Lempert Theorem for $C^2$-smooth strongly linearly convex domains

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This presentation is based on
and
Definition (Strong linear convexity)

A domain $D \subset \mathbb{C}^n$ is strongly linearly convex if

- $D$ has $C^2$-smooth boundary;
- there exists a defining function $r$ of $D$ such that

$$\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k} (a) X_j \overline{X}_k > \left| \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k} (a) X_j X_k \right|,$$

where $a \in \partial D$, $X \in T_D^\mathbb{C}(a)^*$. 
Lempert function

\[ \tilde{k}_D(z, w) = \inf \{ p(\zeta, \xi) : \zeta, \xi \in D \text{ and } \exists f \in \mathcal{O}(D, D) : f(\zeta) = z, f(\xi) = w \}, \quad z, w \in D. \]
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Kobayashi-Royden (pseudo)metric

\[ \kappa_D(z; v) = \inf \{ |\lambda|^{-1}/(1 - |\zeta|^2) : \lambda \in \mathbb{C}_*, \zeta \in D \text{ and } \exists f \in O(D, D) : f(\zeta) = z, f'(\zeta) = \lambda v \}, \quad z \in D, \ v \in \mathbb{C}^n. \]
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If \( z \neq w \) (resp. \( v \neq 0 \)), a mapping for which the infimum is attained we call an extremal (\( \tilde{k}_D \)-extremal or \( \kappa_D \)-extremal)
Carathéodory (pseudo)distance

\[ c_D(z, w) = \sup \{ p(F(z), F(w)) : F \in \mathcal{O}(D, \mathbb{D}) \}, \quad z, w \in D. \]
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\[ \gamma_D(z; v) = \sup \{ |F'(z)v| : F \in \mathcal{O}(D, \mathbb{D}), F(z) = 0 \}, \quad z \in D, \quad v \in \mathbb{C}^n. \]
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**Theorem (Lempert Theorem for \( \mathcal{C}^2 \), Ł. Kosiński, T.W.)**

Let \( D \subset \mathbb{C}^n, \ n \geq 2, \) be a bounded strongly linearly convex domain. Then

\[ c_D = \tilde{\kappa}_D \quad \text{and} \quad \gamma_D = \kappa_D. \]
Definition (Stationary mapping)

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1. $f$ extends to a holomorphic mapping in a neighborhood of $\overline{D}$ (denoted by the same letter);
2. $f(T) \subset \partial D$;
3. there exists a real analytic function $\rho : T \rightarrow \mathbb{R}_{>0}$ such that the mapping $T \ni \zeta \rightarrow \zeta \rho(\zeta) \nu_D(f(\zeta)) \in \mathbb{C}^n$ extends to a mapping holomorphic in a neighborhood of $\overline{D}$ (denoted by $\tilde{f}$).
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\[ f \text{ extends to a } C^{1/2}\text{-smooth mapping on } \overline{\mathbb{D}} \text{ (denoted by the same letter)}; \]
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\[(4) \quad \text{setting } \varphi_z(\zeta) := \langle z - f(\zeta), \nu_D(f(\zeta)) \rangle, \quad \zeta \in \mathbb{T}, \quad \text{we have wind } \varphi_z = 0 \text{ for some } z \in D.\]
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Theorem (Ł. Kosiński, T.W.)

Let \( D \subset \mathbb{C}^n, \ n \geq 2, \) be a bounded strongly linearly convex domain. Then a holomorphic mapping \( f : \mathbb{D} \longrightarrow D \) is an extremal if and only if \( f \) is a weak E-mapping.
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**Theorem (Lempert Theorem)**

Let $D \subset \mathbb{C}^n$, $n \geq 2$, be a bounded strongly linearly convex domain with real analytic boundary. Then

$$c_D = \tilde{k}_D \quad \text{and} \quad \gamma_D = \kappa_D.$$
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**Theorem (Lempert)**

Let $D \subset \mathbb{C}^n$, $n \geq 2$, be a bounded strongly linearly convex domain with real analytic boundary. Then a holomorphic mapping $f : \mathbb{D} \rightarrow D$ is an extremal if and only if $f$ is an $E$-mapping.
Let $D \subset \mathbb{C}^n$, $n \geq 2$, be a bounded strongly linearly convex domain with real analytic boundary. Then a weak stationary mapping $f$ of $D$ is a stationary mapping of $D$ with the same associated mappings $\tilde{f}$, $\rho$.

