Connection between Zernike functions, corneal topography and the voice transform

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Table of Contents

- Motivation – CORNEA Project
- Zernike functions
- The Zernike representation used in ophtamology
- Discrete orthogonality
- Reconstruction of the corneal surface
- Connection to the voice transform
The corneal surface is frequently represented in terms of the Zernike functions.

The optical aberrations of human eyes (for ex. astigma, tilt) and optical systems are characterized with Zernike coefficients.

Abberations are examined with Corneal topographer.

Measurements made by Shack – Hartmann wavefront - sensor.

**Problem:** Approximation of the Zernike coefficients and reconstruction of the corneal surface with minimal error.
Fritz Zernike

- Dutch physicist.
- In 1934 he introduced the two variable orthogonal system – named later Zernike functions.
- They are distinguished from the other orthogonal systems by certain simple invariance properties which can be explained from group theoretical considerations: for ex. they are invariant with respect to rotations of axes about origin.
- In 1953 winner of the Nobel prize for Physics.
Zernike functions

Definition of Zernike functions

\[ Z_n^\ell(\rho, \theta) := \sqrt{\frac{2n + |\ell| + 1}{|\ell| + 2n}} R_{|\ell|+2n}(\rho) e^{i\ell\theta}, \quad \ell \in \mathbb{Z}, \quad n \in \mathbb{N}, \]

The radial terms \( R_{|\ell|+2n}(\rho) \) are related to the Jacobi polynomials in the following way:

\[ R_{|\ell|+2n}(\rho) = \rho^{\ell} P_n^{(0,|\ell|)}(2\rho^2 - 1). \]

Pictures
Orthogonality of Zernike functions

\[ \frac{1}{\pi} \int_{0}^{2\pi} \int_{0}^{1} Z_n(\rho, \phi) \overline{Z_{n'}(\rho, \phi)} \rho d\rho d\phi = \delta_{nn'} \delta_{\ell\ell'}. \]
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Representation of the cornea surface

The corneal surface is described by a two variable function over the unit disc.

\[ g(x, y) \quad \text{or} \quad G(\rho, \phi) = g(\rho \cos \phi, \rho \sin \phi) \]
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Problems


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2. Addition formula for Zernike functions.
Denote by $\lambda_j^N \in (-1, 1), j \in \{1, ..., N\}$ the roots of Legendre polynomials $P_N$ of order $N$,

and for $j = 1, ..., N$, let

$$\ell_j^N(x) := \frac{(x - \lambda_1^N)...(x - \lambda_{j-1}^N)(x - \lambda_{j+1}^N)...(x - \lambda_N^N)}{(\lambda_j^N - \lambda_1^N)...(\lambda_j^N - \lambda_{j-1}^N)(\lambda_j^N - \lambda_{j+1}^N)...(\lambda_j^N - \lambda_N^N)},$$

be the corresponding fundamental polynomials of Lagrange interpolation.
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$$A_j^N := \int_{-1}^{1} \ell_j^N(x) dx,$$

the corresponding Cristoffel-numbers.
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the corresponding Cristoffel-numbers.
Let define the following numbers with the help of the roots of Legendre polynomials of order $N$

$$\rho_k^N := \sqrt{\frac{1 + \lambda_k^N}{2}}, \quad k = 1, \ldots, N,$$

and the set of nodal points:

$$\mathcal{X} := \{z_{jk} := \left(\rho_k^N, \frac{2\pi j}{4N + 1}\right), \quad k = 1, \ldots, N, \quad j = 0, \ldots, 4N\}$$
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$$\nu(z_{jk}) := \frac{A_k^N}{2(4N + 1)}.$$
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Discrete orthogonality

Let introduce the following discrete integral

\[
\int_X f(\rho, \phi) d\nu_N := \sum_{k=1}^{N} \sum_{j=0}^{4N} f(\rho^N_k, \frac{2\pi j}{4N + 1}) \frac{A^N_k}{2(4N + 1)}.
\]
Discrete orthogonality

