Pseudospectral Fourier reconstruction with IPRM

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EuCeTiFA

Outline

1. What is IPRM?
2. IPRM-Algorithm, Condition Number, Optimality
3. Numerical Simulations
4. Variations

IPRM I

- Gibbs phenomenon
- IPRM = Inverse polynomial reconstruction method (Jung and Shizgal, 2003–07)

- Goal: Construct an algebraic polynomial from Fourier coefficients
- Find an approximation of a piecewise smooth function from given Fourier coefficients
- How many Fourier coefficients are required for accurate construction of algebraic polynomial?

- Compression
- Relation between Fourier basis and other bases

- Gottlieb, Shu; Gelb, Tanner; Tadmore; etc.

IPRM II

Given function f on $[-1,1]$ and m consecutive Fourier coefficients

$$\hat{f}(k) = \frac{1}{\sqrt{2}} \int_{-1}^{1} f(x) e^{-i\pi k x}\, dx, \qquad -\lfloor \frac{m-1}{2} \rfloor \leq k \leq \lfloor \frac{m}{2} \rfloor .$$

Find a polynomial p of degree $n-1$ with these Fourier coefficients.
Expand p into normalized Legendre polynomials $\widetilde{P_k}$

$$p = \sum_{l=0}^{n-1} a_l \widetilde{P_l}$$

IPRM: solve the system

$$\sum_{l=0}^{n-1} a_l \widehat{\widetilde{P_l}}(k) = \widehat{f}(k) \qquad k = -\lfloor \frac{m-1}{2} \rfloor, \dots, \lfloor \frac{m}{2} \rfloor .$$

IPRM-Algorithm

Input: m Fourier coefficients $\widehat{f}(k)$,
Let $A_{m,n}$ be $m \times n$ matrix $A_{m,n}$ with entries

$$a_{kl} = \widehat{\widetilde{P}_l}(k) = \sqrt{2}\,(-\imath)^l \sqrt{l+\tfrac{1}{2}}\, j_l(k\pi), \tag{1}$$

$k = -\lfloor \frac{m-1}{2} \rfloor, \dots, \lfloor \frac{m}{2} \rfloor, l = 0, \dots, n-1.$

1. Solve overdetermined least squares problem for approximate Legendre coefficients $\mathbf{c} = [c_0, \dots, c_{n-1}]^t$

$$\min_{\mathbf{c} \in \mathbb{C}^n} \|A_{m,n}\mathbf{c} - [\widehat{f}(d), \dots, \widehat{f}(D)]^t\|_2, \tag{2}$$

   where $d = -\lfloor \frac{m-1}{2} \rfloor$, $D = \lfloor \frac{m}{2} \rfloor$.
2. Approximate f by truncated Legendre series

$$f_n = \sum_{l=0}^{n-1} c_l \widetilde{P}_l. \tag{3}$$

Existence of a Reconstruction

Theorem

Let d and D be integers such that $d \leq 0 \leq D$, and let $p \in \mathcal{P}_M$ have vanishing $D - d + 1$ consecutive Fourier coefficients

$$\widehat{p}(d) = \widehat{p}(d+1) = \ldots = \widehat{p}(D-1) = \widehat{p}(D) = 0. \tag{4}$$

If $D - d + 1 \geq M + 1$, then $p = 0$ identically.

REMARK: $A_{n,n}$ is invertible, and for $m > n$ $A_{m,n}$ has full rank. P_M is the space of algebraic polynomials of degree at most M.

Stability of the Reconstruction

Theorem

For every $\alpha > 1$*, every* $n = 1, 2, \ldots$*, and every integer* $m > \alpha n^2$*, the condition number of the matrix* $A_{m,n}$ *does not exceed* $\sqrt{\frac{\alpha}{\alpha - 1}}$*.*

REMARK: $\alpha > 1$ can be pushed to $\alpha > c$ for some $c \approx 1/2$.

Convergence Rates

Theorem

Let $f = \sum_{l=0}^{\infty} a_l \widetilde{P}_l$ *with Legendre coefficients*

$$|a_l| \leq c e^{-\beta l}, \tag{5}$$

where $c > 0$ *and* $\beta > 0$*, and let* f_n *be the reconstruction by IPRM* (3)*. If* $m > n^2$*, then*

$$\|f - f_n\|_\infty \leq c' n e^{-\beta n}, \tag{6}$$

for another constant $c' > 0$*.*

REMARK: Measured by number of Fourier coefficients $m = \alpha n^2$, the convergence is root-exponential: $\|f - f_n\|_\infty \leq c' \sqrt{m} e^{-\beta \sqrt{m}}$.