Moreover, $f$ is a complex geodesic, that is $c_D(f(\zeta), f(\xi)) = p(\zeta, \xi)$ for any $\zeta, \xi \in \mathbb{D}$.
**Proposition**

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**Proposition (Uniqueness of $E$-mappings)**

For any different $z, w \in D$ (resp. for any $z \in D$, $v \in (\mathbb{C}^n)^*$) there exists a unique $E$-mapping $f : \mathbb{D} \longrightarrow D$ such that $f(0) = z$, $f(\xi) = w$ for some $\xi \in (0, 1)$ (resp. $f(0) = z$, $f'(0) = \lambda v$ for some $\lambda > 0$) (unique = with exactness to $\text{Aut}(\mathbb{D})$).
Definition (Family $\mathcal{D}(c)$)
For a given $c > 0$ let the family $\mathcal{D}(c)$ consist of all pairs $(D, z)$, where $D \subset \mathbb{C}^n$, $n \geq 2$, is a bounded strongly pseudoconvex domain with real analytic boundary and $z \in D$, satisfying

1. $\text{dist}(z, \partial D) \geq 1/c$;
2. The diameter of $D$ is not greater than $c$ and $D$ satisfies the interior ball condition with a radius $1/c$;
3. For any $x, y \in D$ there exist $m \leq 8c^2$ and open balls $B_0, \ldots, B_m \subset D$ of radii $1/(2c)$ such that $x \in B_0$, $y \in B_m$ and the distance between the centers of the balls $B_j, B_{j+1}$ is not greater than $1/(4c)$ for $j = 0, \ldots, m - 1$. 

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(a) for any $w \in \Phi(B \cap \partial D)$ there is a ball of a radius $c$ containing $\Phi(D)$ and tangent to $\partial \Phi(D)$ at $w$ (we call it the “exterior ball condition” with a radius $c$);
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(5) the normal vector $\nu_D$ is Lipschitz with a constant $2c$;

(6) an $\varepsilon$-hull of $D$, i.e. a domain $D_\varepsilon := \{w \in \mathbb{C}^n : \text{dist}(w, D) < \varepsilon\}$, is strongly pseudoconvex for any $\varepsilon \in (0, 1/c]$. 
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**Proposition (Uniform estimates)**

*Fix $c > 0$, let $(D, z) \in \mathcal{D}(c)$ and let $f : (\mathbb{D}, 0) \to (D, z)$ be an E-mapping. Then there is a uniform $C > 0$ (depending only on $c$) such that*
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- $|\rho(\zeta_1) - \rho(\zeta_2)| \leq C \sqrt{|\zeta_1 - \zeta_2|}$, $\zeta_1, \zeta_2 \in \mathbb{T}$;
- $|\tilde{f}(\zeta_1) - \tilde{f}(\zeta_2)| \leq C \sqrt{|\zeta_1 - \zeta_2|}$, $\zeta_1, \zeta_2 \in \overline{\mathbb{D}}$. 
Proposition

A weak E-mapping $f : \mathbb{D} \to D$ of a bounded strongly linearly convex domain $D \subset \mathbb{C}^n$, $n \geq 2$, is a unique $\tilde{k}_D$-extremal for $f(\zeta), f(\xi)$ (resp. a unique $\kappa_D$-extremal for $f(\zeta), f'(\zeta)$), where $\zeta, \xi \in \mathbb{D}$, $\zeta \neq \xi$ (unique = with exactness to $\text{Aut}(\mathbb{D})$). Moreover, $f$ is a complex geodesic.
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The proof is analogous as in the real analytic case.
Lemma

