{Hom}_E\,(W, W')$ we will denote the Banach space of all bounded linear operators from W to W' which commute with the action of E.

LEMMA 3. *Let W and W' be Heisenberg G-modules, and let $T \in \mathrm{Heis}_G\,(W, W')$. If W and W' are viewed as E-modules, then $T \in \mathrm{Hom}_E\,(W, W')$. Thus we have an embedding of the category of Heisenberg G-modules into the category of Hermitian E-modules. In particular, $L^2(G)$ is an irreducible Heisenberg G-module.*

Proof. Let $T \in \mathrm{Heis}_G\,(W, W')$, let $\phi \in C_c(G \times G)$ and let $w \in W$. Then, from the fact that T commutes with the actions of G and $C_\infty(G)$, we have

$$\phi(Tw) = \int \phi(y)y(Tw)\,dy = T\left(\int \phi(y)yw\,dy\right) = T(\phi w).$$

In particular this is true for $\phi \in E$, and so $T \in \mathrm{Hom}_E\,(W, W')$.

If P is the orthogonal projection on a Heisenberg G-submodule of W, then from the above paragraph we see that $P \in \mathrm{Hom}_E\,(W, W)$; so the range of P is also an E-submodule. Since $L^2(G)$ is clearly irreducible as an E-module, it follows that it is irreducible as a Heisenberg G-module. Q.E.D.

LEMMA 4. *Let W be a Hermitian E-module. Then there exist $g \in C_c(G)$ and $w \in W$, both of unit length, such that $\langle g, g\rangle_E w = w$. For such g and w the map Q from $C_c(G)$ into W, defined by*

$$Q(f) = \langle f, g\rangle_E w$$

for $f \in C_c(G)$, extends to an isometric transformation of $L^2(G)$ into W.

Proof. Since W is non-degenerate as an E-module, there must exist g, $g' \in C_c(G)$ and $w_1 \in W$ such that $\langle g', g\rangle_E w_1 \neq 0$. We can assume that g is of unit length. Now $\langle g, g\rangle_E$ is just the orthogonal projection of $L^2(G)$ onto the one-dimensional ray spanned by g. In particular, it is an idempotent element of E, and $\langle g', g\rangle_E = \langle g', g\rangle_E \langle g, g\rangle_E$. It follows that $w_2 = \langle g, g\rangle_E w_1 \neq 0$. Let w be w_2 normalized to unit length. Since $\langle g, g\rangle_E$ is an idempotent, it is clear that $\langle g, g\rangle_E w = w$.

To show that Q is isometric we use the easily verified fact that for f, g, h, $k \in C_c(G)$ we have

$$\langle f, g\rangle_E \langle h, k\rangle_E = \langle \langle f, g\rangle_E h, k\rangle_E .$$

Then with g chosen as in the previous paragraph, we have, using (2),

$$\begin{aligned}\langle Q(f), Q(f)\rangle &= \langle \langle f, g\rangle_E w, \langle f, g\rangle_E w\rangle \\ &= \langle \langle f, g\rangle_E^* \langle f, g\rangle_E w, w\rangle \\ &= \langle \langle \langle g, f\rangle_E f, g\rangle_E w, w\rangle \\ &= \langle \langle \langle f, f\rangle g, g\rangle_E w, w\rangle \\ &= \langle f, f\rangle \langle \langle g, g\rangle_E w, w\rangle = \langle f, f\rangle. \qquad \text{Q.E.D.}\end{aligned}$$

We remark that it is easily seen that Q is an E-homomorphism, but we will not need this fact. Also, if $L^2(G)$ is replaced by any Hilbert space H and if E is taken to be the algebra of all finite rank operators on H, then the above proof of Lemma 4 is just the heart of a proof of the well-known theorem that every $*$-representation of the algebra of all compact operators on H is just a direct sum of copies of its (irreducible) representation on H [12; Theorem 4.10.24], [3; §4.1].