Theorem Pap-Schipp 2005

If \( n + n' + |m| \leq 2N - 1, n + n' + |m'| \leq 2N - 1, n, n' \in \mathbb{N}, m, m' \in \mathbb{Z}, \) then

\[
\int_X Z_n^m(\rho, \phi)Z_{n'}^{m'}(\rho, \phi) d\nu_N = \delta_{nn'}\delta_{mm'}.
\]

For all \( f \in C(\overline{D}) \)

\[
\lim_{N \to \infty} \int_X fd\nu_N = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 f(\rho, \phi) \rho d\rho d\phi.
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The discrete Zernike coefficients

\[ A'_{\ell n} = \int_{\mathcal{X}} G(\rho, \phi) \overline{Z_{n}^{\ell}(\rho, \phi)} d\nu_{N}(\rho, \phi) = \]

\[ \sum_{k=1}^{N} \sum_{j=0}^{4N} G(\rho_{k}^{N}, \frac{2\pi j}{4N + 1}) Z_{n}^{\ell}(\rho_{k}^{N}, \frac{2\pi j}{4N + 1}) \frac{A_{k}^{N}}{2(4N + 1)} \]

The discrete Zernike coefficients of the function \( G \) from \( C(\overline{D}) \) tend to the corresponding continuous Zernike coefficients if \( N \to +\infty \).
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\[ A_{\ell n}' = \int_{\mathcal{X}} G(\rho, \phi) \overline{Z_{n}^{\ell}(\rho, \phi)} d\nu_{N}(\rho, \phi) = \]

\[ \sum_{k=1}^{N} \sum_{j=0}^{4N} G(\rho^{N}_{k}, \frac{2\pi j}{4N+1}) Z_{n}^{\ell}(\rho^{N}_{k}, \frac{2\pi j}{4N+1}) \frac{A_{k}^{N}}{2(4N+1)} \]

The discrete Zernike coefficients of the function \( G \) from \( C(D) \) tend to the corresponding continuous Zernike coefficients if \( N \to +\infty \).
The discrete Zernike coefficients

Let

\[ G_N(\rho, \phi) = \sum_{2n+|m| \leq 2N-1} A_{mn} Z_n^m(\rho, \phi) \]

be an arbitrary linear combination of Zernike polynomials of degree less than \(2N\).

The coefficients \(A_{mn}\) can be expressed in the following two ways:

\[ A_{mn} = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 G_N(\rho', \phi') Z_n^m(\rho', \phi') \rho' d\rho' d\phi', \]
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Zernike representation of some test surfaces


- Computer implementations, experimental results on artificial corneal-like surfaces.
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- Comparison of precision with the measurement methods used by conventional topographers proved that the approximations made using the discrete orthogonality are the best.
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Comparison of precision with the measurement methods used by conventional topographers proved that the approximations made using the discrete orthogonality are the best.
Classical methods

- Approximation of Zernike coefficients is made on a set of points which corresponds to the equidistant divisions along the $Ox$ and $Oy$ of the $[-1, 1] \times [-1, 1]$.

- Equidistant division along the radial line $[0, 1]$ and the angular part $[0, 2\pi]$.
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- The computation of discrete Zernike coefficients can be speeded via FFT.

Future: measurements and reconstructions on real human corneal surface and construction of new topographers based on the measurements on $X$, applications in sight-correcting operations.
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H. G. Feichtinger and K. H. Gröchenig unified the theory of Gábor and wavelet transforms into a single theory. The common generalization of these transforms is the so-called voice transform.

In the construction of the voice-transform the starting point will be a locally compact topological group \((G, \cdot)\).
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Let \(m\) be a left-invariant Haar measure of \(G\):

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\int_G f(x) \, dm(x) = \int_G f(a^{-1} \cdot x) \, dm(x), \quad (a \in G).
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Unitary representation

- **Unitary representation of the group** \((G, \cdot)\): Let us consider a Hilbert-space \((H, \langle \cdot, \cdot \rangle)\).