Let $D \subset \subset \mathbb{B}_n$, $n \geq 2$, be a strongly pseudoconvex domain with $C^2$-smooth boundary. Take $z \in D$ and let $r$ be a defining function of $D$ such that

\begin{align*}
|\nabla r| &= 1 \text{ on } \partial D; \\
\sum_{j, k=1}^{n} \partial^2 r \partial z_j \partial z_k (a) X_j X_k &\geq C |X|^2 \text{ for any } a \in \partial D \text{ and } X \in \mathbb{C}^n \text{ with some constant } C > 0. 
\end{align*}

Suppose that there exist $C^2$-smooth functions $r_m : \mathbb{C}^n \to \mathbb{R}$ such that $\partial |\alpha| r_m \partial x_\alpha \to \partial |\alpha| r \partial x_\alpha$ uniformly on $\mathbb{B}_n$ for $\alpha \in \mathbb{N}_2$ with $|\alpha| \leq 2$. Let $D_m$ be a connected component of the set $\{x \in \mathbb{C}^n : r_m(x) < 0\}$, containing the point $z$. 
Lemma

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- $r \in C^2(\mathbb{C}^n)$;
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- $r \in C^2(\mathbb{C}^n)$;
- $D = \{x \in \mathbb{C}^n : r(x) < 0\}$;
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- \( \mathbb{C}^n \setminus \overline{D} = \{ x \in \mathbb{C}^n : r(x) > 0 \}; \)
- \( |\nabla r| = 1 \) on \( \partial D; \)
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- $\mathbb{C}^n \setminus \overline{D} = \{x \in \mathbb{C}^n : r(x) > 0\}$;
- $|\nabla r| = 1$ on $\partial D$;
- $\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k}(a)X_j\overline{X}_k \geq C|X|^2$ for any $a \in \partial D$ and $X \in \mathbb{C}^n$ with some constant $C > 0$. 
Lemma

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- $\mathbb{C}^n \setminus \overline{D} = \{x \in \mathbb{C}^n : r(x) > 0\}$;
- $|\nabla r| = 1$ on $\partial D$;
- $\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k}(a)X_j \overline{X}_k \geq C|X|^2$ for any $a \in \partial D$ and $X \in \mathbb{C}^n$ with some constant $C > 0$.

Suppose that there exist $\mathcal{C}^2$-smooth functions $r_m : \mathbb{C}^n \rightarrow \mathbb{R}$ such that $\frac{\partial^{|\alpha|} r_m}{\partial x^\alpha} \rightarrow \frac{\partial^{|\alpha|} r}{\partial x^\alpha}$ uniformly on $\overline{\mathbb{B}}_n$ for $\alpha \in \mathbb{N}_0^{2n}$ with $|\alpha| \leq 2$. 
Lemma

Let $D \subset \subset B_n$, $n \geq 2$, be a strongly pseudoconvex domain with $C^2$-smooth boundary. Take $z \in D$ and let $r$ be a defining function of $D$ such that

- $r \in C^2(\mathbb{C}^n)$;
- $D = \{x \in \mathbb{C}^n : r(x) < 0\}$;
- $\mathbb{C}^n \setminus \overline{D} = \{x \in \mathbb{C}^n : r(x) > 0\}$;
- $|\nabla r| = 1$ on $\partial D$;
- $\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial z_k}(a)X_j \overline{X}_k \geq C|X|^2$ for any $a \in \partial D$ and $X \in \mathbb{C}^n$ with some constant $C > 0$.

Suppose that there exist $C^2$-smooth functions $r_m : \mathbb{C}^n \longrightarrow \mathbb{R}$ such that $\frac{\partial^{|\alpha|} r_m}{\partial x^\alpha} \rightarrow \frac{\partial^{|\alpha|} r}{\partial x^\alpha}$ uniformly on $\overline{B}_n$ for $\alpha \in \mathbb{N}^{2n}_0$ with $|\alpha| \leq 2$. Let $D_m$ be a connected component of the set $\{x \in \mathbb{C}^n : r_m(x) < 0\}$, containing the point $z$. 
Then there is $c > 0$ such that $(D_m, z)$ and $(D, z)$ belong to $D(c)$, $m >> 1$. 
Then there is $c > 0$ such that $(D_m, z)$ and $(D, z)$ belong to $\mathcal{D}(c)$, $m \gg 1$.

