Multipliers of $BMO$ in the Bergman Metric with Applications to Toeplitz Operators*

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1. Introduction

Let $\Omega$ be a bounded symmetric domain in $\mathbb{C}^n$ with normalized volume measure $dV$. Let $K(z, w)$ be the Bergman kernel of $\Omega$ associated with $dV$. The Bergman distance $\beta(\cdot, \cdot)$ of $\Omega$ is, by definition, the “integrated form” of the infinitesimal metric

$$G(x) = \frac{1}{2} \frac{\partial^2}{\partial x_i \partial x_j} \log K(z, z).$$

For $a$ in $\Omega$ and $r > 0$, let $E(a, r) = \{z \in \Omega: \beta(a, z) < r\}$ be the open ball in the Bergman metric with center $a$ and radius $r$. Given a function $f$ in $L^2(\Omega, dV)$, the mean oscillation of $f$ in the Bergman metric is the function $MO, f(z)$ defined on $\Omega$ by

$$MO, f(z) = \left( \frac{1}{|E(z, r)|} \int_{E(z, r)} |f(w) - \bar{f}(z)|^2 dV(w) \right)^{1/2},$$

where $|E(z, r)|$ is the $dV$-volume of $E(z, r)$ and

$$\bar{f}(z) = \frac{1}{|E(z, r)|} \int_{E(z, r)} f(w) dV(w)$$

is the mean (or average) of $f$ over $E(z, r)$. $MO, f$ was first introduced and studied in [1-3]. It is clear that $MO, f(z)$ is continuous in $\Omega$ and $MO, f(z) = \left( \|f(z)\|^2 - |\bar{f}(z)|^2 \right)^{1/2}$. Various function spaces on $\Omega$ can be defined by imposing growth conditions on $MO, f(z)$ near the boundary $\partial \Omega$ of $\Omega$ (we use the topological boundary in this paper instead of the distinguished boundary or Shilov boundary). In particular, we define

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BMO,\( (\Omega) \) to be the space of all \( f \) such that \( \text{MO}_r(f) \) is bounded on \( \Omega \). We equip \( \text{BMO},(\Omega) \) with the semi-norm
\[
\| f \| = \sup \{ \text{MO}_r(f(z)): z \in \Omega \} = \sup \{ (\int f(z)^2 - |f(z)|^2)^{1/2} : z \in \Omega \}.
\]

By Theorem 18 of [3], \( \text{BMO},(\Omega) \) is independent of \( r \). Moreover, all the semi-norms \( \| \| \) are mutually equivalent. Thus we'll drop the subscript \( r \) and simply write \( \text{BMO}(\Omega) \) for \( \text{BMO},(\Omega) \). Note that our \( \text{BMO} \) here is different from the classical \( \text{BMO} \) even in the case of the unit disc. The classical \( \text{BMO} \) requires that the mean oscillation be bounded for balls of all sizes, while the definition of our \( \text{BMO} \) here requires this only for balls of a fixed hyperbolic size.

The importance of \( \text{BMO} \) in the Bergman metric was first exhibited in [2, 3], where it was used to characterize the boundedness of Hankel operators on the Bergman spaces. Several other descriptions of \( \text{BMO} \) were also given in [2, 3]. We need the invariant description of \( \text{BMO} \). Suppose \( f \) is in \( L^1(\Omega, dV) \), the Berezin transform of \( f \) is defined by \( \tilde{f}(z) = (1/K(z,z)) \int K(z,w)^2 f(w) dV(w). \) It was proved in [3] that for \( f \) in \( L^2(\Omega, dV) \), we have \( f \in \text{BMO} \) if and only if the function \( |\tilde{f}(z)|^2 - |\tilde{f}(z)|^2 \) is bounded in \( \Omega \). Since the Berezin transform is invariant under the automorphism group \( \text{Aut}(\Omega) \), the semi-norm of \( \text{BMO} \) is defined by
\[
\| f \|_{\text{BMO}} = \sup \{ (|\tilde{f}(z)|^2 - |\tilde{f}(z)|^2)^{1/2} : z \in \Omega \}.
\]

This paper is devoted to the study of the multipliers of \( \text{BMO} \) in the Bergman metric on bounded symmetric domains. We assume that \( \Omega \) is in its standard representation so that \( 0 \in \Omega \) and \( \partial \Omega \) is circular. We state our main results as Theorems A, B, and C.

**Theorem A.** For any bounded symmetric domain \( \Omega \) and \( f \in L^\infty(\Omega, dV) \), the following conditions are all equivalent:

1. \( f \) multiplies \( \text{BMO} \), i.e., \( f \text{BMO} \subset \text{BMO} \);
2. \( \beta(0,z) (|\tilde{f}(z)|^2 - |\tilde{f}(z)|^2)^{1/2} \) is bounded in \( \Omega \);
3. \( \beta(0,z) (|\tilde{f}(z)|^2 - |\tilde{f}(z)|^2)^{1/2} \) is bounded in \( \Omega \) for all \( r > 0 \);
4. \( \beta(0,z) (|\tilde{f}(z)|^2 - |\tilde{f}(z)|^2)^{1/2} \) is bounded in \( \Omega \) for some \( r > 0 \).

Note that when \( \Omega \) is the open unit ball in \( \mathbb{C}^n \), then
\[
\beta(0,z) - \log \frac{1}{1-|z|^2} \sim \log \frac{1}{|E(z,r)|} \quad (|z| \to 1^-)
\]

for any fixed \( r > 0 \) (see Lemma 7). In this case, Theorem A says that for \( f \in L^\infty(\Omega) \), we have \( f \text{BMO} \subset \text{BMO} \) iff
\[
(\log \frac{1}{1-|z|^2}) (|\tilde{f}(z)|^2 - |\tilde{f}(z)|^2)^{1/2}
\]
is bounded, iff
\[
(\log \frac{1}{|E(z,r)|}) \text{MO}_r(f(z))
\]
is bounded. Thus we can say that multipliers of \( \text{BMO} \) are bounded functions with logarithmic mean oscillations in the Bergman metric. The corresponding result for the classical \( \text{BMO} \) on the unit circle was established in [8]. General types of mean oscillation conditions in the classical case were also studied in [5, 6].

