Compact bilinear operators and commutators

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Outline of the talk

- From linear to bilinear theory
  - Bilinear Calderón-Zygmund operators and commutators
  - Compact bilinear operators
  - A nice connection
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Linear multipliers

\[ T(f)(x) = (k \ast f)(x) = \int_{\mathbb{R}^n} k(x - y)f(y) \, dy \]

or

\[ T(f)(x) = \int_{\mathbb{R}^n} \hat{k}(\xi) \hat{f}(\xi) e^{ix \cdot \xi} \, d\xi. \]

Trivially, \( T : L^2 \to L^2 \iff m \in L^\infty. \)

**Theorem (Hörmander-Mihlin)**

If \( |\partial^\alpha m(\xi)| \lesssim |\xi|^{-|\alpha|} \), then \( T : L^p \to L^p \), \( 1 < p < \infty \). Also, \( T : L^1 \to L^{1,\infty} \) and \( T : L^\infty \to BMO. \)

Note: \( |\partial^\alpha m(\xi)| \lesssim |\xi|^{-|\alpha|} \Rightarrow |\partial^\alpha k(y)| \lesssim |y|^{-n-|\alpha|}, y \neq 0. \)

Singular integrals of convolution type: if \( k \) decays as above and satisfies an appropriate cancelation, we have \( L^p \) results for \( T. \)
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Bilinear multipliers

\[ T(f, g)(x) = (k\ast(f \otimes g))(x, x) = \int \int k(x-y, x-z)f(y)g(z) \, dydz \]

or

\[ T(f)(x) = \int \int \hat{k}(\xi, \eta) \hat{f}(\xi)\hat{g}(\eta)e^{ix \cdot (\xi + \eta)} \, d\xi d\eta. \]

Analogously, if

\[ |\partial_\xi^\alpha \partial_\eta^\beta m(\xi, \eta)| \lesssim (|\xi| + |\eta|)^{-|\alpha|-|\beta|} \]

then

\[ |\partial_y^\alpha \partial_z^\beta k(y, z)| \lesssim (|y| + |z|)^{-2n-|\alpha|-|\beta|}. \]
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A classical result

Theorem (Coifman-Meyer)

If \( |\partial_\xi^\alpha \partial_\eta^\beta m(\xi, \eta)| \lesssim (|\xi| + |\eta|)^{-|\alpha|-|\beta|} \), then \( T : L^p \times L^q \to L^r \), \( 1 < p, q < \infty \) and \( 1/2 < r < \infty \) (as well as appropriate end-point results).

- Coifman-Meyer ('78): proof via Littlewood-Paley theory when \( r > 1 \)
- Kenig-Stein ('99), Grafakos-Torres ('02): extension to the range \( r > 1/2 \)

**Proof:** \( T \) is a bilinear operator with Calderón-Zygmund kernel. Also

\[
T(e^{i\xi \cdot}, e^{i\eta \cdot})(x) = m(\xi, \eta) e^{ix \cdot (\xi+\eta)}
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which is in \( L^\infty \) (and thus BMO) uniformly in \( \xi, \eta \). The same applies to the transposes of \( T \). One can then apply the bilinear \( T(1) \) for CZ operators to conclude.
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**Theorem (Coifman-Meyer)**

If $|\partial_\xi^\alpha \partial_\eta^\beta m(\xi, \eta)| \lesssim (|\xi| + |\eta|)^{-|\alpha| - |\beta|}$, then $T : \mathbb{L}^p \times \mathbb{L}^q \rightarrow \mathbb{L}^r$, $1 < p, q < \infty$ and $1/2 < r < \infty$ (as well as appropriate end-point results).

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Bilinear operators with Calderón-Zygmund kernels

\[ T(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y, z) f(y) g(z) \, dy \, dz \]

Away from the diagonal \( x = y = z \), the kernel \( K \) satisfies

\[ |K(x, y, z)| \lesssim (|x - y| + |x - z| + |y - z|)^{-2n} \quad (1) \]

and

\[ |K(x, y, z) - K(x', y, z)| \lesssim \frac{|x - x'|^\varepsilon}{(|x - y| + |x - z| + |y - z|)^{2n + \varepsilon}} \quad (2) \]

for some \( \varepsilon \in (0, 1] \) whenever \( |x - x'| \leq \frac{1}{2} \max\{|x - y|, |x - z|\} \).