We omit the very technical proof. Generally, it relies on studying functions of the form

$$\mathbb{C}^n \ni x \mapsto r_m(x) - t(|x - b|^2 - R^2) \in \mathbb{R},$$

where $t, R \in \mathbb{R}$ and $b \in \mathbb{C}^n$ are fixed.
Proofs of the main Theorems

Losing no generality $0 \in D \subset \subset \mathbb{B}_n$, let $r$ be defining for $D$ as in the lemma.
Proofs of the main Theorems

- Losing no generality $0 \in D \subset \subset \mathbb{B}_n$, let $r$ be defining for $D$ as in the lemma.

- Weierstrass Theorem $\implies \exists$ real polynomials $P_k$ on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ such that $\frac{\partial^{|\alpha|} P_k}{\partial x^\alpha} \to \frac{\partial^{|\alpha|} r}{\partial x^\alpha}$ uniformly on $\overline{B}_n$ for $\alpha \in \mathbb{N}_0^{2n}$ with $|\alpha| \leq 2$. 
Proofs of the main Theorems

- Losing no generality $0 \in D \subset \subset \mathbb{B}_n$, let $r$ be defining for $D$ as in the lemma.

- Weierstrass Theorem $\implies \exists$ real polynomials $P_k$ on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ such that $\frac{\partial^{\alpha} P_k}{\partial x^\alpha} \rightarrow \frac{\partial^{\alpha} r}{\partial x^\alpha}$ uniformly on $\mathbb{B}_n$ for $\alpha \in \mathbb{N}_0^{2n}$ with $|\alpha| \leq 2$.

- $D_{k,\varepsilon} := \{x \in \mathbb{C}^n : P_k(x) + \varepsilon < 0\}$
Proofs of the main Theorems

- Losing no generality $0 \in D \subset \subset B_n$, let $r$ be defining for $D$ as in the lemma.

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- $D_{k,\varepsilon} := \{x \in \mathbb{C}^n : P_k(x) + \varepsilon < 0\}$

- $\varepsilon_m \to 0$, $0 < 3\varepsilon_{m+1} < \varepsilon_m$
Proofs of the main Theorems

- Losing no generality $0 \in D \subset \subset B_n$, let $r$ be defining for $D$ as in the lemma.

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- $D_{k,\varepsilon} := \{ x \in \mathbb{C}^n : P_k(x) + \varepsilon < 0 \}$

- $\varepsilon_m \to 0$, $0 < 3\varepsilon_{m+1} < \varepsilon_m$

- $\forall m \exists k_m : \| P_{k_m} - r \|_{\overline{B}_n} < \varepsilon_m$
Proofs of the main Theorems

- Losing no generality $0 \in D \subset \subset \mathbb{B}_n$, let $r$ be defining for $D$ as in the lemma.

- Weierstrass Theorem $\implies \exists$ real polynomials $P_k$ on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ such that $\frac{\partial^{\lvert \alpha \rvert} P_k}{\partial x^\alpha} \rightarrow \frac{\partial^{\lvert \alpha \rvert} r}{\partial x^\alpha}$ uniformly on $\overline{\mathbb{B}}_n$ for $\alpha \in \mathbb{N}_0^{2n}$ with $\lvert \alpha \rvert \leq 2$.

- $D_{k,\varepsilon} := \{ x \in \mathbb{C}^n : P_k(x) + \varepsilon < 0 \}$

- $\varepsilon_m \rightarrow 0$, $0 < 3\varepsilon_{m+1} < \varepsilon_m$

- $\forall m \exists k_m : \| P_{k_m} - r \|_{\overline{\mathbb{B}}_n} < \varepsilon_m$

- $r_m := P_{k_m} + 2\varepsilon_m \implies r < r_{m+1} < r_m$ in $\overline{\mathbb{B}}_n$
Proofs of the main Theorems

- Losing no generality $0 \in D \subset \subset \mathbb{B}_n$, let $r$ be defining for $D$ as in the lemma.