Let \( V\text{MOO}(\Omega) \) be the closed subspace of \( \text{BMO}(\Omega) \) consisting of functions \( f \) such that \( |\tilde{f}(z)|^2 - |\tilde{f}(z)|^2 \to 0 \) as \( z \to \partial \Omega \). It was shown in [3] that for \( f \) in \( \text{BMO}(\Omega) \), we have \( f \in V\text{MOO}(\Omega) \) if and only if \( |\tilde{f}(z)|^2 - |\tilde{f}(z)|^2 \to 0 \) for all \( r > 0 \). \( V\text{MOO}(\Omega) \) is also invariant under \( \text{Aut}(\Omega) \), and it was used in [1–3] to characterize the compactness of Hankel operators on the Bergman spaces. Several other descriptions of \( V\text{MOO}(\Omega) \) can also be found in [1–3]. Our next result concerns multipliers of the space \( V\text{MOO}(\Omega) \).

**Theorem B.** For any bounded symmetric domain \( \Omega \) and \( f \in L^\infty(\Omega, dV) \), we have \( f \text{MOO} \subset V\text{MOO} \) if and only if \( f \) multiplies \( \text{BMO} \).

Let \( \mathcal{B}(\Omega) \) be the Bloch space of \( \Omega \). See [9] for definition. If \( \Omega = B_n \), the open unit ball, then a holomorphic function \( f \) on \( B_n \) is in \( \mathcal{B}(B_n) \) if and only if \( (1-|z|^2) \text{Vf}(z) \) is bounded in \( B_n \), where \( \text{Vf}(z) = (\text{f1/z1})(z), ..., (\text{f18}) \) is the holomorphic gradient of \( f \). Denote by \( \mathcal{B}_0(B_n) \) the little Bloch space of \( B_n \) consisting of functions in \( \mathcal{B}(B_n) \) with the property that \( (1-|z|^2) \text{Vf}(z) \to 0 \) as \( z \to \partial B_n \). \( \mathcal{B}(B_n) \) is invariant under \( \text{Aut}(B_n) \), \( \mathcal{B}_0(B_n) \) is a closed invariant subspace of \( \mathcal{B}(B_n) \). Let \( H(\Omega) \) be the space of all holomorphic functions in \( \Omega \), then it was shown in [3] that \( \text{BMO}(\Omega) \cap H(\Omega) = \mathcal{B}(\Omega) \). Moreover, the Bloch norm is equivalent to the \( \text{BMO} \)-norm.

Let \( L^p_0(\Omega) \) be the Bergman space of holomorphic functions in \( L^p(\Omega, dV) \). Let \( P \) be the Bergman projection defined by \( Pf(z) = \int K(z,w) f(w) dV(w) \). \( L^p_0(\Omega) \) is a Banach space for all \( 1 \leq p < +\infty \). Given a function \( f \) on \( \Omega \), the Toeplitz operator \( T_f \) is defined by \( T_f g = Pf \). We let \( H^\infty(\Omega) \) denote the space of bounded holomorphic functions in \( \Omega \).
THEOREM C. Suppose \( f \in H(B_n) \), then the following conditions are equivalent:

1. \( f \in BMO \);
2. \( f \in \text{VMO} \);
3. \( f \in \mathfrak{B} \);
4. \( f \in \mathfrak{B}_0 \);
5. \( f \in H^\infty \) and \( (1 - |z|^2)|\nabla f(z)| \log(1/(1 - |z|^2)) \) is bounded in \( B_n \).

As a result of Theorem C, we have a complete characterization for the multipliers of the Bloch space \( \mathfrak{B}(B_n) \) and the little Bloch space \( \mathfrak{B}_0(B_n) \) of the open unit ball in \( \mathbb{C}^n \). For higher rank domains \( \Omega \), \( \text{VMO}(\Omega) \cap H(\Omega) \) consists of just the constant functions \([2, 3]\), and we don’t know a natural way to define the little Bloch space (the closure of the polynomials in \( \mathfrak{B}(\Omega) \) doesn’t behave well in this context).

The extremal problems in Section 2 are of some independent interest. We also give some sufficient conditions for the boundedness of \( T_f \) on \( L^1(B_n) \) in Section 6.

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2. AN EXTREMAL PROBLEM

Recall that a function \( f \) in \( L^1(\Omega, dV) \) is in \( BMO \) if and only if

\[
\| f \|_{BMO} = \sup_{z \in \Omega} \left[ \frac{1}{|\Omega|} \int_{\Omega} |K(z, w)|^2 f(w) dV(w) \right]^{1/2} < +\infty,
\]

where

\[
\hat{f}(z) = \frac{1}{K(z, z)} \int_{\Omega} |K(z, w)|^2 f(w) dV(w)
\]

is the Berezin transform of \( f \). The normalization of \( dV \) implies that \( K(z, 0) = K(0, w) = 1 \) for all \( z \) and \( w \) in \( \Omega \). Thus

\[
\hat{f}(0) = \int_{\Omega} f(w) dV(w).
\]

The following extremal problem is essential to our analysis. We use the notation \( F(z) \sim G(z) (z \to \partial \Omega) \) to mean that the quotient \( |F(z)|/|G(z)| \) stays between two positive constants as \( z \to \partial \Omega \).

**THEOREM 1.** For all \( r > 0 \), we have

1. \( \sup \{ |\hat{f}(z)| : \| f \|_{BMO} \leq 1, f(0) = 0 \} \sim \beta(0, z) (z \to \partial \Omega) \);
2. \( \sup \{ |\hat{f}(z)| : \| f \|_{BMO} \leq 1, f(0) = 0 \} \sim \beta(0, z) (z \to \partial \Omega) \).

**Proof.** Given \( f \) in \( BMO \) and \( z \) in \( \Omega \), we have

\[
|\hat{f}_r(z) - \hat{f}(z)| \leq \frac{1}{|E(z, r)|} \int_{E(z, r)} |f(w) - \hat{f}(z)|^2 dV(w) \frac{1}{J(z)} \leq \frac{1}{|E(z, r)|} \int_{E(z, r)} |f(w) - \hat{f}(z)|^2 dV(w) \frac{1}{J(z)}^{1/2}.
\]

Applying Lemma 8 of \([1]\), we can find a constant \( C_r > 0 \) (depending only on \( r \)) such that

\[
|\hat{f}_r(z) - \hat{f}(z)| \leq C_r \left[ \int_{E(z, r)} |f(w) - \hat{f}(z)|^2 |k_z(w)|^2 dV(w) \right]^{1/2},
\]

where \( k_z(w) = K(w, z)/\sqrt{K(z, z)} \) is the normalized reproducing kernel of \( \Omega \).

Note that the Berezin transform \( \hat{f} \) can also be written as

\[
\hat{f}(z) = \int_{\Omega} f(w) k_z(w)^2 dV(w),
\]

thus we have

\[
\int_{\Omega} |f(w) - \hat{f}(z)|^2 |k_z(w)|^2 dV(w) = |\hat{f}(z)|^2 - |\hat{f}(z)|^2.
\]

Hence

\[
|\hat{f}_r(z) - \hat{f}(z)| \leq C_r \left[ |\hat{f}(z)|^2 - |\hat{f}(z)|^2 \right]^{1/2} \leq C_r \| f \|_{BMO}
\]

for all \( f \in BMO \) and \( z \) in \( \Omega \). This implies that (1) and (2) are equivalent since \( \beta(0, z) \to +\infty \) as \( z \to \partial \Omega \). Next we prove (1).

