APPROXIMATE IDENTITIES AND $H^1(\mathbb{R})$

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Abstract. Let $\varphi(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be a real-valued function with $\int_{\mathbb{R}} \varphi \, dx \neq 0$. For $y > 0$, let $\varphi_y(x) = y^{-1} \varphi(x/y)$. For $f(x) \in L^1(\mathbb{R})$ define

$$f^*_y(x) = \sup_{y > 0, t \in \mathbb{R}} |f \ast \varphi_y(t)|.$$

We investigate the space $H^1_y$ defined to be

$$H^1_y = \{ f \in L^1(\mathbb{R}) : f^*_y \in L^1(\mathbb{R}) \}.$$

1. Introduction. If $\varphi$ is the Poisson kernel, then $H^1_\varphi$ is defined to be $H^1$. Fefferman and Stein [2] showed that $H^1_\varphi = H^1$ for any $\varphi$ that is smooth and dies quickly at infinity; e.g. $\varphi$ can be in the Schwartz class, or Lipschitz continuous (of any order) and compactly supported. However, it is easy to show that $H^1_\varphi = \{0\}$ if $\varphi = \chi_{(0,1]}$ (see [3]), where $\chi_E$ is the characteristic function of a set $E$. G. Weiss asked whether there was an $H^1_\varphi$ that was nontrivial but not $H^1$. In this note, we show the following two results.

Theorem 1. If $H^1_\varphi \neq \{0\}$, then $a(x) \in H^1_\varphi$, where

$$a(x) = \begin{cases} 1 & 0 < x < 1, \\ -1 & -1 < x < 0, \\ 0 & \text{otherwise}. \end{cases}$$

Theorem 2. There exists $\varphi(x) \geq 0$ such that $H^1_\varphi \neq \{0\}$, $H^1_\varphi \neq H^1$.

As a corollary of Theorem 1, we get

Corollary 1. If $H^1_\varphi \neq \{0\}$, then $H^1_\varphi \cap H^1$ is dense in $H^1$.

Comment on notation. To distinguish the “$y$” in $\varphi_y(x) (= y^{-1} \varphi(x/y))$ from the other subindices, in the following we write $(\varphi)_y$ instead of $\varphi_y$. The letter $C$ denotes various constants.

2. Proof of Theorem 1. For $f \in H^1_\varphi$ define

$$\| f \|_{H^1_\varphi} = \| f^*_y \|_{L^1}.$$

This norm makes $H^1_\varphi$ a Banach space. We use two simple facts about $\| \cdot \|_{H^1_\varphi}$.

Fact 1. If $f \in H^1_\varphi$, $g \in L^1$, then $f \ast g \in H^1_\varphi$ with

$$\| f \ast g \|_{H^1_\varphi} \leq \| f \|_{H^1_\varphi} \| g \|_{L^1}.$$
FACT 2. If \( y > 0 \) and \( f \in H^1_y \), then
\[
\| (f) \|_{H^1_y} = \| f \|_{H^1_y}.
\]

Let \( f \in H^1_y \), \( f \not\equiv 0 \) and fix \( f \). In the following part of this section, the constants \( c \) depend on this function \( f \). We may assume that \( f \) is real-valued (since \( \varphi \) is real-valued). We shall construct functions \( p_n, g_n \) \((-\infty < n < \infty)\) satisfying

1. \( \| p_n \|_{H^1_y} \leq c \),
2. \( \sum_{n=-\infty}^{\infty} \| g_n \|_{L^1} < +\infty \),
3. \( a = \sum_{n=-\infty}^{\infty} p_n \ast g_n \),

where the convergence is in \( H^1_y \). This implies the theorem.