- \(U\) denote the set of unitary bijections \(U : H \rightarrow H\). Namely, the elements of \(U\) are bounded linear operators which satisfy \(\langle Uf, Ug \rangle = \langle f, g \rangle \) \((f, g \in H)\).
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- The homomorphism of the group $(G, \cdot)$ on the group $(\mathcal{U}, \circ)$ satisfying
  
  \begin{align*}
  i) \quad & U_{x \cdot y} = U_x \circ U_y \quad (x, y \in G), \\
  ii) \quad & G \ni x \mapsto U_x f \in H \text{ is continuous for all } f \in H
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Definition of the voice transform

Definition

The **voice transform** of $f \in H$ generated by the representation $U$ and by the parameter $\rho \in H$ is the (complex-valued) function on $G$ defined by

$$(V_\rho f)(x) := \langle f, U_x \rho \rangle \quad (x \in G, f, \rho \in H).$$

- Taking as starting point (not necessarily commutative) locally compact groups we can construct in this way important transformations.
- The affine wavelet transform is a voice transform of the affine group.
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The Gábor transform is a voice transform generated by the representation of the Weyl-Heisenberg group:

$$\mathbb{H} := \mathbb{R} \times \mathbb{R} \times T$$

$$(a_1, \omega_1, t_1).(a_2, \omega_2, t_2) := (a_1 + a_2, \omega_1 + \omega_2, t_1 t_2 e^{2\pi i \omega_1 a_2}).$$

The representation of $\mathbb{H}$ on $L^2(\mathbb{R})$:

$$U_{(a,\omega,t)}f(x) := te^{2\pi i \omega x}f(x - a) = tM_\omega T_a f(x)$$

the STFT- Gábor transform

$$V_\varphi f(a, \omega) = \int_{\mathbb{R}} f(t)\varphi(t - a)e^{-2\pi i \omega t}dt = \langle f, M_\omega T_a \varphi \rangle$$
Affine wavelet transform

**The affine group**

\[ G = \{ \ell_{(a,b)}(x) = ax + b : \mathbb{R} \to \mathbb{R} : (a, b) \in \mathbb{R}^* \times \mathbb{R} \} \]

\[ \ell_1 \circ \ell_2(x) = a_1 a_2 x + a_1 b_2 + b_1, \quad (a_1, b_1) \circ (a_2, b_2) = (a_1 a_2, a_1 b_2 + b_1) \]

**The representation of \( G \) on \( L^2(\mathbb{R}) \)**

\[ U_{(a,b)} f(x) = |a|^{-1/2} f(a^{-1}x - b) \]

The affine wavelet transform is a voice transform generated by this representation of affin group:

\[ W_{\psi} f(a, b) = |a|^{-1/2} \int_{\mathbb{R}} f(t) \overline{\psi(a^{-1}t - b)} dt = \langle f, T_b D_a \psi \rangle \]
The voice transform of the Blaschke group

The Blaschke group Let us denote by

\[ B_a(z) := \epsilon \frac{z - b}{1 - \overline{b}z} \quad (z \in \mathbb{C}, a = (b, \epsilon) \in \mathbb{B} := \mathbb{D} \times \mathbb{T}, \overline{b}z \neq 1) \]

the so called Blaschke functions,

\[ \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \}, \quad \mathbb{T} := \{ z \in \mathbb{C} : |z| = 1 \}. \]

If \( a \in \mathbb{B} \), then \( B_a \) is an 1-1 map on \( \mathbb{T}, \mathbb{D} \) respectively.
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The restrictions of the Blaschke functions on the set \( \mathbb{D} \) or on \( \mathbb{T} \) with the operation \((B_{a_1} \circ B_{a_2})(z) := B_{a_1}(B_{a_2}(z))\) form a group.
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\[ B_a(z) := \epsilon \frac{z - b}{1 - \overline{b}z} \quad (z \in \mathbb{C}, a = (b, \epsilon) \in \mathbb{B} := \mathbb{D} \times \mathbb{T}, \overline{b}z \neq 1) \]

the so called **Blaschke functions**,

- \( \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \), \( \mathbb{T} := \{ z \in \mathbb{C} : |z| = 1 \} \).