By Theorem F of \([2, 3]\), \( |\hat{f}(z) - \hat{f}(w)| \leq 2 \sqrt{2} \| f \|_{BMO} \beta(z, w) \) for all \( f \) in \( BMO \) and \( z, w \) in \( \Omega \). In particular, if \( \hat{f}(0) = 0 \) and \( \| f \|_{BMO} \leq 1 \), then

\[
|\hat{f}(z)| \leq 2 \sqrt{2} \beta(0, z)
\]

for all \( z \) in \( \Omega \).
To prove the other direction, let $\lambda = \int_{\Omega} \beta(0, z) dV(z)$, and $f_0(z) = \beta(0, z) - \lambda$. By Theorem E of [2, 3], $\lambda \in (0, +\infty)$. Clearly $\tilde{f}_0(0) = 0$ and by results of [2, 3], $f_0 \in BMO$. By the invariance of the Bergman metric and the triangle inequality, we have

$$\tilde{f}_0(z) = \int_{\Omega} f_0(w) k_z(w)^2 dV(w)$$

$$= \int_{\Omega} f_0(\varphi_z(w)) dV(w)$$

$$= \int_{\Omega} \beta(0, \varphi_z(w)) dV(w) - \lambda$$

$$= \int_{\Omega} \beta(z, w) dV(w) - \lambda$$

$$\geq \int_{\Omega} (\beta(z, 0) - \beta(0, w)) dV(w) - \lambda$$

$$= \beta(0, z) - 2\lambda$$

for all $z$ in $\Omega$, where $\varphi_z$ is the unique element in Aut($\Omega$) with the properties that $\varphi_z \circ \varphi_z = 1d$, $\varphi_z(0) = z$, and $\varphi_z$ has an isolated fixed point. The real Jacobian determinant of $\varphi_z$ is $|k_z|^2$, see [1, 3] for more information on $\varphi_z$. It follows from the above estimate that

$$\sup \{ |\tilde{f}(z)| : \| f \|_{BMO} \leq 1, \tilde{f}(0) = 0 \} \geq \frac{1}{\| f_0 \|_{BMO}} \left| \tilde{f}_0(z) \right|$$

$$\geq \frac{1}{\| f_0 \|_{BMO}} \left[ \beta(0, z) - 2\lambda \right],$$

Since $\beta(0, z) \to +\infty$ as $z \to \partial \Omega$ and $2\lambda$ is just a constant, we can find a compact set $K$ in $\Omega$ such that $\beta(0, z) - 2\lambda \geq \beta(0, z)$ for all $z$ in $\Omega - K$. Thus

$$\sup \{ |\tilde{f}(z)| : \| f \|_{BMO} \leq 1, \tilde{f}(0) = 0 \} \geq \frac{\beta(0, z)}{2 \| f_0 \|_{BMO}}$$

for all $z$ in $\Omega - K$. This completes the proof of Theorem 1.

Recall that $VMO$ is the closed subspace of $BMO$ consisting of functions $f$ such that $|f|^2(z) - |\tilde{f}(z)|^2 \to 0$ as $z \to \partial \Omega$. We look at the corresponding extremal problem for $VMO$.

Theorem 2. For all $r > 0$, we have

1. $\sup \{ |\tilde{f}(z)| : \| f \|_{BMO} \leq 1, \tilde{f}(0) = 0, f \in VMO \} \leq \beta(0, z) (z \to \partial \Omega)$;
2. $\sup \{ |\tilde{f}(z)| : \| f \|_{BMO} \leq 1, \tilde{f}(0) = 0, f \in VMO \} \leq \beta(0, z) (z \to \partial \Omega)$.

Proof. By the proof of Theorem 1, (1) and (2) are equivalent for all $r > 0$. Moreover, since $VMO \subset BMO$, we have $\sup \{ |\tilde{f}(z)| : \| f \|_{BMO} \leq 1, \tilde{f}(0) = 0, f \in VMO \} \leq 2\sqrt{2} \beta(0, z)$ for all $z$ in $\Omega$. So it remains to prove that there exists a constant $\sigma > 0$ and a compact set $K$ of $\Omega$ such that

$$\sup \{ |\tilde{f}(z)| : \| f \|_{BMO} \leq 1, \tilde{f}(0) = 0, f \in VMO \} \leq \sigma \beta(0, z)$$

for all $z$ in $\Omega - K$.

Fix any positive integer $n$, let $S_n = \{ z \in \Omega : \beta(0, z) = n \}$. For any $z_0 \in S_{2n}$, let $a(t): [0, 1] \to \Omega$ be the geodesic (in the Bergman metric) from 0 to $z_0$. Define

$$f_n(t) = \begin{cases} \beta(0, a(t)) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \beta(z_0, a(t)) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

Do this for each and every $z_0 \in S_{2n}$, then $f_n$ is a well-defined function on $E(0, 2n)$ with the property that $f_n(z) = 0$ for all $z$ in $S_{2n}$. Geometrically, $f_n(z) = \beta(0, z)$ in $E(0, n)$ and $f_n$ is geodesically symmetric with respect to $S_n$ in $E(0, 2n)$. Let $f_n(z) = 0$ for all $z \in \Omega - E(0, 2n)$, then $f_n \in VMO$ and $\| f_n \|_{BMO} \leq 2 \| \beta(0, \cdot) \|_{BMO}$ for all $n = 1, 2, \ldots$. Let $g_n = f_n - f_n(0)$, then $g_n \in VMO$, $g_n(0) = 0$, and $\| g_n \|_{BMO} \leq 2 \| \beta(0, \cdot) \|_{BMO}$ for all $n = 1, 2, \ldots$. Moreover,

$$g_n(z) = \int_{E(z, n)} g_n(w) k_z(w)^2 dV(w)$$

$$= \int_{E(z, n)} f_n(w) k_z(w)^2 dV(w) - f_n(0)$$

$$\geq \int_{E(z, n)} f_n(w) k_z(w)^2 dV(w) - f_n(0)$$

$$= \int_{E(z, n)} \beta(z, w) dV(w) - f_n(0)$$

$$\geq \int_{E(z, n)} (\beta(z, 0) - \beta(0, w)) dV(w) - f_n(0)$$

$$= \beta(0, z) E(z, n) - \int_{E(z, n)} \beta(0, w) dV(w) - f_n(0).$$
Let $\lambda = \int Q \beta(0, w) dV(w)$, then $\int_{E(z, n)} \beta(0, w) dV(w) \leq \lambda$ and