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- $\varepsilon_m \to 0$, $0 < 3\varepsilon_{m+1} < \varepsilon_m$

- $\forall m \exists k_m : \| P_{k_m} - r \|_{\mathbb{B}_n} < \varepsilon_m$

- $r_m := P_{k_m} + 2\varepsilon_m \implies r < r_{m+1} < r_m$ in $\overline{\mathbb{B}}_n$

- $D_m :=$ a connected component of $D_{k_m,2\varepsilon_m}$ containing 0.
Proofs of the main Theorems

- Losing no generality $0 \in D \subset \subset \mathbb{B}_n$, let $r$ be defining for $D$ as in the lemma.

- Weierstrass Theorem $\implies \exists$ real polynomials $P_k$ on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ such that $\frac{\partial^{\lvert \alpha \rvert} P_k}{\partial x^{\alpha}} \to \frac{\partial^{\lvert \alpha \rvert} r}{\partial x^{\alpha}}$ uniformly on $\overline{\mathbb{B}}_n$ for $\alpha \in \mathbb{N}^{2n}_0$ with $|\alpha| \leq 2$.

- $D_{k,\varepsilon} := \{x \in \mathbb{C}^n : P_k(x) + \varepsilon < 0\}$

- $\varepsilon_m \to 0$, $0 < 3\varepsilon_{m+1} < \varepsilon_m$

- $\forall m \exists k_m : \|P_{k_m} - r\|_{\overline{\mathbb{B}}_n} < \varepsilon_m$

- $r_m := P_{k_m} + 2\varepsilon_m \implies r < r_{m+1} < r_m$ in $\overline{\mathbb{B}}_n$

- $D_m :=$ a connected component of $D_{k_m,2\varepsilon_m}$ containing 0

- $D_m$ is a bounded strongly linearly convex domain with real analytic boundary and $r_m$ is its defining function for $m >> 1$
Proofs of the main Theorems

Losing no generality $0 \in D \subset \subset \mathbb{B}_n$, let $r$ be defining for $D$ as in the lemma.

Weierstrass Theorem $\implies \exists$ real polynomials $P_k$ on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ such that $\frac{\partial|\alpha|P_k}{\partial x^\alpha} \rightarrow \frac{\partial|\alpha|r}{\partial x^\alpha}$ uniformly on $\overline{\mathbb{B}}_n$ for $\alpha \in \mathbb{N}_0^{2n}$ with $|\alpha| \leq 2$

$D_{k,\varepsilon} := \{x \in \mathbb{C}^n : P_k(x) + \varepsilon < 0\}$

$\varepsilon_m \to 0$, $0 < 3\varepsilon_{m+1} < \varepsilon_m$

$\forall m \exists k_m : \|P_{km} - r\|_{\overline{\mathbb{B}}_n} < \varepsilon_m$

$r_m := P_{km} + 2\varepsilon_m \implies r < r_{m+1} < r_m$ in $\overline{\mathbb{B}}_n$

$D_m := a$ connected component of $D_{km,2\varepsilon_m}$ containing $0$

$D_m$ is a bounded strongly linearly convex domain with real analytic boundary and $r_m$ is its defining function for $m >> 1$

$D_m \subset D_{m+1}, \bigcup_m D_m = D \implies$ Lempert Theorem for $\mathcal{C}^2$. 
Proofs of the main Theorems

It suffices to show that for any different \( z, w \in D \) (resp. \( z \in D, v \in (\mathbb{C}^n)_\ast \)) there is a weak \( E \)-mapping for \( z, w \) (resp. for \( z, v \))
Proofs of the main Theorems

- It suffices to show that for any different \( z, w \in D \) (resp. \( z \in D, \nu \in (\mathbb{C}^n)^* \)) there is a weak \( E \)-mapping for \( z, w \) (resp. for \( z, \nu \))