- If \( a \in \mathbb{B} \), then \( B_a \) is an 1-1 map on \( \mathbb{T}, \mathbb{D} \) respectively.

- The restrictions of the Blaschke functions on the set \( \mathbb{D} \) or on \( \mathbb{T} \) with the operation \( (B_{a_1} \circ B_{a_2})(z) := B_{a_1}(B_{a_2}(z)) \) form a group.
In the set of the parameters $\mathbb{B} := \mathbb{D} \times \mathbb{T}$ let us define the operation induced by the function composition in the following way $B_{a_1} \circ B_{a_2} = B_{a_1 \circ a_2}$.

$(\mathbb{B}, \circ)$ will be the Blaschke group which is isomorphic with the group $(\{B_a, a \in \mathbb{B}\}, \circ)$. 
The voice transform of the Blaschke group

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\end{align*}
\]

- The neutral element of the group $(\mathbb{B}, \circ)$ is $e := (0, 1) \in \mathbb{B}$ and the inverse element of $a = (b, \epsilon) \in \mathbb{B}$ is $a^{-1} = (-b\epsilon, \bar{\epsilon})$. 

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The voice transform of the Blaschke group

- The integral of the function \( f : \mathbb{B} \rightarrow \mathbb{C} \), with respect to this left invariant Haar measure \( m \) of the group \((\mathbb{B}, \circ)\), is given by

\[
\int_{\mathbb{B}} f(a) \, dm(a) = \frac{1}{2\pi} \int_{I} \int_{D} \frac{f(b, e^{it})}{(1 - |b|^2)^2} \, db_1 \, db_2 \, dt,
\]

where \( a = (b, e^{it}) = (b_1 + ib_2, e^{it}) \in \mathbb{D} \times \mathbb{T} \).

- It can be shown that this integral is invariant under the inverse transformation \( a \rightarrow a^{-1} \), so this group is unimodular.
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The representation of the Blaschke group on $H^2(\mathbb{T})$

- Denote by $\epsilon_n(t) = e^{int}$ ($t \in \mathbb{I} = [0, 2\pi]$, $n \in \mathbb{N}$), let consider the Hilbert space $H = H^2(\mathbb{T})$, the closure in $L^2(\mathbb{T})$-norm of the set

\[
\text{span}\{\epsilon_n, n \in \mathbb{N}\}.
\]

- The inner product is given by

\[
\langle f, g \rangle := \frac{1}{2\pi} \int_\mathbb{I} f(e^{it})\overline{g(e^{it})} \, dt \quad (f, g \in H).
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$$ (V_\rho f)(a^{-1}) := \langle f, U_{a^{-1}} \rho \rangle \ (f, \rho \in H^2(\mathbb{T})). $$

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The matrix elements of representations of the Blaschke group

- The matrix elements can be expressed by Zernike functions.
- The matrix elements $v_{mn}(a^{-1}) := \langle \epsilon_n, U_{a^{-1}} \epsilon_m \rangle$ of representation $U$ with respect to the basis $\{\epsilon_n : n \in \mathbb{N}\}$ and $a = (re^{i\varphi}, e^{i\psi})$ are given by

$$v_{mn}(a^{-1}) = \sqrt{1 - r^2} \frac{m + n + 1}{m!} e^{-i((m+1)/2)\psi} Z_{\min\{n, m\}}(r, \varphi).$$

An important consequence of this connection is the addition formula for Zernike functions.
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\[
v_{mn}(a^{-1}) = \frac{\sqrt{1 - r^2}}{\sqrt{m + n + 1}} e^{-i(m+1/2)\psi} (-1)^m Z_{\min\{n,m\}}^{m-n}(r, \varphi).
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- It is known that in general the matrix elements of the representations satisfy the following so called addition formula:

\[
v_{mn}(a_1 \circ a_2) = \sum_k v_{mk}(a_1)v_{kn}(a_2) \ (a_1, a_2 \in \mathbb{B}).
\]