$$f_n(z) = \int \beta(0, z) dV(z) = \int_{E(0, 2n)} \beta(0, z) dV(z)$$

for all $n = 1, 2, \ldots$. Therefore,

$$\tilde{g}_n(z) \geq \beta(0, z) \| E(z, n) \| - 2\lambda$$

for all $z$ in $Q$ and $n = 1, 2, \ldots$. This implies that

$$\sup \{ |\tilde{f}(z)| : \| f \|_{BMO} \leq 1, \tilde{f}(0) = 0, f \in VMO \}$$

\begin{align*}
\geq & \sup \{ \| \tilde{g}_n(z) \| : \| g_n \|_{BMO} \} : n = 1, 2, \ldots \\
\geq & \frac{1}{2 \| \beta(0, \cdot) \|_{BMO}} \sup \{ \beta(0, z) \| E(z, n) \| - 2\lambda : n = 1, 2, \ldots \} \\
= & \frac{\beta(0, z) - 2\lambda}{2 \| \beta(0, \cdot) \|_{BMO}}.
\end{align*}

Since $\beta(0, z) \to +\infty (z \to \partial \Omega)$ and $\lambda$ is just a constant, we can find a compact set $K$ in $\Omega$ such that $\beta(0, z) - 2\lambda > \frac{1}{2} \beta(0, z)$ for all $z$ in $Q - K$. Hence

$$\sup \{ |\tilde{f}(z)| : \| f \|_{BMO} \leq 1, \tilde{f}(0) = 0, f \in VMO \} \geq \frac{\beta(0, z)}{4 \| \beta(0, \cdot) \|_{BMO}}$$

for all $z$ in $Q - K$. This completes the proof of Theorem 2.  

3. **Multipliers of BMO in the Bergman Metric**

A function $f$ on $Q$ is a multiplier of $BMO$ if $fg \in BMO$ for all $g$ in $BMO$. In this case, we write $f BMO \subseteq BMO$. In this section, we characterize all the bounded multipliers of $BMO$.

**Lemma 3.** If $f BMO \subseteq BMO$, then there exists a constant $C > 0$ such that $\| fg \|_{BMO} \leq C \| g \|_{BMO}$ for all $g$ in $BMO$ with $\tilde{g}(0) = 0$.

**Proof.** Let $\| f \|_* = \| f \|_{BMO} + |\tilde{f}(0)|$, then $BMO$ becomes a Banach space with the norm $\| \cdot \|_*$. Note that

$$\| f \|_{L^2(Q, dV)} \leq |\tilde{f}(0)| + \| f \|_{BMO}$$

Thus convergence in $(BMO, \| \cdot \|_*)$ implies $L^2(Q, dV)$-convergence which in turn implies pointwise (a.e.) convergence of a subsequence. In this case, the closed graph theorem applies and we can find a constant $C > 0$ such that

$$\| g \|_{BMO} \leq C \| g \|_*$$

for all $g$ in $BMO$. If $\tilde{g}(0) = 0$, then

$$\| fg \|_{BMO} \leq |\tilde{f}(0)| + \| f \|_{BMO} \leq C \| g \|_*$$

completing the proof of Lemma 3.

**Remark.** It is easy to see from the proof of Lemma 3 that multipliers of $BMO$ have to be bounded functions. In fact, $f BMO \subseteq BMO$ implies $\| fg \|_{BMO} \leq C \| g \|_*$ for some constant $C > 0$ and all $g$ in $BMO$. Since $\| 1 \|_* = 1$, it follows that $\| f \|_* \leq C$, but $\| f \|_{L^2(Q, dV)} \to \infty$ as $\| f \|_*$, so

$$\left[ \int_Q |f(z)|^2 dV(z) \right]^{1/2n} \leq C$$

for all $n = 1, 2, \ldots$. Letting $n \to +\infty$ leads to $\| f \|_{L^\infty(Q, dV)} \leq C$. (The author thanks the referee for this remark.)

**Theorem 4.** For any bounded symmetric domain $Q$ and $f \in L^\infty(Q, dV)$, the following conditions are equivalent:

1. $f BMO \subseteq BMO$;
2. $\beta(0, z)[f(z) \tilde{f} - |\tilde{f}(z)|^{1/2}]$ is bounded in $Q$;
3. $\beta(0, z)[f(z) \tilde{f} - |\tilde{f}(z)|^{1/2}]$ is bounded in $Q$ for all $r > 0$;
4. $\beta(0, z)[f(z) \tilde{f} - |\tilde{f}(z)|^{1/2}]$ is bounded in $Q$ for all $r > 0$.

**Proof.** By Lemma 8 of [1], there exists a constant $M, > 0$ (depending only on $r$) such that

$$|f(z)|^2 - |\tilde{f}(z)|^2 \leq M_r \left[ \int |f(z)|^2 - |\tilde{f}(z)|^2 \right]$$

for all $f \in BMO$ and $z$ in $Q$. Thus (2) immediately implies (3). That (3)
implies (4) is obvious. So it remains to show that (4) implies (1), and (1) implies (2).

We first prove the implication (1) \( \Rightarrow \) (2). Suppose \( f \in \text{BMO} \subset \text{BMO} \), then by Lemma 3, there is a constant \( C > 0 \) (depending on \( f \)) such that \( \|f\|_{\text{BMO}} \leq C \|g\|_{\text{BMO}} \) for all \( g \in \text{BMO} \) with \( \hat{g}(0) = 0 \). Write

\[
 f(z) g(z) - \hat{f}(a) g(a) - \hat{g}(a) g(z) - \hat{f}(a) g(a) + \hat{f}(a) g(a) - \hat{g}(a) \\
= f(z) g(z) - \hat{f}(a) g(a) + \hat{g}(a) f(z) - \hat{f}(a) g(a) + \hat{f}(a) g(a) - \hat{g}(a).
\]

It follows that

\[
\|\hat{g}(a)\| f(z) - \hat{f}(a) \leq f(z) g(z) - \hat{f}(a) g(a) + \|f\|_{\infty} \|g(z) - \hat{g}(a)\|
\]

\[
+ \|f\|_{\infty} \|g(z) - \hat{g}(a)\|.
\]