The construction of \( p_n \) and \( g_n \). Since \( f \not\equiv 0 \) and since \( f \) is real-valued, we may assume there exist \( r > 1 \) and \( \varepsilon > 0 \) such that
\[
|\hat{f}(\xi)| > \varepsilon \quad \text{on } [-r, -r^{-1}] \cup [r^{-1}, r].
\]

Let \( \psi(x) \in \mathcal{S}(\mathbb{R}) \) be a real-valued even function such that
\[
\text{supp } \psi \subset [-r, -r^{-1}] \cup [r^{-1}, r], \quad \sum_{k=-\infty}^{\infty} \hat{\psi}(r^k \xi)^2 = 1 \quad \text{for any } \xi \neq 0.
\]

Now we invoke Wiener's Lemma: Let \( f_1(x), f_2(x) \in L^1(\mathbb{R}) \). If there exist an \( \varepsilon > 0 \) and an interval \( I \subset \mathbb{R} \) for which \( |\hat{f}_1(\xi)| > \varepsilon, \xi \in I, \) and \( \text{supp } f_2 \subset I \), then there is an \( h(x) \in L^1(\mathbb{R}) \) such that \( \hat{f}_2(\xi) = \hat{h}(\xi) \hat{f}_1(\xi) \).

Applying Wiener's Lemma to \( f(x) \) and \((\hat{\psi} \chi_{(0,\infty)})^\vee\), we get \( h_i(x) \in L^1(\mathbb{R}) \) such that
\[
\hat{\psi}(\xi) \chi_{(0,\infty)}(\xi) = \hat{h}_i(\xi) \hat{f}(\xi).
\]

Set \( \hat{h}(\xi) = \hat{h}_i(\xi) + \overline{\hat{h}_i(-\xi)} \). Then \( \hat{\psi}(\xi) = \hat{h}(\xi) \hat{f}(\xi) \), and
\[
\| \psi \|_{H^1_y} \leq \| h \|_{L^1} \| f \|_{H^1_y} \leq c \| f \|_{H^1_y}.
\]

We now define
\[
p_n(x) = (\psi)_{r^n}(x), \quad g_n(x) = a \ast (\psi)_{r^n}(x).
\]

Then (1) follows from (4). By taking Fourier transforms, we see that \( a = \sum_{n=-\infty}^{\infty} p_n \ast g_n \) in \( \mathcal{S}' \). To estimate \( \| g_n \|_{L^1} \), we divide into two cases.

Case 1. \( n \geq 0 \). We write
\[
| g_n(x) | = \left| \int_{-1}^{1} r^{-n} \psi(r^{-n}(x-t))a(t) \, dt \right|
\]
\[
= r^{-n} \int_{-1}^{1} (\psi(r^{-n}(x-t)) - \psi(r^{-n}x))a(t) \, dt
\]
\[
\leq cr^{-2n} \sup_{|r^{-n}x-t| < r^{-n}} |\psi'(t)| \leq cr^{-2n}R(r^{-n}x),
\]
where $R(x) = \sup_{|x-y|<1} |\psi'(y)|$. Therefore,
\[
\|g_n\|_{L^1} \leq cr^{-n} \int R(r^{-n} x) \, dx \leq cr^{-n}.
\]

Case 2. $n < 0$. We distinguish three subcases.

Subcase 1. $|x| > 3$.
\[
|g_n(x)| = \left| \int_{-1}^{1} r^{-n} \psi(r^{-n}(x-t))a(t) \, dt \right| \leq cr^{-n} / (r^{-n} |x|)^4
\]
($\psi$ is rapidly decreasing). Thus, \( \int_{|x|>3} |g_n(x)| \leq cr^{3n} \).

Subcase 2. $|x| \leq 3$, \( \min(|x|, |x+1|, |x-1|) \geq r^{n/2} \). These $x$’s are away from the discontinuities of $a(x)$. We have
\[
|g_n(x)| \leq \left| \int_{|x-t|<r^{n/2}} r^{-n} \psi(r^{-n}(x-t))a(t) \, dt \right| + \left| \int_{|x-t|>r^{n/2}} \cdots \, dt \right|
\]
The second term can be estimated as in the first subcase. The first term equals zero or it equals \( \int_{|t|>r^{-n/2}} \psi(t) \, dt \) (because \( \int \psi(t) \, dt = 0 \)). This is dominated by \( cr^n \), since $\psi$ is rapidly decreasing.