For symmetry and interpolation purposes we require that \( K^1(x, y, z) = K(y, x, z) \) and \( K^2(x, y, z) = K(z, y, x) \) also satisfy (2).
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For simplicity, in the following we will replace the size and regularity conditions (1)-(2) on $K$ with:

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|\partial^{\beta}K(x, y, z)| \lesssim \left(|x - y| + |y - z| + |z - x|\right)^{-2n-|\beta|}, \quad |\beta| \leq 1.
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- We say that such a kernel $K(x, y, z)$ is a \textit{bilinear Calderón-Zygmund (CZ) kernel}.
- An operator $T$ is a \textit{bilinear CZ operator} if it extends to a bounded operator from $L^{p_0} \times L^{q_0}$ into $L^{r_0}$ for some $1 < p_0, q_0 < \infty$ and $1/p_0 + 1/q_0 = 1/r_0 \leq 1$. 
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Bilinear commutators

For a bilinear operator $T$, and $b, b_1, b_2$ some appropriately smooth functions, we are interested in the following three bilinear commutators:

$$(C1)[T, b]_1(f, g) = T(bf, g) - bT(f, g),$$

$$(C2)[T, b]_2(f, g) = T(f, bg) - bT(f, g),$$

$$(C3)[[T, b_1]_1, b_2]_2(f, g) = [T, b_1]_1(f, b_2g) - b_2[T, b_1]_1(f, g).$$

Formally, if $T$ has kernel $K$, then

$$(C1) = \int \int K(x, y, z)(b(y) - b(x))f(y)g(z) \, dydz,$$

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Theorem

If $T$ is a bilinear CZ operator with kernel $K$ and $b, b_1, b_2 \in BMO$, then $T$ is bounded from $L^p \times L^q \rightarrow L^r$ with $1/p + 1/q = 1/r$ for all $1 < p, q < \infty$. Moreover, the following estimates hold:

$$\|[T, b]_1(f, g)\|_{L^r}, \|[T, b]_2(f, g)\|_{L^r} \lesssim \|b\|_{BMO} \|f\|_{L^p} \|g\|_{L^q} \quad (3)$$

$$\|[[T, b_1], b_2](f, g)\|_{L^r} \lesssim \|b_1\|_{BMO} \|b_2\|_{BMO} \|f\|_{L^p} \|g\|_{L^q}. \quad (4)$$

- Perez-Torres ('03), Tang ('08),
- Lerner-Ombrosi-Pérez-Torres-Trujillo González ('09),
- Perez-Pradolini-Torres-Trujillo Gonzalez ('11)

Question

Do these bilinear commutators behave “better” if the multiplicative functions $b, b_1, b_2$ are assumed to be smoother?
Boundedness of commutators

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Do these bilinear commutators behave “better” if the multiplicative functions $b, b_1, b_2$ are assumed to be smoother?
The main question asked before has a positive answer in the linear case. Denote by $CMO$ the closure of $C_c^\infty$ in the $BMO$ topology.

**Theorem (Uchiyama, ’78)**

*If $T$ is a linear CZ operator and $b \in CMO$, then $[T, b]$ is a compact operator from $L^p \to L^p$, $p > 1$."

Some relevant applications:

- Coifman-Lions-Meyer-Semmes (’93): Compensated compactness
- Iwaniec-Sbordone (’98): A Fredholm alternative for equations with $CMO$ coefficients in all $L^p$ spaces with $1 < p < \infty$
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Compact bilinear operators

Definition (Calderón, ’64)

Let $X$, $Y$, $Z$ be three normed spaces and $T : X \times Y \rightarrow Z$ a bilinear operator. $T$ is called compact if $\{ T(x, y) : \|x\|, \|y\| \leq 1 \}$ is precompact in $Z$.

- B.-Torres (’12): (a) Several natural equivalent statements; (b) If $Z$ is Banach, the space of compact bilinear operators is a closed linear subspace of the space of $X \times Y \rightarrow Z$ bounded operators.
- Note that if $T$ is bilinear compact, then the sections $T_x$, $T_y$ are linear compact operators for all $x \in X$, $y \in Y$.
- $T : C[0, 1] \times C[0, 1] \rightarrow C[0, 1]$ (endowed with the supremum norm), $T(f, g) = f \cdot g$, is bounded but not compact; for example, because $T_{f=1} = I_d$ is not compact (by Riesz’s theorem).
- $S : C[0, 1] \times C[0, 1] \rightarrow C[0, 1]$ (endowed with the supremum norm), $S(f, g)(x) = \int_0^x f(t)g(t) \, dt$, is bilinear compact (via Arzelà-Ascoli’s theorem).
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• The notion of compactness in multilinear setting was only considered in the context of interpolation (Fernandez-da Silva, ’10)

But now we have:

**Theorem (B.-Torres, ’12)**

Let $T$ be a bilinear CZ operator. If $b \in \text{CMO}$, $1/p + 1/q = 1/r$, $1 < p, q < \infty$ and $1 \leq r < \infty$, then $[T, b]_1 : L^p \times L^q \to L^r$ is compact. Similarly, if $b_1, b_2$ are also in CMO, then $[T, b]_2$ and $[T, b_1]_1, b_2]_2$ are compact for the same range of exponents.
The notion of compactness in multilinear setting was only considered in the context of interpolation (Fernandez-da Silva, ’10). But now we have:

**Theorem (B.-Torres, ’12)**

Let $T$ be a bilinear CZ operator. If $b \in \text{CMO}$, $1/p + 1/q = 1/r$, $1 < p, q < \infty$ and $1 \leq r < \infty$, then $[T, b]_1 : L^p \times L^q \to L^r$ is compact. Similarly, if $b_1, b_2$ are also in CMO, then $[T, b]_2$ and $[T, b_1]_1, b_2]_2$ are compact for the same range of exponents.
The proof for the first commutator

• Relies on the Fréchet-Kolmogorov theorem characterizing the pre-compactness of a set in $L^r$
• It is enough to show that for $b, f, g \in C_c^\infty$ the following estimates hold:
  (a) Given $\epsilon > 0$, there exists an $A > 0$ ($A = A(\epsilon)$ but independent of $f$ and $g$) with the property that
  $$\left( \int_{|x| > A} |[T, b]_1(f, g)(x)|^r \, dx \right)^{1/r} \lesssim \epsilon \|f\|_{L^p} \|g\|_{L^q}.$$
  (b) Given $\epsilon \in (0, 1)$ there exists a sufficiently small $t_0$ ($t_0 = t_0(\epsilon)$ but independent of $f$ and $g$) such that for all $0 < |t| < t_0$,
  $$|[T, b]_1(f, g)(\cdot) - [T, b]_1(f, g)(\cdot + t)|_{L^r} \lesssim \epsilon \|f\|_{L^p} \|g\|_{L^q}.$$
• The previous estimates emphasize the cancelation phenomenon of the commutators.
The proof of estimate (a)

- Pick $A > 1$ (large), $A > 2 \max\{|y| : y \in \text{supp} \, b\}$. Let $|x| > A$.

\[
|\langle T, b \rangle_1(f, g)(x) | \leq \int \int_{y \in \text{supp} \, b} |K(x, y, z)| |b(y)||f(y)||g(z)| \, dy \, dz
\]
\[
\leq \|b\|_{L^\infty} \int \int_{y \in \text{supp} \, b} \frac{|f(y)||g(z)|}{(|x - y| + |x - z|)^{2n}} \, dy \, dz
\]
\[
\leq \int \int_{y \in \text{supp} \, b} \frac{|f(y)|}{|x - y|^n} \int \frac{|g(z)|}{(|x - y| + |x - z|)^n} \, dz \, dy
\]
\[
\leq 2^n |x|^{-n} \int_{y \in \text{supp} \, b} |f(y)| \left( \int (|x - y| + |x - z|)^{-nq'} \, dz \right)^{1/q'} \, dy \|g\|_{L^q}
\]
\[
\leq 2^n |x|^{-n} |\text{supp} \, b|^{1/p'} \|f\|_{L^p} \left( \int (1/2 + |z|^{-nq'}) \, dz \right)^{1/q'} \|g\|_{L^q}
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\lesssim |x|^{-n} |\text{supp} \, b|^{1/p'} \|f\|_{L^p} \|g\|_{L^q}.
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Now integrate $|\langle T, b \rangle_1(f, g)(x) |^r$ over $|x| > A$. 15
The proof of estimate (a)

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Now integrate $|[T, b]_1(f, g)(x)|^r$ over $|x| > A$. 

[End of page]
The proof of estimate (b)

- More involved: a further decomposition is required.
- Controlling each term in this decomposition uses:
  1. a variant of the maximal truncated bilinear singular integral
  2. the smoothness estimate of the kernel
  3. the $L^p$ boundedness of the Hardy-Littlewood maximal function
- The second commutator is handled similarly.
- The second order commutator $[[T, b_1], b_2]$ is harder to study in general for symbols in $BMO$. It is in fact easier when the symbols are in $CMO$ because of extra cancelations!
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We say that a symbol \( \sigma(x, \xi, \eta) \) belongs to \( BS_{\rho,\delta}^m \) if

\[
|\partial_x^\alpha \partial_{\xi}^\beta \partial_{\eta}^\gamma \sigma(x, \xi, \eta)| \lesssim (1 + |\xi| + |\eta|)^{m+\delta|\alpha|-\rho(|\beta|+|\gamma|)}
\]

for all \((x, \xi, \eta) \in \mathbb{R}^{3n}\) and all multi-indices \(\alpha, \beta, \gamma\).