But

\[
|\hat{f}(a) g(a) - \hat{g}(a)| = \left| \int_{\Omega} f(z)(g(z) - \hat{g}(a))|k_{z}(z)|^{2} dV(z) \right|
\]

\[
\leq \|f\|_{\infty} \left[ \int_{\Omega} |g(z) - \hat{g}(a)|^{2} |k_{z}(z)|^{2} dV(z) \right]^{1/2}
\]

\[
= \|f\|_{\infty} \left[ \int_{\Omega} |g(z) - \hat{g}(a)|^{2} dV(z) \right]^{1/2} \leq \|f\|_{\infty} \|g\|_{\text{BMO}},
\]

so

\[
|\hat{g}(a)\| f(z) - \hat{f}(a) \leq \|f\|_{\infty} \|g\|_{\text{BMO}} + \|f\|_{\infty} \|g(z) - \hat{g}(a)\|
\]

\[
+ \|f\|_{\infty} \|g(z) - \hat{g}(a)\|.
\]

This implies that

\[
|\hat{g}(a)| \left[ \int_{\Omega} |f(z) - \hat{f}(a)|^{p} d\mu(z) \right]^{1/p}
\]

\[
\leq \|f\|_{\infty} \|g\|_{\text{BMO}} + \|f\|_{\infty} \left[ \int_{\Omega} |g(z) - \hat{g}(a)|^{p} d\mu(z) \right]^{1/p}
\]

\[
+ \left[ \int_{\Omega} |f(z) g(z) - \hat{f}(a) g(a)|^{p} d\mu(z) \right]^{1/p}
\]

for all \( 1 \leq p < +\infty \) and any probability measure \( \mu \) on \( \Omega \). Let \( p = 2 \) and \( d\mu(z) = |k_{z}(z)|^{2} dV(z) \), then we get

\[
|\hat{g}(a)| \left[ \int_{\Omega} |f(z) - \hat{f}(a)|^{2} dV(z) \right]^{1/2}
\]

\[
\leq \|f\|_{\infty} \|g\|_{\text{BMO}} + \|f\|_{\infty} \|g\|_{\text{BMO}} + \|f\|_{\text{BMO}}
\]

\[
\leq (2 \|f\|_{\infty} + C) \|g\|_{\text{BMO}}
\]

This means that

\[
\text{BMO in the Bergman Metric}
\]

for all \( g \in \text{BMO} \) with \( \hat{g}(0) = 0 \) and \( a \) in \( \Omega \). Taking the supremum of the above inequality over \( g \) with \( \|g\|_{\text{BMO}} \leq 1 \) and \( \hat{g}(0) = 0 \), and applying Theorem 1, we conclude that \( \beta(0, a)[|f(z) - \hat{f}(a)|^{2}]^{1/2} \) is bounded in \( \Omega \). This proves (1) \( \Rightarrow \) (2).

Next we prove that (4) \( \Rightarrow \) (1). Let

\[
M = \sup_{z \in \Omega} \beta(0, z)[|f(z) - \hat{f}(z)|^{2}]^{1/2} < +\infty.
\]

We wish to prove that \( f \in \text{BMO} \subset \text{BMO} \). Given \( g \) in \( \text{BMO} \), with \( \hat{g}(0) = 0 \), we can write

\[
f(z) g(z) - \hat{f}(a) g(a) = f(z)(g(z) - \hat{g}(a)) + \hat{g}(a)(f(z) - \hat{f}(a))
\]

\[
+ \hat{g}(a) \hat{f}(a) - \hat{f}(a).
\]

It follows that

\[
|f(z) g(z) - \hat{f}(a) g(a)| \leq \|f\|_{\infty} \|g(z) - \hat{g}(a)\| + \|g(a)\| |f(z) - \hat{f}(a)|
\]

\[
+ \|g(a)\| |f(z) - \hat{f}(a)|.
\]

Note that

\[
|\hat{g}(a)\| f(z) - \hat{f}(a) \leq \|f\|_{\infty} \|g\|_{\text{BMO}} + \|f\|_{\infty} \|g(z) - \hat{g}(a)\|
\]

\[
+ \|f\|_{\infty} \|g(z) - \hat{g}(a)\|.
\]

This implies that

\[
|\hat{g}(a)| \left[ \int_{\Omega} |f(z) - \hat{f}(a)|^{p} d\mu(z) \right]^{1/2}
\]

\[
\leq \|f\|_{\infty} \|g\|_{\text{BMO}} + \|f\|_{\infty} \|g\|_{\text{BMO}} + \|g\|_{\text{BMO}}
\]

\[
\leq (2 \|f\|_{\infty} + C) \|g\|_{\text{BMO}}
\]

for all \( 1 \leq p < +\infty \) and any probability measure \( \mu \) on \( \Omega \). Let \( p = 2 \) and \( d\mu(z) = |k_{z}(z)|^{2} dV(z) \), then we get

\[
|\hat{g}(a)| \left[ \int_{\Omega} |f(z) - \hat{f}(a)|^{2} dV(z) \right]^{1/2}
\]

\[
\leq \|f\|_{\infty} \|g\|_{\text{BMO}} + \|f\|_{\infty} \|g\|_{\text{BMO}} + \|f\|_{\text{BMO}}
\]

\[
\leq (2 \|f\|_{\infty} + C) \|g\|_{\text{BMO}}
\]

This implies
By Theorem 1, there exists a constant $C > 0$ and a compact set $K$ of $Q$ such that

$$|\hat{g}(a)| \leq C \| g \|_{\text{BMO}} \beta(0, a)$$

for all $a \in Q - K$. Therefore,

$$[ |f g|^2 (a) - |\hat{f} g|^2 (a)]^{1/2} \leq 2 \| f \|_{\infty} \| g \|_{\infty} + C \| g \|_{\text{BMO}} M$$

for all $a$ in $Q - K$. This implies that $\| f g \| < +\infty$. By the Corollary to Theorem 18 of [3], we have $f g \in \text{BMO}$. This completes the proof of Theorem 4.

**Corollary.** Suppose $X$ is a closed subspace of $\text{BMO}$ with the property that

$$\sup \{ |\hat{f}(z)| : \| f \|_{\text{BMO}} \leq 1, \hat{f}(0) = 0, f \in X \} \sim \beta(0, z) \quad (z \to \partial Q),$$

then we have

$$g \text{BMO} \subset \text{BMO} \iff g X \subset \text{BMO}$$

if $g$ is in $L^\infty(Q, d\nu)$.