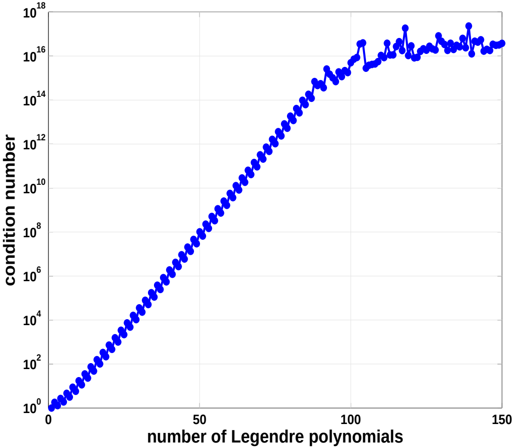


Figure: Condition numbers of the square matrix $A_{n,n}$ for $n = 1, \ldots, 150$.

Computation versus Proof

Experimentally: Smallest singular value $\lambda_{\min}(n)$ of $A_{n,n}$ decays exponentially (equivalently: condition number of the square matrix grows exponentially)

Current estimate: $\lambda_{\min} \leq 0.65$

Needed: Behavior of Bessel functions J_ν in the non-asymptotic region $\nu \leq x \leq \nu^2$.

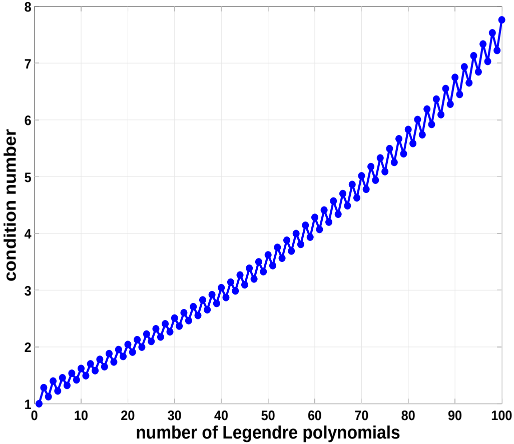


Figure: Condition numbers of the matrix $A_{\lceil n^{\frac{3}{2}} \rceil, n}$ for $n = 1, \dots, 100$.

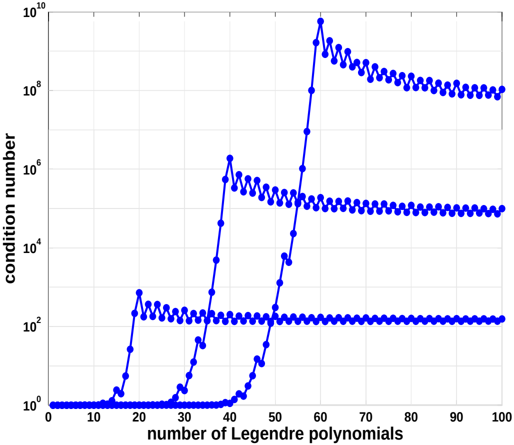


Figure: Condition numbers of the matrix $A_{\lceil \alpha n^2 \rceil, n}$ for $\alpha = \frac{1}{20}, \frac{1}{40}, \frac{1}{60}$.

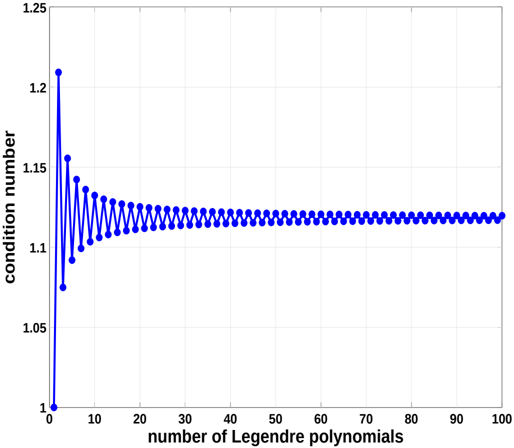


Figure: Condition numbers of the matrix $A_{n^2,n}$ for $n = 1, \dots, 100$.

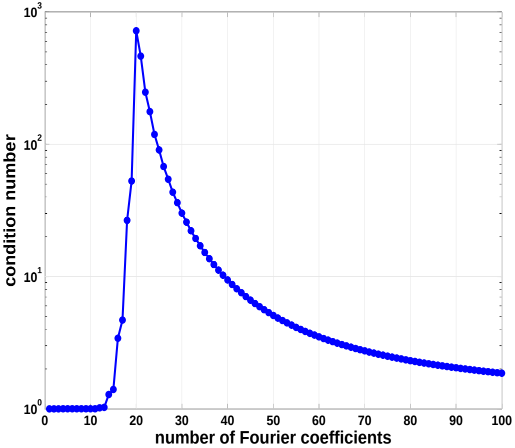


Figure: Condition numbers of the matrix $A_{m,20}$ with m Fourier coefficients for $m = 1, \ldots, 100$.