Associated to such a bilinear symbol, we have a bilinear pseudodifferential operator

\[
T_\sigma(f, g)(x) = \int \int \sigma(x, \xi, \eta) \hat{f}(\xi) \hat{g}(\eta) e^{ix \cdot (\xi + \eta)} d\xi d\eta
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- The class \( BS_{1,0}^0 \) includes \( x \)-dependent symbols that generalize the bilinear multipliers of Coifman-Meyer.
We say that a symbol $\sigma(x, \xi, \eta)$ belongs to $BS^m_{\rho, \delta}$ if

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Grafakos-Torres ('02): Bilinear CZ kernels correspond to bilinear pseudodifferential symbols in the class $BS_{1,1}^0$.

B.-Torres ('03): This class is “forbidden”: in general, for $\sigma \in BS_{1,1}^0$, $T_\sigma : L^2 \times L^2 \not\rightarrow L^1$; substitute results on products of Sobolev spaces.

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Corollary

Let $\sigma \in BS^0_{1,\delta}$, $0 \leq \delta < 1$, and $T_\sigma$ the associated BPSDO. If $b \in CMO$, $1/p + 1/q = 1/r$, $1 < p, q < \infty$ and $1 \leq r < \infty$, then $[T, b]_1 : L^p \times L^q \to L^r$ is compact. Similarly, if $b_1, b_2$ are also in $CMO$, then $[T, b]_2$ and $[T, b_1]_1, b_2]_2$ are compact for the same range of exponents.
A more general scheme

- Bilinear CZ operators are just a family (corresponding to $\alpha = 0$) that belongs to a more general class of bilinear operators $\{ T_\alpha \}_{\alpha \geq 0}$, where

$$T_\alpha(f, g)(x) = \int_{\mathbb{R}^{2n}} K_\alpha(x, y, z)f(y)g(z) \, dydz;$$

The kernel $K_\alpha$ satisfies

$$|\partial^\beta K_\alpha(x, y, z)| \lesssim (|x - y| + |y - z| + |x - z|)^{-2n+\alpha-|\beta|}, |\beta| \leq 1.$$

Example

$$K_\alpha(x, y, z) = (|x - y| + |x - z|)^{-2n+\alpha}, 0 < \alpha < 2n;$$

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Theorem (B.-Moen-Torres, ’12)

Suppose $0 \leq \alpha < n$, $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} \leq 1$ and

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{\alpha}{n}.$$ 

Let $T_{\alpha}$ be the bilinear operator whose kernel $K_{\alpha}$ satisfies size and regularity conditions as above, and $b \in \text{CMO}$. Then

$[b, T_{\alpha}]_1, [b, T_{\alpha}]_2 : L^p \times L^q \to L^r$ are compact.

The same statement holds for the second order commutator.
The proof of estimate (a)

\[ |[b, T_\alpha]_1(f, g)(x)| \lesssim \|b\|_\infty \int \int_{y \in \text{supp } b} \frac{|f(y)||g(z)|}{(|x-y| + |x-z|)^{2n-\alpha}} \, dy \, dz \]

\[ \lesssim \frac{\|b\|_\infty}{|x|^{2n-\alpha}} \int_{y \in \text{supp } b} |f(y)| \int \frac{|g(z)|}{\left(\frac{1}{2} + \frac{|x-z|}{|x|}\right)^{2n-\alpha}} \, dz \, dy \]

\[ \lesssim \frac{\|b\|_\infty \|g\| \|L^q\}}{|x|^{2n-\alpha}} \int_{y \in \text{supp } b} |f(y)| \left(\int \left(\frac{1}{2} + \frac{|x-z|}{|x|}\right)^{-(2n-\alpha)q'} dz\right)^{1/q'} \]

\[ = \frac{\|b\|_\infty \|g\| \|L^q\}}{|x|^{2n-\alpha - \frac{n}{q'}}} \int_{\text{supp } b} |f(y)| \left(\int \left(\frac{1}{2} + |z|\right)^{-(2n-\alpha)q'} dz\right)^{1/q'} \, dy \]

\[ \lesssim \frac{\|b\|_\infty \|g\| \|f\| \|L^p\}}{|x|^{2n-\alpha - \frac{n}{q'}}} |\text{supp } b|^{1/p'} . \]

We used \((2n - \alpha)q' > n \iff 1 + \frac{1}{q} > \frac{\alpha}{n}\) and

\[ r(2n - \alpha - \frac{n}{q'}) > n \iff \frac{1}{q} + 1 - \frac{\alpha}{n} > \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{\alpha}{n} . \]
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Thank you!