**Proof.** $\Rightarrow$ if obvious. Suppose $g X \subset \text{BMO}$, then the closed graph theorem (similar to the proof of Lemma 3) shows that $M_g : X \to \text{BMO}$ is a bounded operator. Now the proof of $(1) \Rightarrow (2)$ in the theorem implies that $\beta(0, z)[ |g|^2 (z) - |\hat{g}(z)|^2]^{1/2}$ is bounded in $Q$, thus $g \text{BMO} \subset \text{BMO}$ by the theorem. □

## 4. Multipliers of $\text{VMO}$ in the Bergman Metric

We determine in this section the bounded multipliers of $\text{VMO}$ in bounded symmetric domains. The main result is

**Theorem 5.** For any bounded symmetric domain $Q$ and $f \in L^\infty(Q, d\nu)$, we have $f \text{VMO} \subset \text{VMO}$ if and only if $f \text{BMO} \subset \text{BMO}$.

**Proof.** Suppose $f \text{VMO} \subset \text{VMO}$, then in particular, $f \text{VMO} \subset \text{BMO}$. By Theorem 2 and Corollary 2 to Theorem 4, we have $f \text{BMO} \subset \text{BMO}$. To prove the other half of the theorem, we need the following

**Lemma 6.** If $f \in \text{VMO}$, then

$$\lim_{z \to a} \hat{f}(z) = 0.$$ 

**Proof.** By Theorem 14 and Corollary 2 to Theorem F in [3], $f \in \text{VMO}$ implies that

$$\lim_{a \to a} \sup \{ |\hat{f}(z)| : z \in E(a, r) \} = 0 \quad \text{for all } r > 0.$$ 

Given $r > 0$, choose $R > 0$ such that $\sup \{ |\hat{f}(z)| : z \in E(a, r) \} < r$. Note that

$$N = \beta(0, z) + 1,$$

for all $\beta(0, z) > R$ (note that $\beta(0, a) \to +\infty$). Fix $a$ with $\beta(0, a) > R$, let $a(t)$ be the geodesic (in the Bergman metric) from $0$ to $a$. Let $t_0$ be the unique number between $0$ and $1$ such that $\beta(0, a(t_0)) = R$. Let $M = \sup \{ |\hat{f}(z)| : \beta(0, z) = R \}$ and $N = \beta(0, a) + 1$ (here $\lfloor x \rfloor$ denotes the largest integer $\leq x$), then $M < +\infty$ because $\{z : \beta(0, z) = R\}$ is compact. Divide $[t_0, 1]$ into $N$ equal subintervals, $t_0 < t_1 < \cdots < t_N = 1$, then we have

$$|\hat{f}(a)| \leq |\hat{f}(a(t_0))| + \sum_{k=0}^{N-1} |\hat{f}(a(t_{k+1})) - \hat{f}(a(t_k))| \leq M + \epsilon N$$

for all $\beta(0, a) > R$. Note that

$$N \leq \beta(a(t_0), a) + 1 \leq \beta(0, a) + 1,$$

thus

$$\frac{|\hat{f}(a)|}{\beta(0, a)} \leq \frac{M}{\beta(0, a)} + \frac{\epsilon}{\beta(0, a)}$$

for all $\beta(0, a) > R$. Let $a \to \partial Q$, we get

$$\lim_{a \to \partial Q} |\hat{f}(a)| / \beta(0, a) \leq \epsilon.$$
Since $\varepsilon$ is arbitrary, we must have
\[ \lim_{a \to \partial B} \beta(0, a) = 0. \]
This finishes the proof of Lemma 6. \hfill \Box

We can now prove the remaining part of Theorem 5. Suppose $f \in BMO$ and $a \in \Omega$. Let $C = \sup \{ \beta(0, z) \mid \beta(0, z) \leq 1 \} \geq 0$. Then the equality
\[ f(z) g(z) - f(a) g(z) = f(z) (g(z) - g(a)) + g(a) (f(z) - f(a)) \]
and the proof of Theorem 3 imply that
\[ \left[ \| f \|_\infty \| g \|_\infty \right]^2 \leq 2 \| f \|_\infty \left[ \| g \|_\infty^2 - \| g(a) \|_\infty^2 \right]^{1/2} + C \| g(a) \|_\beta(0, a) \]
for all $g \in \text{MO}$ and $a \in \partial \Omega$. If $g$ is in $VMO$, then $\| g \|_\infty^2 - \| g(a) \|_\infty^2 \to 0$, and by Lemma 6, $\| g(a) \|_\beta(0, a) \to 0$ as $a \to \partial \Omega$. Thus $\| f \|_\infty^2 - \| f(a) \|_\infty^2 \to 0$ as $a \to \partial \Omega$. This implies that $fg \in VMO$ if $g \in VMO$. We have completed the proof of Theorem 5. \hfill \Box

5. The Rank 1 Case

In this section we specialize to the case where $\Omega = B_n$, the open unit ball in $C^n$. In this case,
\[ K(z, w) = \frac{1}{\left( 1 - \langle z, w \rangle \right)^n} + 1 \]
and
\[ \beta(0, z) = \left( \frac{n + 1}{8} \right)^{1/2} \log \frac{1 + |z|}{1 - |z|} \]
It follows that $\beta(0, z) \sim \log K(z, z) \sim \log(1/(1 - |z|^2))$ as $z \to \partial B_n$. By 1.4.10 of \cite{7}, we have
\[ \int_{B_n} |K(z, w)| \, d\mu(w) \sim \log \frac{1}{1 - |z|^2} \quad (z \to \partial B_n). \]
By Lemma 8 of \cite{1} with $a = z$, we have $|E(z, r)| \sim 1/K(z, z)$ for all $r > 0$. Thus we also have $\log(1/(1/E(z, r))) \sim \log(1/(1 - |z|^2))$. We state these results as

\[ \begin{array}{l}
\text{LEMMA 7.} \\
\text{For $Q = B_n$ and $r > 0$, the following are equivalent as $z \to \partial B_n$:}
\end{array} \]
\[ \begin{array}{l}
(1) \int_{B_n} |K(z, w)| \, d\mu(w);
(2) \beta(0, z);
(3) \log K(z, z);
(4) \log(1/E(z, r));
(5) \log(1/(1 - |z|^2)).
\end{array} \]
Recall that the Bloch space $\mathcal{B}(B_n)$ consists of holomorphic functions $f$ on $B_n$ with
\[ \| f \|_{\mathcal{B}} = \sup \{ \frac{1}{\| f \|_\infty} \mid \| f \|_\infty < \infty \}. \]
The little Bloch space $\mathcal{B}_0(B_n)$ is the subspace of $\mathcal{B}(B_n)$ generated by polynomials. $f \in \mathcal{B}_0(B_n)$ if and only if
\[ \lim_{z \to \partial B_n} (1 - |z|^2) \| f \|_\infty = 0. \]
Let $H(B_n)$ be the space of all holomorphic functions in $B_n$ and $H^\infty(B_n)$ be the space of bounded holomorphic functions. We have $\mathcal{B}(B_n) = BMO(B_n) \cap H(B_n)$ and $\mathcal{B}_0(B_n) = VMO(B_n) \cap H(B_n)$, see \cite{3}. The Bloch norm $\| f \|_{\mathcal{B}}$ is equivalent to the $BMO$-norm $\| f \|_{BMO}$ for holomorphic functions \cite{2, 3}. Since $\mathcal{B}$ and $\mathcal{B}_0$ are complete in the Bloch norm, they are closed subspaces of $BMO$.