- \( z, w \in D_m \) (resp. \( z \in D_m \)) for \( m \gg 1 \iff \exists \) an \( E \)-mapping \( f_m \) of \( D_m \) s.t. \( f_m(0) = z, f_m(\xi_m) = w, \xi_m \in (0, 1) \)
Proofs of the main Theorems

- It suffices to show that for any different $z, w \in D$ (resp. $z \in D, v \in (\mathbb{C}^n)_*$) there is a weak $E$-mapping for $z, w$ (resp. for $z, v$)

- $z, w \in D_m$ (resp. $z \in D_m$) for $m >> 1 \iff \exists$ an $E$-mapping $f_m$ of $D_m$ s.t. $f_m(0) = z, f_m(\xi_m) = w, \xi_m \in (0, 1)$ (resp. $f_m(0) = z, f'_m(0) = \lambda_m v, \lambda_m > 0$)
Proofs of the main Theorems

- It suffices to show that for any different \( z, w \in D \) (resp. \( z \in D, v \in (\mathbb{C}^n)_* \)) there is a weak \( E \)-mapping for \( z, w \) (resp. for \( z, v \))

- \( z, w \in D_m \) (resp. \( z \in D_m \)) for \( m \gg 1 \) \( \implies \exists \) an \( E \)-mapping \( f_m \) of \( D_m \) s.t. \( f_m(0) = z, f_m(\xi_m) = w, \xi_m \in (0, 1) \) (resp. \( f_m(0) = z, f'_m(0) = \lambda_m v, \lambda_m > 0 \))

- \( D_m \subset D_{m+1} \subset \mathbb{B}_n \) and \( f_m \) are complex geodesics \( \implies \exists \) compact \( K \subset (0, 1) \) s.t. \( \{\xi_m\} \subset K \)
Proofs of the main Theorems

- It suffices to show that for any different \( z, w \in D \) (resp. \( z \in D, \nu \in (\mathbb{C}^n)_* \)) there is a weak E-mapping for \( z, w \) (resp. for \( z, \nu \))

- \( z, w \in D_m \) (resp. \( z \in D_m \)) for \( m \gg 1 \) \( \implies \) \( \exists \) an E-mapping \( f_m \) of \( D_m \) s.t. \( f_m(0) = z, f_m(\xi_m) = w, \xi_m \in (0,1) \) (resp. \( f_m(0) = z, f'_m(0) = \lambda_m \nu, \lambda_m > 0 \))

- \( D_m \subset D_{m+1} \subset \mathbb{B}_n \) and \( f_m \) are complex geodesics \( \implies \) \( \exists \) compact \( K \subset (0,1) \) s.t. \( \{\xi_m\} \subset K \) (resp. \( \exists \) compact \( \widetilde{K} \subset (0,\infty) \) s.t. \( \{\lambda_m\} \subset \widetilde{K} \)
Proofs of the main Theorems

- It suffices to show that for any different \( z, w \in D \) (resp. \( z \in D, \nu \in (\mathbb{C}^n)_* \)) there is a weak \( E \)-mapping for \( z, w \) (resp. for \( z, \nu \))

- \( z, w \in D_m \) (resp. \( z \in D_m \)) for \( m >> 1 \) \( \implies \exists \) an \( E \)-mapping \( f_m \) of \( D_m \) s.t. \( f_m(0) = z, f_m(\xi_m) = w, \xi_m \in (0, 1) \)
  (resp. \( f_m(0) = z, f_m'(0) = \lambda_m \nu, \lambda_m > 0 \))

- \( D_m \subset D_{m+1} \subset \mathbb{B}_n \) and \( f_m \) are complex geodesics \( \implies \exists \) compact \( K \subset (0, 1) \) s.t. \( \{\xi_m\} \subset K \)
  (resp. \( \exists \) compact \( \tilde{K} \subset (0, \infty) \) s.t. \( \{\lambda_m\} \subset \tilde{K} \))