Subcase 3. \( \min(|x|, |x+1|, |x-1|) < r^{n/2} \). Here the best we can do is \( |a * \psi_n(x)| \approx c \). But the measure of this set is \( \approx 6r^{n/2} \).

Combining the three subcases yields for $n < 0$, \( \|g_n\|_{L^1} \leq cr^{-n/2} \). We therefore have (2).

3. Proof of Corollary 1. It is well known that the dual space of $H^1$ is the space BMO (see [2]). This is the space of locally integrable functions $h(x)$ that satisfy
\[
\sup_I |I|^{-1} \int_I |h(x) - h_I| \, dx = \|h\|_\ast < \infty.
\]
The supremum is over all intervals $I \subset \mathbb{R}$; $h_I$ denotes the average of $h(x)$ over $I$.

Clearly $a(x) \in H^1$. Also $H^1$ and $H^1_\psi$ are closed under translations and dilations. If $H^1_\psi \cap H^1$ is not dense, then there is an $h \in \text{BMO}$ such that $\|h\|_\ast = 1$ but $\int h(x)g(x) \, dx = 0$, for any $g \in H^1_\psi \cap H^1$. The same must hold for any dilation or translation of $a(x)$. This implies that $h$ is constant and $\|h\|_\ast = 0$.

4. Proof of Theorem 2. An examination of the proof of Theorem 1 shows that it works because of the relative smoothness of $a(x)$. In this section, we exhibit an $H^1_\varphi$ that is not trivial or $H^1$, by building functions $b(x) \in H^1$ and $\varphi(x)$, each of which has “large” high frequency terms in its Fourier series. The high frequencies of $\varphi(x)$ almost cancel out when $\varphi(x)$ is convolved with $a(x)$, but they match up with those of $b(x)$ to make $b(x) \in H^1_\varphi$.

For $n = 1, 2, 3, \ldots$, define
\[
\mu_n(x) = \sum_{k=1}^{n} \sin(2^k \pi x) \chi_{[1,2]}(x).
\]
We estimate $|a * (\mu_n)_f(x)|$ as follows.
Case 1. $y < 1$.

$$|a \ast (\mu_n)_y (x)| \leq C \sum_{k=1}^{n} (1/y)(y/2^k) \leq C.$$ 

Case 2. $y > 2^n$.

$$|a \ast (\mu_n)_y (x)| \leq C \sum_{k=1}^{n} (1/y)(2^k/y) \leq C2^n/y^2.$$ 

Case 3. $1 \leq y \leq 2^n$.

$$|a \ast (\mu_n)_y (x)| \leq C \sum_{\log_2 y < k < n} (1/y)(y/2^k) + C \sum_{1 \leq k \leq \log_2 y} (1/y)(2^k/y) \leq C/y.$$ 

Now observe that $a \ast (\mu_n)_y (t) = 0$ if $y \leq (t - 1)/2$ or $y < (-t - 1)/2$. Thus

$$a_{\mu_n}^*(x) \leq \begin{cases} C & \text{if } |x| \leq 1, \\ C/|x| & \text{if } 1 \leq |x| \leq 6 \cdot 2^n, \\ C2^n/|x|^2 & \text{if } 6 \cdot 2^n \leq |x|. \end{cases}$$ 

This yields $\|a_{\mu_n}^*\|_{L^1} \leq Cn$.

If $\alpha > 1$, then by

$$a \ast (\mu_n(\alpha \cdot))(t) = a^{-1}a \ast (\mu_n)_y(\alpha \cdot)(t),$$

and by similar observations as above, we get

(5) $$\|a_{\mu_n(\alpha \cdot)}^*\|_{L^1} \leq Cn,$$

where $C$ does not depend on $\alpha > 1$.