- From this relation we obtain the following addition formula for Zernike functions: if \( a_j := (r_j e^{i\varphi_j}, e^{i\psi_j}), j \in \{1, 2\} \) and \( a := (re^{i\varphi}, e^{i\psi}) = a_1 \circ a_2 \) then

\[
\sqrt{1 - r^2} e^{i(m+1/2)\psi} Z_{\min\{m,n\}}(r, \varphi) = \\
\sqrt{(n + m + 1)(1 - r^2_1)(1 - r^2_2)} \\
\sum_k (-1)^k e^{-i(m+1/2)\psi_1} e^{-i(k+1/2)\psi_2} Z_{\min\{m,k\}}(r_1, \varphi_1) Z_{\min\{k,n\}}(r_2, \varphi_2).
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\[
\frac{\sqrt{1 - r^2}}{\sqrt{(n + m + 1)(1 - r_1^2)(1 - r_2^2)}} e^{-i(m+1/2)\psi} Z_{\text{min}\{m,n\}}^{|n-m|}(r, \varphi) = \\
\sum_k (-1)^k \frac{e^{-i(m+1/2)\psi_1} e^{-i(k+1/2)\psi_2}}{\sqrt{(m + k + 1)(n + k + 1)}} Z_{\text{min}\{m,k\}}^{|k-m|}(r_1, \varphi_1) Z_{\text{min}\{k,n\}}^{|n-k|} (r_2, \varphi_2) .
\]
### Discrete Laguerre functions

Let \( \varphi = 1 \) be the *mother wavelet*, the shift operator:

\[
(S\varphi)(z) = z\varphi(z) \quad (\varphi \in H^2(\mathbb{D}), \ z \in \mathbb{D} \cup \mathbb{T}).
\]

Then the *discrete Laguerre* functions

\[
\varphi_{a,m}(z) := (U_{a-1}S^m\varphi)(z) = \sqrt{\epsilon(1 - |b|^2)} \left( \frac{\epsilon(z - b)}{1 - \bar{b}z} \right)^m.
\]

Let \( V_{\epsilon_m}f(a^{-1}) = \langle f, U_{a^{-1}\epsilon_m} \rangle \) and let define the following projection operator

\[
Pf(a, z) := \sum_{m=0}^{\infty} (V_{\epsilon_m}f)(a^{-1})\varphi_{a,m}(z),
\]
Properties

Reconstruction formula

**Theorem**  For every $f \in H^2(\mathbb{T})$, for every $z = r_1 e^{it} \in \mathbb{D}$ and for every $a \in \mathbb{B}$

$$\lim_{r_1 \to 1} Pf(a, z) = f(e^{it})$$

a.e. $t \in I$ and in $H^2$ norm. If $f \in C(\mathbb{T})$, then the convergence is uniform.
Properties

Infinite series representation of voice – transform

Inner product generated by the weight $w = (1 - r^2)$ on $B$

$$\langle\langle F, G \rangle\rangle := \frac{1}{2\pi} \int_0^1 \int_\pi^0 \frac{r F(re^{i\varphi}) G(re^{i\varphi})}{1 - r^2} d\varphi dr. \quad (1)$$

**Theorem** Let us consider $\rho \in H^2(\mathbb{T})$, let us denote by $b_n := \langle\rho, \epsilon_n \rangle$ and suppose that $\sum_{n=0}^\infty |b_n| < \infty$, then for all $f \in H^2(\mathbb{T})$ and $a = (re^{i\varphi}, 1) \in B$

$$V_\rho f(a^{-1}) = (V_\rho f)(re^{i\varphi}) = \sqrt{1 - r^2} \sum_{\ell=-\infty}^{\infty} r^\ell |\ell| e^{-i\ell\varphi} \sum_{n=0}^\infty c^\ell_n P^\ell_n(r^2),$$

where the infinite series is absolute convergent and convergent in norm induced by $(1)$. 
Properties