- \( \exists c > 0 : (D_m, z) \in \mathcal{D}(c), m >> 1 \)
Proofs of the main Theorems

- It suffices to show that for any different $z, w \in D$ (resp. $z \in D, v \in (\mathbb{C}^n)_*$) there is a weak $E$-mapping for $z, w$ (resp. for $z, v$)
- $z, w \in D_m$ (resp. $z \in D_m$) for $m \gg 1 \implies \exists$ an $E$-mapping $f_m$ of $D_m$ s.t. $f_m(0) = z, f_m(\xi_m) = w, \xi_m \in (0, 1)$ (resp. $f_m(0) = z, f'_m(0) = \lambda_m v, \lambda_m > 0$)
- $D_m \subset D_{m+1} \subset \mathbb{B}_n$ and $f_m$ are complex geodesics $\implies \exists$ compact $K \subset (0, 1)$ s.t. $\{\xi_m\} \subset K$ (resp. $\exists$ compact $\tilde{K} \subset (0, \infty)$ s.t. $\{\lambda_m\} \subset \tilde{K}$)
- $\exists c > 0 : (D_m, z) \in \mathcal{D}(c), m \gg 1$
- $f_m, \tilde{f}_m$ and $\rho_m$ satisfy the uniform estimates
Proofs of the main Theorems

- Arzela-Ascoli Theorem + passing to subsequences \implies
  \[ f_m \to f \text{ uniformly on } \overline{D} \implies f \in O(D) \cap C^{1/2}(D) \]
Proofs of the main Theorems

- **Arzela-Ascoli Theorem + passing to subsequences**

  - $f_m \to f$ uniformly on $\overline{D} \implies f \in O(D) \cap C^{1/2}(\overline{D})$
  - $\tilde{f}_m \to \tilde{f}$ uniformly on $\overline{D} \implies \tilde{f} \in O(D) \cap C^{1/2}(\overline{D})$

$D$ is strongly pseudoconvex $\implies f(D) \subset D$

The conditions (3') and (4) from the definition of a weak $E$-mapping follow from the uniform convergence of suitable functions $f$ is a weak $E$-mapping of $D$.
Proofs of the main Theorems

- Arzela-Ascoli Theorem + passing to subsequences $\implies$
  - $f_m \to f$ uniformly on $\overline{D} \implies f \in \mathcal{O}(\mathbb{D}) \cap C^{1/2}(\mathbb{D})$
  - $\tilde{f}_m \to \tilde{f}$ uniformly on $\overline{D} \implies \tilde{f} \in \mathcal{O}(\mathbb{D}) \cap C^{1/2}(\mathbb{D})$
  - $\rho_m \to \rho$ uniformly on $\mathbb{T} \implies \rho \in C^{1/2}(\mathbb{T}), \rho > 0$
Proofs of the main Theorems

- Arzela-Ascoli Theorem + passing to subsequences $\implies$
  - $f_m \to f$ uniformly on $\overline{D} \implies f \in \mathcal{O}(D) \cap C^{1/2}(\overline{D})$
  - $\tilde{f}_m \to \tilde{f}$ uniformly on $\overline{D} \implies \tilde{f} \in \mathcal{O}(D) \cap C^{1/2}(\overline{D})$
  - $\rho_m \to \rho$ uniformly on $T \implies \rho \in C^{1/2}(T), \rho > 0$
  - $\xi_m \to \xi \in (0, 1)$ (resp. $\lambda_m \to \lambda > 0$)
### Proofs of the main Theorems

- **Arzela-Ascoli Theorem + passing to subsequences** \(\implies\)
  - \(f_m \to f\) uniformly on \(\overline{D}\) \(\implies\) \(f \in \mathcal{O}(\mathbb{D}) \cap C^{1/2}(\mathbb{D})\)
  - \(\tilde{f}_m \to \tilde{f}\) uniformly on \(\overline{D}\) \(\implies\) \(\tilde{f} \in \mathcal{O}(\mathbb{D}) \cap C^{1/2}(\mathbb{D})\)
  - \(\rho_m \to \rho\) uniformly on \(\mathbb{T}\) \(\implies\) \(\rho \in C^{1/2}(\mathbb{T}), \rho > 0\)
  - \(\xi_m \to \xi \in (0, 1)\) (resp. \(\lambda_m \to \lambda > 0\))