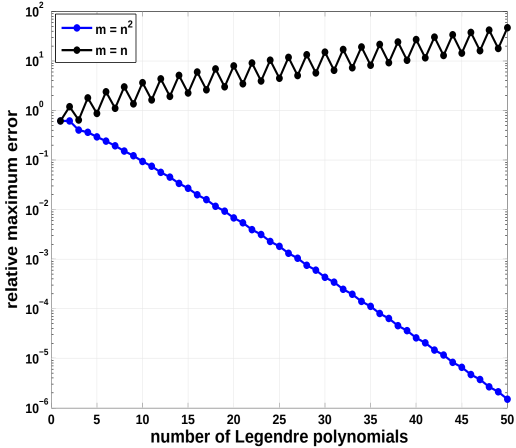
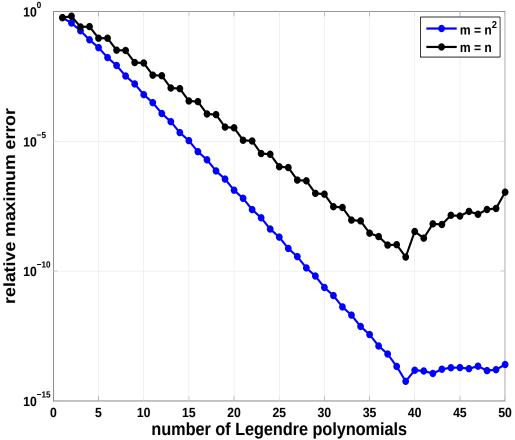


Figure: Relative maximum reconstruction errors for the function $\frac{1}{x-0.3\imath}$ on the interval $[-1, 1]$ with the two versions of IPRM.



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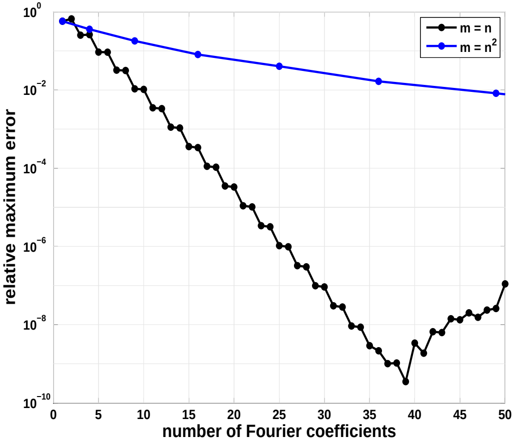

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Piecewise Polynomials from Fourier Coefficients

Fix nodes

$$-1 = a_0 < a_1 < \ldots < a_{L-1} < a_L = 1, \tag{7}$$

and consider

$$\mathcal{P}_{M,\mathbf{a}} = \{f : f|_{(a_{j-1},a_j)} \text{ is polynomial of degree } M\}$$

$$\dim \mathcal{P}_{\mathbf{a},M} = L(M+1)$$

Theorem

Let d and D be integers such that $d \leq 0 \leq D$, and let $p \in \mathcal{P}_{M,\mathbf{a}}$ have vanishing $D-d+1$ consecutive Fourier coefficients

$$\widehat{p}(d) = \widehat{p}(d+1) = \ldots = \widehat{p}(D-1) = \widehat{p}(D) = 0. \tag{8}$$

If $D-d+1 \geq L(M+1)$, then $p = 0$ identically.

Piecewise Constant Functions with free nodes

$$p = \sum_{j=1}^{L} p_j \, \chi_{(t_{j-1}, t_j)} . \tag{9}$$

Theorem

Let p a step function on $[-1, 1]$ with at most $L - 1$ points of discontinuity, and let d, and $D \in \mathbb{Z}$ be such that $d \leq 0 \leq D$. If $D - d + 1 \geq 2L - 1$, then p is uniquely determined by its $D - d + 1$ consecutive Fourier coefficients $\widehat{p}(d), \widehat{p}(d + 1), \ldots, \widehat{p}(D - 1), \widehat{p}(D)$.

Reconstruction by Prony's spectral estimator,
Used in compressed sensing by M. Vetterli as "Occam's razor"

To Do List

- Optimality of order of condition number
- Open question: Is

$$\lim_{n\to\infty} \kappa(A_{\alpha n^2,n}) = e^{\beta/\alpha}$$

- Condition numbers for piecewise polynomials with fixed nodes
- Variable degrees for piecewise polynomials with fixed nodes
- Method for piecewise polynomials with free nodes
- Reconstruction from arbitrary frequencies, from random frequencies

Summary

- Rigorous convergence analysis of IPRM
- First proof of existence of the square IPRM (invertibility of $A_{n,n}$)
- $n \times n$ IPRM is acceptable for entire functions
- $n^2 \times n$ IPRM is reliable for meromorphic functions
- $n^2 \times n$ IPRM useful in applications because it handles noisy signals and uses all available Fourier coefficients

Thank you!

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