Coefficients

The coefficients $c^\ell_n$ can be expressed using the trigonometric Fourier coefficients of $f$ and $\rho$:

$$c^\ell_n := \langle f, \epsilon_n \rangle \overline{\langle \rho, \epsilon_{n+\ell} \rangle} \ (\ell \geq 0), \quad c^\ell_n := \langle f, \epsilon_{n-\ell} \rangle \overline{\langle \rho, \epsilon_n \rangle} \ (\ell < 0),$$

furthermore

$$\left\langle \left\langle V_\rho f, V_\rho f \right\rangle \right\rangle = \frac{1}{2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{|\langle f, \epsilon_n \rangle|^2 |\langle \rho, \epsilon_m \rangle|^2}{n + m + 1}.$$

$$c^\ell_n = \frac{2n + \ell + 1}{2\pi} \int_0^1 \int_{\mathbb{I}} (V_\rho f)(re^{i\varphi}) r |\ell| e^{i\ell\varphi} P^\ell_n(r^2) \frac{r}{\sqrt{1 - r^2}} d\varphi dr \ (\ell \in \mathbb{Z}, \ n \in \mathbb{N})$$
Representation of the voice – transform as a differential operator

Let fix a polynomial

\[ \kappa(z) := c_0 + c_1 z + \cdots + c_N z^N \quad (z \in \mathbb{C}) \]

and a complex number \( b \in \mathbb{C} \) and let denote by \( \mathcal{A} \) the set of analytic functions on \( \mathbb{D} \). Denote by \( \alpha_b(z) := 1 - \bar{b}z \quad (z \in \mathbb{C}) \). For every \( f \in \mathcal{A} \) let be

\[ L_b^\kappa f := \sum_{n=0}^{N} \frac{\overline{c_n}}{n!} (\alpha_b^n f)^{(n)}. \]
Properties

Representation of the voice – transform as a differential operator

For an arbitrary function

\[ f(e^{it}) = \sum_{n=-\infty}^{\infty} a_n e^{int} \quad (t \in \mathbb{I}) \]

let denote by

\[ f^*(z) := \sum_{n=0}^{\infty} a_n z^n, \quad f_*(z) = \sum_{n=0}^{\infty} a_{-n-1} z^n \quad (z \in \mathbb{D}). \]

Then \( f^*, f_* \in H^2(\mathbb{D}) \) and

\[ f(e^{it}) = f^*(e^{it}) + e^{-it} f_*(e^{-it}) \quad (\text{for almost every } t \in \mathbb{I}). \]
Properties

Representation of the voice – transform as a differential operator

**Theorem.** For every function $f \in L^2(\mathbb{T})$ and for every trigonometric polynomial $\rho \in L^2(\mathbb{T})$ the voice transform $V_{\rho}f$ of $f$ can be represented as

$$V_{\rho}f(a^{-1}) = \sqrt{1 - |b|^2}[(L_{\rho^*}^b f^*)(b)+(L_{\rho^*}^b f)(\bar{b})] \quad (a = (b,1) \in \mathbb{B}).$$
Properties

Admissible functions

**Theorem** Let suppose that $\rho$ is a real trigonometric polynomial and $\rho^*$ is an odd or even algebraic polynomial which vanishes in 0, namely

\[ b_0 = 0, \ b_k = \overline{b_{-k}} \ (N \in \mathbb{N}^*, k \leq N), \ \rho^*(-z) = \pm \rho^*(z) \ (z \in \mathbb{D}). \]

Then $\rho$ is an admissible function for the representation $U_a$ which means that $V_\rho \rho \in L_m^2(\mathbb{B})$. 

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References


References


References


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László Lovász reminded us that "Mathematics is queen and servant of science" (Eric Temple Bell). The different branches of science get increasingly more distance from each other with the deepening of the knowledge, the dialogue becomes increasingly heavier between them. For the world’s accurate understanding at the same time is necessary the contact of disciplines.

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