- \(f\) passes through \(z, w\) (resp. \(f(0) = z, f'(0) = \lambda v\)), \(f(\overline{D}) \subset \overline{D}, f(\mathbb{T}) \subset \partial D\)
Proofs of the main Theorems

- **Arzela-Ascoli Theorem + passing to subsequences** \implies
  - $f_m \to f$ uniformly on $\overline{D} \implies f \in O(D) \cap C^{1/2}(\overline{D})$
  - $\tilde{f}_m \to \tilde{f}$ uniformly on $\overline{D} \implies \tilde{f} \in O(D) \cap C^{1/2}(\overline{D})$
  - $\rho_m \to \rho$ uniformly on $T \implies \rho \in C^{1/2}(T)$, $\rho > 0$
  - $\xi_m \to \xi \in (0, 1)$ (resp. $\lambda_m \to \lambda > 0$)

- $f$ passes through $z, w$ (resp. $f(0) = z$, $f'(0) = \lambda v$), $f(\overline{D}) \subset \overline{D}$, $f(T) \subset \partial D$

- $D$ is strongly pseudoconvex \implies $f(D) \subset D$
## Proofs of the main Theorems

- **Arzela-Ascoli Theorem + passing to subsequences** →
  - \( f_m \to f \) uniformly on \( \overline{D} \) → \( f \in \mathcal{O}(D) \cap \mathcal{C}^{1/2}(\overline{D}) \)
  - \( \tilde{f}_m \to \tilde{f} \) uniformly on \( \overline{D} \) → \( \tilde{f} \in \mathcal{O}(D) \cap \mathcal{C}^{1/2}(\overline{D}) \)
  - \( \rho_m \to \rho \) uniformly on \( \mathbb{T} \) → \( \rho \in \mathcal{C}^{1/2}(\mathbb{T}), \rho > 0 \)
  - \( \xi_m \to \xi \in (0, 1) \) (resp. \( \lambda_m \to \lambda > 0 \))

- \( f \) passes through \( z, w \) (resp. \( f(0) = z, f'(0) = \lambda v \)), \( f(\overline{D}) \subset \overline{D}, f(\mathbb{T}) \subset \partial D \)

- \( D \) is strongly pseudoconvex → \( f(D) \subset D \)

- The conditions (3’) and (4) from the definition of a weak \( E \)-mapping follow from the uniform convergence of suitable functions
Proofs of the main Theorems

- Arzela-Ascoli Theorem + passing to subsequences \( \implies \)
  - \( f_m \to f \) uniformly on \( \overline{D} \) \( \implies f \in \mathcal{O}(\mathbb{D}) \cap C^{1/2}(\mathbb{D}) \)
  - \( \tilde{f}_m \to \tilde{f} \) uniformly on \( \overline{D} \) \( \implies \tilde{f} \in \mathcal{O}(\mathbb{D}) \cap C^{1/2}(\mathbb{D}) \)
  - \( \rho_m \to \rho \) uniformly on \( \mathbb{T} \) \( \implies \rho \in C^{1/2}(\mathbb{T}), \rho > 0 \)
  - \( \xi_m \to \xi \in (0, 1) \) (resp. \( \lambda_m \to \lambda > 0 \))

- \( f \) passes through \( z, w \) (resp. \( f(0) = z, f'(0) = \lambda v \)), \( f(\overline{D}) \subset \overline{D}, f(\mathbb{T}) \subset \partial D \)

- \( D \) is strongly pseudoconvex \( \implies f(\mathbb{D}) \subset D \)

- The conditions (3') and (4) from the definition of a weak \( E \)-mapping follow from the uniform convergence of suitable functions

- \( f \) is a weak \( E \)-mapping of \( D \)
Thank you for attention.