LEMMA 5. *Let W be a Heisenberg G-module, which we view as a Hermitian E-module. If Q is defined as in Lemma 4, then $Q \in \mathrm{Heis}_G\,(L^2(G), W)$.*

Proof. Let g and w be chosen as in the statement of Lemma 4. Now for any $f \in C_c(G)$ we have

$$\langle f, g\rangle_E(y, x) = f(x)g^*(x^{-1}y) = f(x)\bar{g}(y^{-1}x)\Delta(y^{-1}x).$$

If we define $h \in C_c(G)$ by $h(z) = \bar{g}(z)\Delta(z)$ for $z \in G$, then we see that, viewing $\langle f, g\rangle_E$ as an element of $C_c(G, C_c(G))$, we have

$$\langle f, g\rangle_E(y) = f(yh)$$

(pointwise product). Then, viewing h as an element of $C_\infty(G)$ and using (1), we obtain

$$\begin{aligned}Q(f) = \langle f, g\rangle_E w &= \int f(yh(y^{-1}w))\, dy \\ &= \int f(y(hw))\, dy\end{aligned}$$

(which is essentially the first equation in the proof of [12; Theorem 10.6]). Then for $x \in G$ we have, again using (1),

$$\begin{aligned} x(Q(f)) &= x \int f(y(hw))\, dy = \int x(f(y(hw)))\, dy \\ &= \int (xf)(xy(hw))\, dy = \int (xf)(y(hw))\, dy \\ &= Q(xf), \end{aligned}$$

and for $F \in C_\infty(G)$ we have

$$\begin{aligned} F(Q(f)) &= F \int f(y(hw))\, dy = \int F(f(y(hw)))\, dy \\ &= \int (Ff)(y(hw))\, dy = Q(Ff). \end{aligned}$$

Since these equalities hold for any f in the dense subspace $C_c(G)$ of $L^2(G)$, they hold for all $f \in L^2(G)$. Q.E.D.

We can now complete the proof of the theorem. What we have found is that any Heisenberg G-module contains a submodule which is unitarily equivalent to the Schrödinger Heisenberg G-module. But the orthogonal complement of this submodule will also be a Heisenberg G-module (if it is not trivial) and so will also contain a submodule unitarily equivalent to the Schrödinger Heisenberg G-module. An application of Zorn's lemma completes the proof. Q.E.D.

We remark that it is not at all difficult to verify that in fact the category of Heisenberg G-modules is isomorphic to the category of Hermitian E-modules.

References

1. Richard Beals, *Topics in Operator Theory*, Chicago, 1971.
2. Pierre Cartier, *Analyse spectrale et théorème de prédiction statistique de Wiener*, Séminaire Bourbaki, No. 218, Paris, 1960/61.
3. Jacques Dixmier, *Les C*-algèbres et leurs représentations*, Paris, 1964.
4. Edward G. Effros and Frank Hahn, *Locally compact transformation groups and C*-algebras*, Mem. Amer. Math. Soc., No. 75(1967).
5. Gerard G. Emch, *Algebraic Methods in Statistical Mechanics and Quantum Field Theory*, New York, 1972.
6. James Glimm, *Families of induced representations*, Pacific J. Math., vol. 12(1962), pp. 885–911.
7. Edwin Hewitt and Kenneth A. Ross, *Abstract Harmonic Analysis, II*, Berlin, 1970.
8. Daniel Kastler, *The C*-algebras of a free Boson field*, Comm. Math. Phys., vol. 1(1965), pp. 14–48.
9. A. A. Kirillov, *Unitary representations of nilpotent Lie groups*, Uspehi Mat. Nauk, vol. 17(1962), pp. 57–110.
10. Lynn H. Loomis, *Note on a theorem of Mackey*, Duke Math. J., vol. 19(1952), pp. 641–645.
11. George W. Mackey, *A theorem of Stone and von Neumann*, Duke Math. J., vol. 16(1949), pp. 313–326.

12. CHARLES EARL RICKART, *General Theory of Banach Algebras*, Princeton, 1960.
13. MARC A. RIEFFEL, *Induced representations of C^*-algebras*, Bull. Amer. Math. Soc., vol. 78(1972), pp. 606–609.
14. IRVING E. SEGAL, *Algebraic integration theory*, Bull. Amer. Math. Soc., vol. 71(1965), pp. 419–489.
15. IRVING E. SEGAL AND RAY A. KUNZE, *Integrals and Operators*, New York, 1968.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT BERKELEY, BERKELEY, CALIFORNIA 94720