\[ \begin{array}{l}
\text{LEMMA 8.} \quad \text{If $\Omega = B_n$, then we have}
\end{array} \]
\[ \begin{array}{l}
(1) \sup \{ |f(z)| : \| f \|_{\mathcal{B}} \leq 1, f(0) = 0 \} \sim \beta(0, z) (z \to \partial B_n);
(2) \sup \{ |f(z)| : \| f \|_{\mathcal{B}} \leq 1, f(0) = 0, f \in \mathcal{B}_0 \} \sim \beta(0, z) (z \to \partial B_n).
\end{array} \]
\[ \begin{array}{l}
\text{Proof.} \quad \text{Note that $\hat{f}(z) \equiv f(z)$ for $f$ holomorphic and in $L^1(\mu, d\mu)$, and that $\mathcal{B} \subset \mathcal{B} \subset BMO$. By Theorem 1, it suffices to show that there exists a constant $\sigma > 0$ and a compact set $K$ in $B_n$ such that}
\end{array} \]
\[ \sup \{ |f(z)| : \| f \|_{\mathcal{B}} \leq 1, f(0) = 0, f \in \mathcal{B}_0 \} \geq \sigma \beta(0, z) \]
for all $z \in B_n - K$.
Fix $z$ in $B_n$, let $g_z(w) = K(w, z)$ (the principal branch with $\log 1 = 0$), then $g_z \in \mathcal{B}_0$ since $g_z$ is continuous on $\partial B_n$. We also have $g_z(0) = 0$ since
$K(0, z) = 1$ by the normalization of $dV$. It is easy to compute that
\[ \| g \|_\infty \leq 2(n+1) \text{ for all in } B_n. \]
It follows that
\[ \sup \{ |f(z)| : \| f \|_\infty \leq 1, f(0) = 0, f \in \mathcal{B}_0 \} \leq \frac{1}{2(n+1)} \log K(z, z). \]

Now the desired result follows from Lemma 7.

**Theorem 9.** Suppose $f \in L^\infty(B_n, dV)$, then the following conditions are all equivalent:

1. $f \mathcal{BMO} \subset \mathcal{BMO}$;
2. $f \mathcal{B} \subset \mathcal{B}$;
3. $f \mathcal{B}_0 \subset \mathcal{B}_0$;
4. $f \mathcal{B}_0 \subset \mathcal{VMO}$.

**Proof.** The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious, (3) $\Rightarrow$ (1) follows from Lemma 8 and the corollary to Theorem 4. Thus (1), (2), and (3) are equivalent. It is clear that (4) implies (3). Since $\mathcal{B}_0 \subset \mathcal{VMO}$, thus (1) implies (4) by Theorem 5. This completes the proof of Theorem 9.

Next we further specialize to the case where the function $f$ is holomorphic in $B_n$. In this case, it is not necessary to assume that $f$ is bounded as shown by the following lemma.

**Lemma 10.** Suppose $f$ is holomorphic in $B_n$ and $f \mathcal{B} \subset \mathcal{B}$ or $f \mathcal{B}_0 \subset \mathcal{B}_0$, then $f \in H^\infty(B_n)$.

**Proof.** By the closed graph theorem, there exists a constant $C > 0$ such that
\[ \| fg \|_\infty \leq C \| g \|_\infty \]
for all $g \in \mathcal{B}$ (or $\mathcal{B}_0$) with $g(0) = 0$. This implies that
\[ |f(z)g(z)| = |\langle fg, K(\cdot, z) \rangle| \leq M \| fg \|_\infty \| K(\cdot, z) \|_1 \leq MC \| g \|_\infty \| K(\cdot, z) \|_1, \]
where the first inequality and the constant $M > 0$ follow from the $L^1$-$\mathcal{B}$ duality (this is only true for rank 1 domains, see [10]). Taking the supremum of the above inequality we get
\[ |f(z)| \sup \{ |g(z)| : \| g \|_\infty \leq 1, g(0) = 0 \} \leq MC \| K(\cdot, z) \|_1. \]

If $f \mathcal{B}_0 \subset \mathcal{B}_0$, we get
\[ |f(z)| \leq MC \| K(\cdot, z) \|_1. \]

Now the desired result follows from Lemmas 7 and 8.

**Theorem 11.** Suppose $f \in H(B_n)$, then the following conditions are equivalent:

1. $f \mathcal{BMO} \subset \mathcal{BMO}$;
2. $f \mathcal{B} \subset \mathcal{B}$;
3. $f \mathcal{B}_0 \subset \mathcal{B}_0$;
4. $f \in H^\infty$ and $(1 - |z|^2) \nabla f(z) (1/((1 - |z|^2)))$ is bounded in $B_n$.

**Proof.** The implication (1) $\Rightarrow$ (2) follows from the fact that $\mathcal{BMO}(B_n) \cap H(B_n) = \mathcal{B}(B_n)$ (see [2, 3]). If $f \mathcal{B} \subset \mathcal{B}$ (this implies $f \mathcal{BMO}$), then by Lemma 10, $f \in L^\infty(B_n)$, thus by Theorem 9 $(2) \Rightarrow (4)$, $f \mathcal{B}_0 \subset \mathcal{VMO}$. Since $f$ is holomorphic and $\mathcal{VMO}(B_n) \cap H(B_n) = \mathcal{B}_0(B_n)$, we must have $f \mathcal{B}_0 \subset \mathcal{B}_0$. This proves (2) $\Rightarrow$ (3). If $f \mathcal{B}_0 \subset \mathcal{B}_0$, then $f \in L^\infty(B_n)$ and $f \mathcal{B}_0 \subset \mathcal{VMO}$, thus by Theorem 9, we have $f \mathcal{BMO} \subset \mathcal{BMO}$, therefore, (1), (2), and (3) are all equivalent. Next we prove that (2) and (4) are equivalent.