Define

$$a_n = 2^{2^n}, \quad \eta(x) = \sum_{n=1}^{\infty} n^{-2-\epsilon_0} \mu_n(\alpha_n x),$$

where $\epsilon_0 > 0$ is a small number. Then, by (5) we have

(6) $$\|a_n^*\|_{L^1} \leq \sum_n n^{-2-\epsilon_0} \|a_{\mu_n(\alpha_n \cdot)}^*\|_{L^1} \leq C \sum n^{1-\epsilon_0} < +\infty.$$ 

Let $\epsilon > 0$ be a small number. Define

$$b(x) = -\sum_{k=1}^{\infty} k^{-1+\epsilon} \sin(2^k \pi x) \chi_{[-2^{-1},1]}(x).$$ 

From the fact that $b \in L^2$, $\int b \, dx = 0$ and $\text{supp } b \subset [-2, -1]$, it follows that $b \in H^1$ (see [1]).

We claim that for $n > N_\epsilon$ and $0 \leq i \leq n/2$,

$$\left| \int b(x) \mu_n(2^{-i}(-x - 1) + 1) \, dx \right| \geq C_n \epsilon^i.$$
This is because the left-hand side equals
\[
\left| \int b(x) \sum_{k=i+1}^{n} \sin(2^k \pi (2^{-i}(-x - 1) + 1)) \, dx \right|
+ \int b(x) \sum_{k=1}^{i} \sin(2^k \pi (2^{-i}(-x - 1) + 1)) \, dx.
\]

The first integral equals
\[
\frac{1}{2} \sum_{k=i+1}^{n} (k - i)^{-1+\epsilon} \geq C_\epsilon n^\epsilon.
\]
The second integral is no larger than
\[
\|b\| \left\| \sum_{k=1}^{i} \left( \sin(2^k \pi (2^{-i}(-x - 1) + 1)) \right) \right\|_{\infty} \leq C \sum_{k=1}^{i} 2^{-k} \leq C
\]
(since \( \int b \, dx = 0 \)). Thus
\[
\left| \int b(x) \mu_n(2^{-i}(-x - 1) + 1) \, dx \right| \geq C_\epsilon n^\epsilon - C \geq C_\epsilon' n^\epsilon,
\]
if
\[ (7) \quad 0 \leq i \leq n/2 \quad \text{and} \quad n > N_\epsilon. \]

Therefore, if (7) holds,
\[
b^* \eta_{2'^{-1} - 1} = (2^{'-1} \alpha_n)^{-1} n^{-2-\epsilon_0} \int b(x) \mu_n(2^{-1}(2^i - 1 - x)) \, dx
\]
\[
\geq C_\epsilon' (2^{'-1} \alpha_n)^{-1} n^{-2-\epsilon_0+\epsilon}.
\]
Thus, \( b^*_\eta(x) \geq C_\epsilon' (2^{'-1} \alpha_n)^{-1} n^{-2-\epsilon_0+\epsilon} \) on \( E_{n,i} = \{ x : 2^i \alpha_n < |x| < 2^i \alpha_n - (2^i - 1) \} \).

Thus,
\[
\int_{E_{n,i}} b^*_\eta \, dx \geq C_\epsilon' n^{-2-\epsilon_0+\epsilon},
\]
which yields, upon summing for \( 0 < i \leq n/2 \),
\[
\int_{|x| < 2^{n/2} \alpha_n} b^*_\eta \, dx \geq C_\epsilon' n^{-1-\epsilon_0+\epsilon}.
\]
Therefore
\[ (8) \quad \|b^*_\eta\|_{L^1} \geq C_\epsilon' \sum_{n} n^{-1-\epsilon_0+\epsilon} = + \infty, \]
if \( \epsilon_0 < \epsilon \).

Take \( \nu(x) \in \mathcal{S} \) such that \( \nu(x) + \eta(x) \geq 0 \) for any \( x \in \mathbb{R} \). Then the kernel \( \varphi = \nu + \eta \) is nonnegative and \( a^*_\varphi \in L^1 \) and \( b^*_\varphi \not\in L^1 \), by (6) and (8). Thus
\[
H^1_\varphi \neq \{0\} \quad \text{and} \quad H^1_\varphi \neq H^1.
\]
REFERENCES


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