Suppose $f \mathcal{B} \subset \mathcal{B}$, then $f \in H^\infty(B_n)$ by Lemma 10 and there exists a constant $C > 0$ (by the closed graph theorem) such that $\| fg \|_\infty \leq C \| g \|_\infty$ for all $g \in \mathcal{B}$ with $g(0) = 0$. Observe that
\[ \nabla(fg)(z) = f(z) \nabla g(z) + g(z) \nabla f(z), \]
It follows that
\[ |g(z)| \| \nabla f(z) (1 - |z|^2) \|_\infty \leq \| f \|_\infty \| g \|_\infty + C \| g \|_\infty \]
for all $g \in \mathcal{B}$ with $g(0) = 0$ and $z \in B_n$. Taking the supremum of the above inequality over $\| g \|_\infty \leq 1$ and $g(0) = 0$ and using Lemmas 7 and 8, we conclude that $\| \nabla f(z) (1 - |z|^2) \| (1/(1 - |z|^2))$ is bounded in $B_n$.

On the other hand, if (4) holds, then for all $g \in \mathcal{B}$ with $g(0) = 0$, we have
\[ \| \nabla(fg)(z) \| = \| f(z) \nabla g(z) + g(z) \nabla f(z) \| \leq \| f \|_\infty \| \nabla g(z) \| + \| g(z) \| \| \nabla f(z) \| \leq \| f \|_\infty \| \nabla g(z) \| + \| g \|_\infty \| \beta(0, z) \| \| \nabla f(z) \|, \]
See (3.13) of [9] for the last inequality. This implies that $(1 - |z|^2) \| \nabla(fg)(z) \| \leq \| f \|_\infty (1 - |z|^2) \| \nabla g(z) \| + \| g \|_\infty (1 - |z|^2) \| \nabla f(z) \| \beta(0, z)$ is bounded in $B_n$ since $\beta(0, z) \sim \log(1/(1 - |z|^2))(z \to \partial B_n)$. We have completed the proof of Theorem 11.
6. Toeplitz Operators on the Bergman Space $L_a^1(B_n)$

We investigate the boundedness of Toeplitz operators on the Bergman space $L_a^1(B_n)$. Recall that the Toeplitz operator $T_f$ with symbol $f$ is defined by $T_f g = P(fg)$, where $P$ is the Bergman projection. It is interesting to know when $T_f$ is bounded for $f \in L^\infty(\Omega)$. We give some sufficient conditions for the boundedness of $T_f$ on $L_a^1(B_n)$. If $f$ is anti-holomorphic, then we also obtain necessary and sufficient conditions.

Let $\mu$ be a positive Borel measure on $\Omega$. We say that $\mu$ is a Carleson measure on Bergman spaces if

$$\sup \left\{ \frac{\int_\Omega |f(z)|^p \, dm(z)}{\int_\Omega |f(z)|^p \, dV(z)} : f \in L^p_a \right\} < +\infty.$$

It turns out that Carleson measures on Bergman spaces are independent of $\mu$. In fact, $\mu$ is a Carleson measure on Bergman spaces if and only if

$$\sup \left\{ \frac{\mu(E(a, r))}{|E(a, r)|} : a \in \Omega \right\} < +\infty$$

for all (or some) $r > 0$. See [3].

If $f$ is a positive function in $L^2(\Omega, dV)$, then $T_f$ is a bounded operator on $L^2_a(\Omega)$ if and only if $f(z) \, dV(z)$ is a Carleson measure on Bergman spaces [3]. Our first result is a sufficient condition for the boundedness of Toeplitz operators on $L_a^1(B_n)$ in terms of Carleson measures.

**Proposition 12.** Suppose a function $g$ on $B_n$ satisfies the condition

$$\sup_{z \in B_n} \frac{\log(1/(1-|z|^2))}{|E(z, r)|} \int_{E(z, r)} |g(w)| \, dV(w) < +\infty$$

for some $r > 0$, then $T_g$ and $T_{\bar{g}}$ are bounded on $L_a^1$.

**Proof.** Suppose $g$ satisfies the above conditions, then by Lemmas 6 and 8 of [1], we have

$$\sup_{z \in B_n} \frac{1}{|E(z, r)|} \int_{E(z, r)} |g(w)| \log(1/(1-|w|^2)) \, dV(w) < +\infty.$$

So $|g(w)| \log(1/(1-|w|^2)) \, dV(w)$ is a Carleson measure on Bergman spaces. This implies that there exists a constant $C > 0$ such that

$$\int_{B_n} |f(z)| \, dV(z) < C \int_{B_n} |f(z)| \, dV(z)$$

for all $f$ in $L_a^1(B_n)$. Applying Lemma 7, we get another constant $C > 0$ such that

$$\int_{B_n} |f(z)| \, dV(z) \leq C \int_{B_n} |f(z)| \, dV(z)$$

for all $f$ in $L_a^1(B_n)$. Fubini's theorem gives

$$\int_{B_n} dV(w) \int_{B_n} f(z) \, dV(z) = C \int_{B_n} f(z) \, dV(z).$$

This implies that

$$\|T_g f\|_1 = \int_{B_n} \left| \int_{B_n} f(z) \, dV(z) \right| g(z) \, dV(z) \leq C \int_{B_n} f(z) \, dV(z).$$

for all $f$ in $L_a^1(B_n)$. Hence $T_g$ is bounded on $L_a^1(B_n)$. Similarly, $T_{\bar{g}}$ is bounded on $L_a^1(B_n)$.

**Remark.** The condition in Proposition 12 is not necessary even for positive functions $g$ on $B_n$. For example, if $g \equiv 1$, then $T_g$ is bounded on $L_a^1$, but $g$ doesn't satisfy the condition in the proposition.

**Lemma 13.** The Bergman operator $P$ is a bounded projection from $BMO(B_n)$ onto $B(B_n)$.

**Proof.** See Theorems 8 and 13 of [10].

**Remark.** Lemma 13 only holds for rank 1 domains. It even fails for the polydiscs. See [10].

**Theorem 14.** Suppose $g \in BMO(B_n)$, then $T_g$ and $T_{\bar{g}}$ are bounded on $L_a^1(B_n)$.

**Proof.** If $g$ multiplies $BMO(B_n)$, then the closed graph theorem implies that the multiplication operator $M_g : BMO \rightarrow BMO$ is bounded. In particular, $\|M_g \|_{BMO} \leq C \|g\|_{BMO}$ for all $f \in BMO$ with $f(0) = 0$. It follows that $\|P(M_g) f\|_{BMO} \leq C \|g\|_{BMO} \leq C \|f\|_{BMO}$ by Lemma 13. Thus $T_g$ is bounded on the Bloch space. Similarly, $T_{\bar{g}}$ is bounded on the Bloch space. Since $L^1_a(B_n)^* \cong B(B_n)$ [10] and $T_g = T_{\bar{g}}$ under this duality, $T_g$ and $T_{\bar{g}}$ must be bounded on $L_a^1(B_n)$. This completes the proof of Theorem 14.
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